CONSTRUCTIVE BAIRE CATEGORY THEOREM

THIERRY COQUAND

Introduction

We recall the point-free definition of Borel sets in [?] of Cantor space. We define then what it means to be of empty interior, and we prove Baire category theorem for Cantor space: a countable union of closed subset of empty interior is of empty interior.

1. Borel sets

We let B the Boolean algebra of closed open subset of Cantor space Ω. It is possible to describe B directly in a purely syntactical way. We write $b, b_1, a, \dots$ the elements of B. Following [?] we introduce Borel subsets of Ω as infinitary expressions $X, Y, A, \dots$ built from basic sets $b \in B$ by formal countable union and intersection. We define then the following one-sided sequent calculus, where the axioms are finite sets Γ such that Γ contains $b_1, \dots, b_n$ such that $\vee b_i = 1 \in B$ and the rules are

$$\frac{\dots \ \Gamma, X_n \ \dots}{\Gamma, \bigcap X_n} \qquad \frac{\Gamma, \bigcup X_n, X_{n_0}}{\Gamma, \bigcup X_n}$$

An alternative treatment will be to allow countable sets as sequents. The rules will then be

$$\frac{\dots \ \Gamma, X_n \ \dots}{\Gamma, \bigcap X_n} \qquad \frac{\Gamma, \dots, X_n, \dots}{\Gamma, \bigcup X_n}$$

As usual we use the notation Γ, Δ for $\Gamma \cup \Delta$ and Γ, X for $\Gamma \cup \{X\}$.

It is direct to define the formal complement X' of a Borel set by induction on the construction of X. We define $X \subseteq Y$ as the fact that the sequent X', Y is derivable.

Theorem 1.1. *If both Γ, X and Δ, X' are derivable then so is Γ, Δ.*

Proof. See [?]. □

Corollary 1.2. *The relation $\subseteq$ is transitive on Borel sets.*

2. Baire Category Theorem

We say that X is of *empty interior* iff whenever the sequent $X, b_1, \dots, b_n$ is derivable then $\vee b_i = 1 \in B$.

Theorem 2.1. *Let F_n be a sequence of closed subsets such that each F_n is of empty interior. Then $X = \bigcup F_n$ is of empty interior.*

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Proof. We prove that if $X, F_{n_1}, \dots, F_{n_k}, b_1, \dots, b_n$ is derivable then $\vee b_i = 1 \in B$ by induction on the derivation of $X, F_{n_1}, \dots, F_{n_k}, b_1, \dots, b_n$. This is clear if this is an axiom. Otherwise either we have n_0 such that the last rule is

$$\frac{X, F_{n_0}, F_{n_1}, \dots, F_{n_k}, b_1, \dots, b_n}{X, F_{n_1}, \dots, F_{n_k}, b_1, \dots, b_n}$$

and then $\vee b_i = 1 \in B$ by induction, or we have $F_{n_i} = \bigcap a_p$ and for all p

$$X, F_{n_1}, \dots, F_{n_k}, a_p, b_1, \dots, b_n$$

In this case, by induction hypothesis we have

$$a_p \vee \vee b_i = 1 \in B$$

for all p. But this implies that

$$F_{n_i}, b_1, \dots, b_n$$

is derivable, and hence, since F_{n_i} is of empty interior we have $\vee b_i = 1 \in B$. □

References

[1] P. Martin-Löf. *Notes on Constructive Mathematics.* Almqvist and Wiksell, Stockholm, 1968.

Department of Computer Science, Chalmers University of Technology and Gothenburg University, SE-412 96 Göteborg, Sweden. E-mail: coquand@cs.chalmers.se