We let D be the set of all untyped, maybe open, λ-terms, with β-conversion as equality. We let c_n be the lambda term $\lambda x \lambda f\ f^n\ x$. We consider the following types

$$T \quad ::= \quad \alpha \mid T \rightarrow T \mid (\Pi\alpha)T$$

where in the quantification, T has to be built using only α and $\rightarrow$.
We use the notation $T_1 \rightarrow T_2 \rightarrow T_3$ for $T_1 \rightarrow (T_2 \rightarrow T_3)$ and similarly $T_1 \rightarrow T_2 \rightarrow \ldots \rightarrow T_n$ for $T_1 \rightarrow (T_2 \rightarrow (\ldots \rightarrow T_n))$.
Let us give some examples to illustrate the restriction on quantification. We can have $T = (\Pi\alpha)[\alpha \rightarrow \alpha]$ or $(\Pi\alpha)[\alpha \rightarrow (\alpha \rightarrow \alpha) \rightarrow \alpha]$ or even $(\Pi\alpha)[((\alpha \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha]$ but a type such as $(\Pi\alpha)[[(\Pi\beta)[\alpha \rightarrow \beta]] \rightarrow \alpha]$ is not allowed.
We have the following typing rules

$$\frac{}{\Gamma \vdash x : T} \quad x : T \in \Gamma$$

$$\frac{\Gamma, x : T \vdash t : U}{\Gamma \vdash \lambda x\ t : T \rightarrow U} \qquad \frac{\Gamma \vdash u : V \rightarrow T \quad \Gamma \vdash v : V}{\Gamma \vdash u\ v : T}$$

$$\frac{\Gamma \vdash t : (\Pi\alpha)T}{\Gamma \vdash t : T[U]} \qquad \frac{\Gamma \vdash t : T}{\Gamma \vdash t : (\Pi\alpha)T}$$

where Γ is a finite set of type declaration $x : T$, and in the last rule, α does not appear free in any type of Γ.
We let N be the type $(\Pi\alpha)[\alpha \rightarrow (\alpha \rightarrow \alpha) \rightarrow \alpha]$. We have $\vdash c_n : N$ for each n. The goal of this note is to provide a finitary proof of the following result.
Theorem: If $\vdash t : N \rightarrow N$ then for each n there exists m such that $t\ c_n\ x\ f = f^m\ x$ for x, f variables.

Let L, M range over 1st-order *contexts*. Let S be the following poset: the elements of S are 1st-order contexts, and the order is reversed inclusion. We let H be the set of downward closed subsets of S. H is thought of as a set of generalised truth-values.
If T is a first-order type, we define $X_T : D \rightarrow H$ by

$$X_T(t) = \{L \in S \mid L \vdash t : T\}$$

If $X, Y : D \rightarrow H$ we define

$$(X \rightarrow Y)(t) = \{L \mid (\forall M \supseteq L)(\forall u) M \in X(u) \rightarrow M \in Y(t\ u)\}$$

Lemma 1. If $L \subseteq M$ and $L \vdash t : T$ then $M \vdash t : T$.
Lemma 2. If $L, x : T \vdash t : T_1$ and x is not free in t then $L \vdash t : T_1$.

By these two lemmas, we derive

Lemma 3. (Yoneda) We have $X_{T_1} \rightarrow X_{T_2} = X_{T_1 \rightarrow T_2}$.

It is indeed clear by lemma 1 that if $L \vdash t : T_1 \rightarrow T_2$ and $M \vdash u : T_1$ and $M \supseteq L$ then $M \vdash t\ u : T_2$. Conversely if $L \in (X_{T_1} \rightarrow X_{T_2})(t)$ then for x not free in t we have $L, x : T_1 \in X_{T_1}(x)$ and hence $L, x : T_1 \in X_{T_2}(t\ x)$ and so $L, x : T_1 \vdash t\ x : T_2$. It follows from this that we have $L, x : T_1 \vdash t : T_1 \rightarrow T_2$ and hence by lemma 2, $L \vdash t : T_1 \rightarrow T_2$.

If T is a first-order type and we have an assignment $\rho(\alpha) : D \rightarrow H$ defined for at least all the free type variables α occuring in T, we define the interpretation $T\rho$ by induction as usual: $\alpha\rho = \rho(\alpha)$ and $(T_1 \rightarrow T_2)\rho = T_1\rho \rightarrow T_2\rho$.

Lemma 4. If $x_1 : T_1, \ldots, x_n : T_n \vdash t : T$ then for any assigment ρ, if $L \in T_1\rho(u_1), \ldots, L \in T_n\rho(u_n)$ then $L \in T\rho(t(u_1, \ldots, u_n))$.

Corollary. If $M = x_1 : T_1, \ldots, x_n : T_n$ and $M \vdash t : T(\alpha)$ and $T(\alpha)$ is a first-order type using only α as a free type variable, which does not appear in $T_1, \ldots, T_n$ then $M \in T(\alpha = X)(t)$ for any $X : D \rightarrow H$.

For this we take $\rho(\beta) = X_\beta$ for β free in $T_1, \ldots, T_n$ and $\rho(\alpha) = X$. By lemma 3 we have $T_i\rho = X_{T_i}$ and since $M \in X_{T_i}(x_i)$ we get by lemma 4 $M \in T\rho$ that is, $M \in T(\alpha = X)(t)$.

Main lemma. If $T(\alpha)$ is a first-order type using only α as a free type variable, then $\bigcap_{X:D\rightarrow H} T(\alpha = X)(t)$ is exactly $\{L \mid L \vdash T(\beta)\}$ for β fresh w.r.t. L.

This shows that we have a finitary model of our fragment of system F.

We can now prove the theorem by considering the following $X_0 : D \rightarrow H$ which is defined in a finitary way

$$X_0(u) = \{L \mid (\exists m)\ u\ x\ f = f^m\ x\}$$

We have clearly $L \in X_0(x)$ and $L \in X_0(f\ u)$ if $L \in X_0(u)$. If $\vdash t : N \rightarrow N$ we have $\vdash t\ c_n : N$ and hence $L \in X_0(t\ c_n\ x\ f)$ for all L. This shows that there exists m such that $t\ c_n\ x\ f = f^m\ x$ as desired.