A NOTE ON MEASURES WITH VALUES IN A PARTIALLY ORDERED VECTOR SPACE

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Abstract. The goal of this note is to present an alternative, and we think simpler, proof of the following generalisation of the Riesz representation theorem due to J.D.M. Wright [8]: any positive linear map $\phi : C(X) \to V$ can be represented by a V-valued measure on Baire subsets of X, where X is compact Hausdorff and V is a monotone σ-complete ordered vector space, not necessarily a lattice. Our proof suggests a purely inductive approach to measure theory, in the spirit of Borel's original definition of measure of Borel sets [1, 2].

Introduction

The goal of this note is to present an alternative, and we think simpler, proof of the following generalisation of the Riesz representation theorem due to J.D.M. Wright [8]: any positive linear map $\phi : C(X) \to V$ can be represented by a V-valued Borel measure, where X is compact Hausdorff and V is a monotone σ-complete ordered vector space, not necessarily a lattice. As noticed by Wright [7], in general one cannot expect the resulting measure m to be regular (even if V is a lattice) and so, the usual Daniell-Bourbaki approach, centred around the definition of an outer measure, cannot be used here. Our approach is based on a universal characterisation of the Riesz space $B(X)$ of bounded Baire functions on X, and a general lemma about existence of binary sups which holds in any monotone σ-complete space. Our construction suggests also a purely inductive approach to measure theory, in the spirit of Borel's original definition of measure of Borel sets [1, 2][1].

1. Universal characterisation of bounded Baire functions

We shall use the terminology of [3]. A *Riesz space* is an ordered vector space which is a lattice. We shall only consider Riesz space with a *strong unit* 1: for any element x there exists n such that $-n.1 \leq x \leq n.1$. A map of Riesz space, or Riesz homomorphism, $f : V_1 \to V_2$ is a map of ordered vector space that preserves l.u.b. and strong unit. A Riesz space is *Dedekind σ-complete* iff it is also monotone σ-complete: any increasing bounded

[1]Even for real valued measures, or in the case where V is a lattice, we believe that our approach gives essentially new proofs of basic results. Being purely inductive, it is an *alternative* to the use of outer measure, which, since Lebesgue's work through Daniell, Caratheodory, Bourbaki, is the usual way to define the measure of Borel subsets. In particular, and in contrast to Wright's work [7, 8], which relies for instance on the usual Riesz representation theorem, our proofs are developped *independently* of measure theory, and relies *only* on the inductive characterisation of the space of bounded Baire functions given in lemma 1.1 and some general properties of ordered vector spaces.

sequence has a l.u.b. A Riesz homomorphism $f : V_1 \to V_2$ is *σ-continuous* iff it preserves l.u.b. of increasing bounded sequences.

We can now define the notion of σ-completion of a Riesz space V. It is a Dedekind σ-complete Riesz space W with a Riesz homomorphism $i : V \to W$ such that, for any other Dedekind σ-complete Riesz space W_1 and Riesz homomorphism $f : V \to W_1$ there is a unique σ-continuous Riesz homomorphism $f_1 : W \to W_1$ such that $f_1 \circ i = f$. This condition characterises uniquely (up to isomorphism) the pair $W, i : V \to W$.

This definition may surprise readers used to Boolean algebras: notice that, and this is an essential point, we do not require the map $i : V \to W$ to preserve the sequential suprema already existing in V. The justification of our definition in the framework of our paper is given by the next result.

Lemma 1.1. *Let X be a compact Hausdorff space, and $C(X)$ the Riesz space of continuous functions over X, with the constant function 1 as strong unit. The Riesz space $B(X)$ of bounded Baire functions on X, with the inclusion function $i : C(X) \to B(X)$, is the σ-completion of $C(X)$.*

Proof. We use standard results, which go back to Stone [5, 6]. First, any Dedekind σ-complete Riesz space W_1 is of the form $C(Y)$, where Y is the representative space of a σ-complete Boolean algebra. Also [6], any bounded Baire function α in $B(Y)$ determines a unique $H(\alpha) \in C(Y)$ such that $\alpha(y) = H(\alpha)(y)$ except on a meager set. Furthermore the map $H : B(Y) \to C(Y)$ is σ-continuous.

Any map $f : C(X) \to C(Y)$ corresponds to a continuous map $Y \to X$, which extends to a σ-continuous map $G : B(X) \to B(Y)$. Composing this map with $H : B(Y) \to C(Y)$, we get the desired extension $H \circ G : B(X) \to C(Y)$ of f, which is uniquely determined. ☐

2. Reformulation of Riesz representation theorem

The interest of the lemma 1.1 is that it suggests a natural way to prove Wright's result (and Riesz representation theorem) if we reformulate it in the following way.

Theorem 2.1. *Let $\phi : C(X) \to V$ be a positive linear map, where V is a monotone σ-complete ordered vector space, not necessarily a lattice. Then ϕ has a unique extension to a σ-continuous map $B(X) \to V$.*

Since the σ-complete Boolean algebra of components [3] of $B(X)$ can be seen as the algebra of Baire subsets of X, the extension of ϕ to $B(X)$ will indeed defined a measure on Baire subsets of X. Notice that this reformulation contains as direct corollary the monotone (and hence the bounded) convergence theorem. The theorem will follow directly from lemma 1 and the following remarks.

We let M be the ordered space of linear maps $l : C(X) \to V$ that are bounded by ϕ: there exists n such that $-n.\phi(\alpha) \le l(\alpha) \le n.\phi(\alpha)$ for $\alpha \ge 0$. The space M is clearly monotone σ-complete, but may not be a lattice. The evaluation map

$$ev : l \longmapsto l(1),\ \ M \to V$$

clearly preserves l.u.b. of increasing bounded sequences. The map

$$I : C(X) \to M$$

defined by $I(\alpha)(\beta) = \phi(\alpha.\beta)$ is monotone. Even though M may not be a lattice, we have the following result [4].

Lemma 2.2. *If $\alpha_1, \alpha_2 \in C(X)$ then $I(\alpha_1 \vee \alpha_2)$ is the l.u.b. in M of $I(\alpha_1)$ and $I(\alpha_2)$.*

Proof. It is clear that we have $I(\alpha_i) \leq I(\alpha_1 \vee \alpha_2)$. Let us fix $J \in M$ such that $I(\alpha_1) \leq J$ and $I(\alpha_2) \leq J$. We show that we have $I(\alpha_1 \vee \alpha_2) \leq J$.

For this, we fix $\beta \in C(X),\ \beta \geq 0$ and we show that we have

$$I(\alpha_1 \vee \alpha_2)(\beta) \leq J(\beta) + \epsilon\phi(\beta)$$

for any $\epsilon > 0$. This will show

$$I(\alpha_1 \vee \alpha_2)(\beta) \leq J(\beta)$$

and hence $I(\alpha_1 \vee \alpha_2) \leq J$.

We write $\alpha = \alpha_1 - \alpha_2$, and we take

$$e_1 = 1 \wedge n\alpha^+,\ e_2 = 1 - e_1$$

with $n \geq 1/\epsilon$. We have $e_1\alpha = e_1\alpha^+$ and

$$\alpha_1 \vee \alpha_2 = \alpha_2 + \alpha^+ = \alpha_2 e_2 + \alpha_1 e_1 + \alpha^+ e_2$$

since

$$\alpha^+ e_2 \leq 1/n$$

this implies

$$\alpha_1 \vee \alpha_2 \leq \alpha_2 e_2 + \alpha_1 e_1 + \epsilon$$

It follows that we have

$$(\alpha_1 \vee \alpha_2)\beta \leq \beta\alpha_2 e_2 + \beta\alpha_1 e_1 + \beta\epsilon$$

and hence

$$I(\alpha_1 \vee \alpha_2)(\beta) \leq J(\beta e_1) + J(\beta e_2) + \epsilon\phi(\beta) = J(\beta) + \epsilon\phi(\beta)$$

as desired.

The key point is that this argument does not require V to be a lattice. Indeed, the only property needed for V is to be an ordered vector space. □

Lemma 2.3. *Let a, b be two elements of M then $a \vee b$ exists iff $a \wedge b$ exists (and in this case $a \vee b + a \wedge b = a + b$).*

Lemma 2.4. *If b_m is a bounded decreasing sequence, and $a \vee b_m$ exists for all m then $a \vee \bigwedge b_m$ exists. Furthermore, it is equal to $\bigwedge_m (a \vee b_m)$.*

Proof. It is enough by the previous lemma to show that $a \wedge \bigwedge b_m$ exists. By hypothesis, $a \vee b_m$ and hence by the previous lemma, $a \wedge b_m$ exists for all m. Since $a \wedge b_m$ is a bounded decreasing sequence, $\bigwedge_m (a \wedge b_m)$ exists and is $a \wedge b$.

Since $a \vee b_m$ is decreasing and bounded, $\bigwedge_m (a \vee b_m)$ exists. By general distributivity property valid in any ordered space [3], we have $a \vee \bigwedge b_m = \bigwedge_m (a \vee b_m)$. □

Corollary 2.5. *If R a subspace of M such that $a \vee b$ exists and belongs to R if $a, b \in R$. Then the least monotone σ-complete ordered subspace containing R is a Dedekind σ-complete Riesz space.*

Proof. We let M_1 be the smallest subset $\subseteq M$ containing R which satisfies: if $a_n \in M_1$ is a bounded increasing (resp. decreasing) sequence then $\bigvee_n a_n \in M_1$ (resp. $\bigwedge_n a_n \in M_1$). We claim that M_1 is a subspace of M and that, if $a, b \in M_1$ then $a \vee b$ exists and belongs to M_1. This will show that M_1 is actually the least monotone σ-complete ordered subspace containing R and that, being also a lattice, it is a Dedekind σ-complete Riesz space. The proof of this claim is straightforward by induction from lemma 2.4. Since this is a key point, we give explicitely the argument proving the second closure property.

We first show that if $a \in R$ and $b \in M_1$ then $a \vee b$ exists and belongs to M_1. Indeed the set M_2 of elements $b \in M_1$ such that $a \vee b$ exists and is in M_1 contains R by hypothesis. If $b_m \in M_2$ is an increasing bounded sequence then $a \vee \bigvee b_m = \bigvee (a \vee b_m)$ exists and belongs to M_1. Similarly if $b_m \in M_2$ is a decreasing bounded sequence then $a \vee \bigwedge b_m = \bigwedge (a \vee b_m)$ exists by lemma 2.4 and belongs to M_1. This shows that $M_2 = M_1$.

Next, we consider the set M_3 of elements $a \in M_1$ such that if $b \in M_1$ then $a \vee b$ exists and belongs to M_1. We have just shown that this set contains R. We show next that M_3 is closed by sups and infs of bounded monotone sequence.

If $a_n \in M_3$ is an increasing bounded sequence and $b \in M_1$ then $a_n \vee b$ exists and belongs to M_1 for all n and hence $(\bigvee a_n) \vee b = \bigvee (a_n \vee b)$ exists and belongs to M_1. This shows that $\bigvee a_n \in M_3$.

Similarly if $a_n \in M_3$ is a decreasing bounded sequence and $b \in M_1$ then $a_n \vee b$ exists and belongs to M_1 for all n. By the lemma 2.4 we have that $(\bigwedge a_n) \vee b = \bigwedge (a_n \vee b)$ exists and belongs to M_1. Hence $\bigwedge a_n \in M_3$. □

This last formal result is valid for any monotone σ-complete space M, with the same argument. It can be compared to the theorem that a family of pairwise commuting operators have a common spectral decomposition. The condition of being pairwise commuting $ab = ba$ is here replaced by the condition that $a \vee b$ exists. Since R is a subset of a Dedekind σ-complete Riesz space M_1, all elements of R have a spectral decomposition w.r.t. the σ-complete Boolean algebra of components of M_1 [3].

Theorem 2.1 is then a direct consequence of lemmas 1.1,2.2 and corollary 2.5: the image $I(C(X))$ is contained a Dedekind σ-complete Riesz space $M_1 \subseteq M$ and hence by initiality the map $I : C(X) \to I(C(X))$ can be extended in a unique way to a σ-continous map $B(X) \to M_1$. By composing this map with the evaluation map $ev : M \to V$ we get the desired σ-continuous extension of the map $\phi : C(X) \to V$.

3. Inductive measure theory

One possible interest of the previous construction is that it suggests a purely inductive and representation-free approach to measure theory, in the spirit of Borel's original definition of measure of Borel sets [1, 2]. Since this is not the main purpose of this note, we limit ourselves here to state the following representation-free version of our construction, which has a similar justification to the one of theorem 2.1.

In a representation-free approach, we start not from a compact space but from a Riesz space C over the rationals. This vector space is thought of as a pointfree presentation of a space $C(X)$. We can then *define* the corresponding Riesz space of bounded Baire functions as being the σ-completion of C. This definition is justified by lemma 1.1. A *measure* is then defined to be a positive linear map on C. The representation-free formulation of measure theory, which contains not only Riesz representation theorem, but also the monotone and bounded convergence theorem, can then be formulated as follows.

Theorem 3.1. *Let C be a Riesz space over the rationals. Any positive linear map from C to a monotone σ-complete ordered vector space has a unique σ-continuous extension to the σ-completion of C.*

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References

[1] E. Borel. *Leçons sur la Théorie des Fonctions.* Gauthier-Villars, quatrième édition 1950, 1ere édition 1898.
[2] N. Lusin. *Leçons sur les ensembles analytiques.* Gauthier-Villars, 1930.
[3] W.A. Luxemburg and A.C. Zaanen. *Riesz Spaces I.* North-Holland, 1971.
[4] M. Riesz. Sur la décomposition des opérations fonctionelles linéaires. Atti del Congr. Internaz. dei Mat., Bologna 1928, 3 (1930), 143-148.
[5] M.H. Stone. A General Theory of Spectra II. Proc. Nat. Acad. Sci., 27, 1941, p. 83-87.
[6] M.H. Stone. Boundedness Properties in Function-Lattices. Canadian Journal of Mathematics 1, 1949, p. 177-186.
[7] J.D. M. Wright. Stone-algebra-valued measures and integrals. Proc. London Math. Soc. (3) 19 1969 107–122.
[8] J.D. M. Wright. Measures with values in a partially ordered vector space. Proc. London Math. Soc. (3) 25 (1972), 675–688.

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