Lewis Carroll, Gentzen and Entailment Relations

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1 Introduction

This note summarizes some remarks about entailment relations. Basically these are elementary facts about propositional logic, but I find it suggestive to formulate it in the simple framework of inference rules (no logical connective is needed!).

2 Entailment Relation

Let S be a set, we think of its elements as abstract "statements" or propositions. We denote by $X, Y, Z, \ldots$ arbitrary finite subsets of S. We write X, Y for $X \cup Y$ and X, s for $X \cup \{s\}$.

Definition 2.1 An entailment relation $\rightarrow$ on the set S is a relation satisfying the following conditions of reflexivity, monotonicity and transitivity:

$$X \rightarrow Y \textit{ if } X \cap Y \textit{ is inhabited} \tag{R}$$

$$\frac{X \rightarrow Y}{X, X' \rightarrow Y, Y'} \tag{M}$$

$$\frac{X \rightarrow s, Z \qquad X, s \rightarrow Z}{X \rightarrow Z} \tag{T}$$

Notice that this definition is "symmetric": the converse of an entailment relation is also an entailment relation.

In presence of (M) the rule (T) is actually equivalent to the following rule:

$$\frac{X \rightarrow s, Z \qquad X', s \rightarrow Z'}{X, X' \rightarrow Z, Z'}$$

As emphasised by Scott [Sco74], this notion of entailment relation can be seen as an abstract generalisation of Gentzen's multi-conclusion sequent calculus.

3 Completeness

Scott proves a completeness theorem that can be seen as the purest form of completeness theorem for a logic (recall that S is *any* set, not necessarily a set of syntactical formulae).

Theorem: An entailment is a semantical consequence of a given set of rules iff it can be deduced by the three rules above.

This is actually equivalent to the completeness of Quine's method for enumerating all prime implicants of a given formule in conjunctive normal forms.

It is quite remarkable that we can enumerate in this way all interesting consequences of a given set of axioms. This is quite similar to Knuth-Bendix algorithm, that enumerates interesting equational consequences of a given set of equations.

Let us give Quine's proof: we suppose that we have only finitely many atoms. Suppose then that $A \to B$ is not derived. Then A and B are *disjoint* by the rule (R). If all atoms are in A, B then we have a valuation (all As set to 1 and all Bs set to 0) that shows that $A \to B$ is not a consequence of the set of rules. Otherwise, let x be an atom not in A, B. It follows from the rule (C) that *either* $A \to B, x$ *or* $A, x \to B$ is not derived. Eventually we get back to the cases where all atoms are in A, B.

It may be interesting to note that there is a purely intuitionistic version of this result. Let a *model* of a rule $A \to B$ be a subset $M \subseteq S$ such that $M \cap B$ is inhabited whenever $A \subseteq M$. Then $A \to B$ is a consequence of a set of rules iff for any model M of the rules is a model of $A \to B$.

4 Gentzen

The first published work of Gentzen who was the first to formulate the rule (T) was concerned with the problem of finding independent axiomatisations of a given set of formula. Gentzen was working in the restricted system where there is at most one atom in the right side of an entailment.

In this framework he gives an example of an infinite system that has no independent axiomatisation. The set S is the set of atoms $b, c, a_0, a_1, \ldots$ and the axioms are

$$a_0, b \to c \qquad a_1, b \to c \qquad a_2, b \to c \qquad \ldots$$

and $a_i \to a_j$ for $i < j$.

This infinite set of axioms is not independent: for instance $a_0, b \to c$ is a consequence of $a_0 \to a_2$ and $a_2, b \to c$.

Theorem: This system has no independent axiomatisation.

Notice that we can characterise the theorems that are consequence of this theory: either $X \to c$ with X containing either c or b and one of the a_i, or $X \to a_j$ with X containing $a_i,\ i \leq j$ or $X \to b$ with X containing b. The theorem is a simple consequence of this remark.

5 Problems of Lewis Carroll

There are two contributions: one is a quick method to guess what may be an interesting consequence of a set of rules, the second is the *method of trees* to find if a given set of rules is inconsistent.

Most problems analysed by Lewis Carroll (in Symbolic Logic Part I and II) can be presented in the following way. We have a finite sets of entailment, and we are asked to find an interesting consequence. An example is the following:

1. $h, m, k \to$

2. $\to d, e, c$

3. $h \to k, a$

4. $b, l \to h$

5. $c, k \to m$

6. $h, e \to c$

7. $b, a \to k$

Like Gentzen, Lewis Carroll spends some time at looking at the linear case, where all rules are of the form $a \to b$ and the binary case, where the rules may also be of the form $a, b \to$ or $\to a, b$. The case of rules that are ternary or more is clearly much more difficult and constitutes the part II of his book of Symbolic logic. Lewis Carroll usually did not formulate his rules with letters. Here is a typical (simple) example: what can you deduce from the clauses

1. Babies are illogical;

2. Nobody is despised who can manage a crocodile;

3. Illogical persons are despised?

However in this note, I will limit myself to examples with letters.

Lewis Carroll makes an interesting remark: among the letters we can distinguish the *retinends* that occur only on one side of the entailment: here b, l occuring only at the left and d only at the right, and the *eliminands* tha occur both in the left and right side.

We thus expect that $b, l \to k$ should be a consequence. In order to prove this we rewrite the clauses by taking away all occurences of the eliminands:

1. $h, m, k \to$

2. $\to e, c$

3. $h \to k, a$

4. $\to h$

5. $c, k \to m$

6. $h, e \to c$

7. $a \to k$

We see then that h has to hold and we can take away all the occurence of h :

1. $m, k \to$

2. $\to e, c$

3. $\to k, a$

4. $c, k \to m$

5. $e \to c$

6. $a \to k$

The problem is then to test if this set of rules is inconsistent. The method consists in chosing a rule with a minimum number of atoms and to make a case analysis. For instance from the second rule we either have $e = 1$ or $e = 0, c = 1$. The first case gives

1. $m, k \to$

2. $\to k, a$

3. $c, k \to$

4. $\to c$

5. $a \to k$

which is contradictory. The second case gives

1. $m, k \to$

2. $\to k, a$

3. $k \to m$

4. $a \to k$

which is also contradictory.

6 Improvement and Ternary Case

We can recognize consequence of binary clauses: if we have $\to a, b$ and $a \to b$ then $\to b$ is a consequence. Similarly $b \to$ is a consequence of $b \to a$ and $b, a \to$.

Also, it is standard that the general case can be coded in the ternary case. For instance $D \equiv A \vee B \vee C$ can be represented as $d \to u$, $u \to d$ and $a \to v$, $b \to v$, $v \to a, b$, expressing $v \equiv a \vee b$ and $v \to u$, $c \to u$ and $u \to c, v$ expressing $u \equiv c \vee v$.

7 Genealogical Tree

Beside the observation about retinends and eliminands, the method that is used to test if a given set of rules is inconsistent is quite remarkable.

In this approach we build a tree of possibilities, similar to Beth trees. Carroll calls this method the method of trees or the "Genealogical Method" and was quite proud of it. He had also a justification of the fact that the trees should grow *head-downwards*. "A *Genealogical* tree always grows downwards: then why may not a *Logical* tree do likewise?"

Let us give an abstract version of the method. We have a set of rules $A_i \rightarrow B_i$ and a subset $X \subseteq S$. We want to see if X is inconsistent, that is if $X \rightarrow$ can be derived. We have the unique rule

$$\frac{X, b_1 \quad \ldots \quad X, b_m}{X}$$

whenever $B_i = b_1, \ldots, b_m$ and $A_i \subseteq X$. Dually, we may analyse if a subset $Y \subseteq S$ is such that $\rightarrow Y$ is derivable. The unique rule is

$$\frac{Y, a_1 \quad \ldots \quad Y, a_n}{Y}$$

whenever $A_i = a_1, \ldots, a_n$ and $B_i \subseteq Y$.

We can see Carroll's method as a combination of these two rules we have

$$\frac{X, b_1 \rightarrow Y \quad \ldots \quad X, b_m \rightarrow Y}{X \rightarrow Y}$$

whenever $B_i = b_1, \ldots, b_m$ and $A_i \subseteq X$ and

$$\frac{X \rightarrow Y, a_1 \quad \ldots \quad X \rightarrow Y, a_n}{X \rightarrow Y}$$

whenever $A_i = a_1, \ldots, a_n$ and $B_i \subseteq Y$. The axioms are $X \rightarrow Y$ if $X \cap Y$ is inhabited. The theorem is then that this system of rules is *complete*.

References

[Carroll96] L. Carroll. Symbolic Logic, Part I and II edited by W.W. Bartley, Harvester Press.

[Sco74] D. Scott. Completeness and axiomatizability. *Proceedings of the Tarski Symposium*, pages 411–435, 1974.