NOTIONS INVARIANT BY CHANGE OF BASES

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Introduction

One importance of theory of sheaves for constructive mathematics is that it provides a general expression of the algebraic method of "adding new indeterminates". The method of "change of base", or moving in a suitable topos of sheaves over a formal space, can indeed be interpreted as the addition of new unknown quantities submitted to some given constraints. This method compensates the absence of choice (an example is provided in [6]), and gives a constructive way to add ideal elements such as prime ideals.

This method requires some care however: adding these quantities may add new sequences for instance, and some facts that did hold before the addition of these quantities may not hold anymore after. It is thus quite important to recognize the concepts that are invariant under such an operation. This motivates for instance the notion of "geometrical statement": if a concept is expressed by a geometrical statement, it is invariant.

The purpose of this note is to analyze two mathematical concepts that are invariant, but are not (directly?) expressed in a geometrical way. These concepts are: for a space to be *compact*, and for a ring to be *Noetherian*. We present in each case a direct inductive proof that these concepts are invariant. In the compact case, this is known already [5]. However it may be interesting to have an elementary presentation of these results (which were obtained directly by an analysis of the proof in [1], after a talk presenting some results of [5]). As an application, we give a direct proof of Tychonov's theorem, which seems even more direct than the proof in [4].

1. Compact Space

A space X will be given as a meet-semi-lattice with a covering relation [3] $Cov(x)$, $x \in X$. Each $A \in Cov(x)$ is a downward closed of X and we have $yA \in Cov(y)$ if $y \leq x$. If U is a downward closed subset of X we define $x \triangleleft U$ (read: U *covers* x) inductively by

- $x \in U$ or
- for some $A \in Cov(x)$ we have $z \triangleleft U$ for all $z \in A$.

We say that U is a covering iff $1 \triangleleft U$ iff $x \triangleleft U$ for all $x \in X$. Corresponding to each covering U we have a principle of U-induction: if $\phi(x)$ holds for each $x \in U$ and is hereditary, if $\phi(x)$ holds whenever for some $A \in Cov(x)$ we have $\phi(z)$ for all $z \in A$, then $\phi(x)$ holds for all x.

If X is a space, we let $Sp(X)$ be the set of all its points.

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We say that X is *compact* iff whenever $1 \lhd U$ there exists $x_1, \ldots, x_n \in U$ such that $1 \lhd x_1, \ldots, x_n$. We shall write also $X = x_1, \ldots, x_n$ for expressing that $x_1, \ldots, x_n$ is a covering.

We want to show that this notion is stable under addition of new quantities. First we analyze what it means. Let Y be another space, and let us introduce a new generic quantity $\beta \in Sp(Y)$. We now consider a covering $U(\beta)$ of X which depends in a continuous way of β. Another way of looking at $U(\beta)$ is that we have a relation $y \Vdash x \in U$, which for each $x \in X$ defines an open of the space $Sp(Y)$.

We can then define $y \Vdash x \lhd U$ inductively by

1. $y \Vdash x \in U$ or
2. for some $B \in Cov(y)$ we have $y_1 \Vdash x \lhd U$ for all $y_1 \in B$ or
3. for some $A \in Cov(x)$ we have $x_1 \Vdash x_1 \lhd U$ for all $x_1 \in A$.

However, this definition expresses only $x \lhd U$ in the sense of $Sh(Y)$ and it will be actually more convenient to reason directly in $Sh(Y)$.

We want to show that if $1 \lhd U$ in $Sh(Y)$ then we have

$$(\exists K \subseteq_{fin} U) 1 \lhd K$$

in the sense of Y. For this, we shall reason in $Sh(Y)$ introducing the predicate $C(M)$ on finite subsets of X that is defined by $y \Vdash C(M)$ iff

- $1 \lhd M$ or
- for some $B \in Cov(y)$ we have $y_1 \Vdash C(M)$ for all $y_1 \in B$

Notice that, a priori, $y \Vdash C(M)$ is stronger than $y \Vdash 1 \lhd M$.

Theorem 1.1. *Let us define*

$$x \prec U = (\forall M \subseteq_{fin} X)[C(M, x) \to (\exists K \subseteq_{fin} U) C(M, K)]$$

then we have

- *if* $x \in U$ *then* $x \prec U$
- *if* $A \in Cov(x)$ *and* $x_1 \prec U$ *for all* $x_1 \in A$ *then* $x \prec U$

In particular if $1 \lhd U$ *then* $1 \prec U$ *and hence* $(\exists K \subseteq_{fin} U) C(K)$,

Proof. We have two cases

1. $x \in U$ we can then take $K = \{x\}$.
2. For some $A \in Cov(x)$ we have $x_1 \prec U$ for all $x_1 \in A$. If $C(M, x)$ then we can find $x_1, \ldots, x_n \in A$ such that $C(M, x_1, \ldots, x_n)$ because X is compact. Since $\phi(x_1)$ holds we can find $K_1 \subseteq_{fin} U$ such that $C(M, K_1, x_2, \ldots, x_n)$ holds. We can then find $K_2 \subseteq_{fin} U$ such that $C(M, K_1, K_2, x_3, \ldots, x_n)$ holds. In this way we find $K = K_1 \cup K_2 \cup \cdots \subseteq_{fin} U$ such that $C(M, K)$ holds.

Hence we have $x \prec U$ if $x \lhd U$. If we take $x = 1$ and $M = \emptyset$ we get the existence of $K \subseteq_{fin} U$ such that $C(K)$ holds. ∎

Notice that this argument, being intuitionistic, can be relativised to any sheaf models.

2. Compact Spaces on Distributive Lattices

It may be interesting to abstract what is going on in the previous argument. Here is one possible analysis. Let D be a distributive lattice and define $x \prec U$ for $x \in D$ and $U \subseteq D$ by

$$(\forall a)[a \vee x = 1 \rightarrow (\exists K \subseteq_{fin} U)[a \vee \vee_{u \in K} u = 1]]$$

We have then the following property.

Theorem 2.1. *The relation $x \prec U$ satisfies*

- *if $x \prec U$ and $u \prec V$ for all $u \in U$ then $x \prec V$*
- *if $x \prec U$ and $x_1 \leq x$ then $x_1 \prec U$*
- *if $x \in U$ then $x \prec U$*
- *if $x \prec U$ then $yx \prec yU$*
- *$1 = x_1 \vee \cdots \vee x_n$ iff $1 \prec x_1, \ldots, x_n$*
- *if $1 \prec U$ then there exists $K \subseteq_{fin} U$ such that $1 \prec K$*

Hence the relation $x \prec U$ defines a covering relation and hence a locale which is compact. Furthermore let $\triangleleft$ be any other covering relation on D which satisfies the two last conditions, we have $\triangleleft \subseteq \prec$.

Proof. We prove the first property, to show that it contains the core of the previous argument. We assume $x \prec U$ and $u \prec V$ for all $u \in U$. If $a \vee x = 1$ we can then find $u_1, \ldots, u_n \in U$ such that $a \vee u_1 \vee \cdots \vee u_n = 1$. Since $u_1 \prec V$ we can find $K_1 \subseteq_{fin} V$ such that

$$a \vee \vee K_1 \vee u_2 \cdots \vee u_n = 1$$

We can then find $K_2 \subseteq_{fin} V$ such that

$$a \vee \vee K_1 \vee \vee K_2 \vee u_2 \cdots \vee u_n = 1$$

In this way we find $K = K_1 \cup K_2 \cup \cdots \subseteq_{fin} U$ such that $a \vee \vee K = 1$. □

The importance of this theorem is due to the fact that *any* compact frame is of the form $D, \triangleleft$ where D is a distributive lattice and $\triangleleft$ a covering relation such that $\triangleleft \subseteq \prec$. Indeed, if we start from any space X, Cov we let D be the distributive lattice of finite subsets of X with the order relation $K \leq L$ iff $x \triangleleft L$ for all $x \in K$ (this is similar to the Stone cover operation defined in [7]).

As an application, let us give a short proof of Tychonov's theorem. If we have a family of distributive lattices D_i, $i \in I$, where we do not assume decidability of equality on I, we consider the coproduct D of D_i with injection maps $\sigma_i : D_i \to D$.

Lemma 2.2. *If $u \in D_i$ and $a \vee \sigma_i u = 1 \in D$ then there exists $v \in D_i$ such that $\sigma_i v \leq a$ and $u \vee v = 1 \in D_i$.*

Proof. If $a \in D$ then a can be written as a conjunction $\wedge m_k$ of disjunctions of elements of the form $\sigma_l a_l$. If we have $a \vee \sigma_i u$ then for each k we have $m_k \vee \sigma_i u = 1$. Using the results of [2], the disjunction m_k should contain $\vee_{v \in L_k} \sigma_i v$ for some $K \subseteq_{fin} D_i$ such that $u \vee \vee L_k = 1 \in D_i$. We can then take $v = \wedge \vee L_k$. □

Corollary 2.3. *If $u \prec U$ in D_i then $\sigma_i u \prec \sigma_i U$.*

Proof. If $a \vee \sigma_i u = 1 \in D$ then by the lemma, we have $v \in D_i$ such that $v \vee u = 1 \in D_i$ and $\sigma_i v \leq a$. Since $u \prec U$ we find $K \subseteq_{fin} U$ such that $v \vee \vee K = 1$ and then $\sigma_i(K) \subseteq_{fin} \sigma_i(U)$ and $a \vee \sigma_i(K) = 1$. □

Corollary 2.4. *A coproduct of compact frames is compact.*

Proof. Let X_i, Cov_i be a family of compact frame, and D_i be the distributive lattice of finite subsets associated to X_i. On each D_i we have a covering relation $\lhd_i \subseteq \prec$. The coproduct of X_i can be described as the frame $D, \lhd$ where D is the coproduct of D_i and $\lhd$ is the least covering relation such that $\sigma_i u \lhd \sigma_i U$ whenever $u \lhd_i U$. In order to prove that $\lhd$ is compact, it is enough to show $\lhd \subseteq \prec$ which thus amounts to show that $\sigma_i u \prec \sigma_i U$ whenever $u \lhd_i U$. Since $\lhd_i \subseteq \prec$ this follows from the previous corollary. □

3. Noetherian Ring

Given a ring R we let $G(a_0 \ldots a_{n-1})$ to mean that there exists $i < n$ such that $a_i \in (a_0, \ldots, a_{i-1})$. If $a_0 \ldots a_{n-1}$ is the empty sequence $G()$ is the absurd proposition $\perp$. We define inductively the predicate $G|$ on finite sequences ($G|\sigma$ reads: G *bars* the sequence σ) by

1. $G(\sigma)$ or
2. $G|\sigma a$ for all $a \in R$

We then say that R is Noetherian iff $G|()$.

We want now to analyze what this notion becomes in $Sh(Y)$ if Y is a given space. We define inductively a new relation $y \Vdash G|\sigma$ by

1. $G(\sigma)$ or
2. $y \Vdash G|\sigma a$ for all $a \in R$ or
3. for some $B \in Cov(y)$ we have $y_1 \Vdash G|\sigma$ for all $y_1 \in B$

Intuitively if $y \Vdash G|\sigma$ then all "local observers" $\beta \in y$ see that $G|\sigma$. The following result expresses then that if A is Noetherian, it stays Noetherian in $Sh(Y)$.

Theorem 3.1. *If $G|\sigma$ then $y \Vdash G|\sigma$ for all y. In particular if $G|()$ then $1 \Vdash G|()$.*

Proof. By induction on the proof that $G|\sigma$. □

Had we express Noetherian as: for any infinite sequence $a_0, a_1, \ldots$ there exists n such that $a_n \in (a_0, \ldots, a_{n-1})$, it would not seem possible to prove the invariance under change of bases: the addition of $\beta \in Sp(Y)$ may introduces new infinite sequences.

We can remark also that if Y has a positivity predicate (is open surjective) the converse direction holds.

Theorem 3.2. *If Y has a positivity predicate and $y \Vdash G|\sigma$ for some positive y then $G|\sigma$ holds. In particular, if $1 \Vdash G|()$ then $G|()$.*

In this case also, the arguments are intuitionistic, and hence can be relativized to any sheaf models.

References

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