Formal Topology with Posets

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Introduction

We explore basic notions of formal topology with Dragalin's definition of formal spaces [1]. This definition appears also in the exercice 5, chapter IX of [3], as the notion of *covering system*, and seems to be well-known among people in theory of locales. The point of this note is simply to define some basic notions of topology in this framework. In particular, we define the notion of points of a space, of continuous maps between and of products of two spaces.

0.1 Informal Presentation

We define a (generalised) *Post system* as a structure (X, Cov) such that for each $x \in X$, Cov_x is a family of subsets of X.

Given such a structure, and U a subset of X, we define inductively $u \lhd U$ by the clauses:

- (C1) $u \lhd U$ if $u \in U$,
- (C2) $u \lhd U$ if $S \lhd U$ for one $S \in \mathsf{Cov}_u$,

where $S \lhd U$ means $s \lhd U$ for all $s \in S$.

In term of Post systems, $u \lhd U$ means that u can be deduced from the set of "axioms" U.

One important thing here is that we have a natural example of a generalised inductive definition. If we represent a derivation of $u \lhd U$ as a tree, then the branching of the tree is determined by the "size" of the sets in Cov_x for $x \in X$.

0.2 Predicative Complete Posets

Given a Post system (X, Cov) the operation $U \longmapsto \lhd U$ defines an operator on subsets of X. The poset of fixed point of this operator form a complete lattice, with the sup $\bigvee U_i = \lhd \bigcup U_i$ for any set indexed family of subsets U_i.

0.3 Representation in Type Theory

We discuss the representation of this in Type Theory. We must have for each $x \in X$, a set C_x and for each $x_1 \in C_x$ a set D_{xx_1} together with a map $f : (\Sigma_{x \in X} \Sigma_{x_1 \in C_x} D_{xx_1}) \rightarrow X$.

It may be interesting that Kent Petersson and Dan Synek have introduced a similar structure, the *tree type*, in Type Theory [6], motivated by considerations from language theory. What seems to me important here is that formal spaces are closely connected to the theory of generalised inductive definitions, since, as noted by Per Martin-Löf, tree types are the natural expressions of Post systems in Type Theory.

This construction has the following suggestive interpretation (suggested by P. Martin-Löf). We think of the elements of X as *stage of knowledge*. The set C_x represents the set of *possible experiments* that one can do in the stage of knowledge $x \in X$. The set D_{xy} is the set of *possible outcomes* of the experiment $y \in C_x$. When doing the experiment $y \in C_x$, one gets eventually an outcome $z \in D_{xy}$ and, as a result, is transported in a new stage of knowledge $f(x, y, z)$.

This interpretation can be supplemented by some game-theoretic considerations. We interpreted (X, C, D, f) as specifying a game between nature and an observer. The element of X are then position in this game. A winning strategy for nature gives answers to any experiment in such a way that any game goes on forever.

0.4 Complete Posets in Type Theory

In intensional type theory, there may be problems in working with only the tree set structure. For instance, the definition of the positivity predicate does not seem possible over a tree set (A, B, C, d) without using a notion of equality over A.

If $\sqsubseteq$ is a relation over X, we extend it to a relation over subsets of X by defining $T \sqsubseteq S$ to mean that, for any $t \in T$ there exists $s \in S$ such that $t \sqsubseteq s$.

We say that a relation $\sqsubseteq$ over X is a *simulation* iff whenever $y \sqsubseteq x$ and $S \in C_x$ there exists $T \in C_y$ such $T \sqsubseteq S$.[1]

In absence of a general notion of equality, the following notion seems to be a natural candidate for the representation of *complete posets* in *type theory*. A complete poset is given by a structure $(X, \sqsubseteq, \mathsf{Cov})$ such that

1. $\sqsubseteq$ is reflexive and transitive,
2. (X, Cov) is a Post system,
3. $\sqsubseteq$ is a simulation for (X, Cov).

Intuitively, $\sqsubseteq$ plays the rôle of a refinement of the equality/equivalence relation (and symmetry is not needed).

We can now define the operator $\lhd$ in type theory. Given such a structure, and U a subset of X, we define inductively $u \lhd U$ by the clauses:

- (C1) $u \lhd U$ if $u \in U$,
- (C2) $u \lhd U$ if $u \sqsubseteq v$ and $v \lhd U$,
- (C3) $u \lhd U$ if $S \lhd U$ for one $S \in \mathsf{Cov}_u$,

where $S \lhd U$ means $s \lhd U$ for all $s \in S$.

Proposition: The relation $\lhd$ on subset of X is a reflexive and transitive relation. Any set indexed family U_i has a least upper bound $\bigvee U_i = \lhd(\bigcup U_i)$.

P. Aczel has formulated a notion of *set based* posets, and shown essentially a representation theorem, that any such complete poset can be represented by a structure $(X, \mathsf{Cov}, \sqsubseteq)$.

QUESTION: may be, following von Plato, it will be more elegant to start with an inequality relation on A instead?? Intuitively, we observe when two approximations differ. This is for instance the natural notion for Böhm trees.

0.5 Product of two complete posets

We start with two structures (X, A_x, B_{xx_1}, f) with $f(x, x_1, x_2) \in X$ for $x \in X$, $x_1 \in A_x$, $x_2 \in B_{xx_1}$ and (Y, C_y, D_{yy_1}, g) with $g(y, y_1, y_2) \in Y$ for $y \in Y$, $y_1 \in C_y$, $y_2 \in D_{yy_1}$. We define then the structure $E_{(x,y)} = A_x + C_y$ and $F_{(x,y)x_1} = A_{xx_1}$, $F_{(x,y)y_1} = B_{yy_1}$, while $h(z, z_1, z_2)$ for $z \in X \times Y$, $z_1 \in E_z$, $z_2 \in F_{zz_1}$ is defined by the equations $h((x,y), x_1, x_2) = (f(x, x_1, x_2), y)$ and $h((x,y), y_1, y_2) = (x, g(y, y_1, y_2))$. It is then direct that the condition $h(z, z_1, z_2) \sqsubseteq z$ holds if this condition is satisfied both by f and g.

If we start with two complete posets $(X, \mathsf{Cov}, \sqsubseteq)$ and $(Y, \mathsf{Cov}, \sqsubseteq)$, we can furthermore consider the relation $(x_1, y_1) \sqsubseteq (x_2, y_2)$ on the product $X \times Y$ defined by $x_1 \sqsubseteq x_2$ and $y_1 \sqsubseteq y_2$.

[1] Tree sets can be seen as a "interactive" generalisation of transition system, and this notion of simulation generalises the usual notion of simulation over a transition system.

0.6 Positivity predicate

We can define in well-founded type theory a coinductive notion, the notion of positivity on any complete poset $(A, B, C, d, \sqsubseteq)$.

We want define $\mathsf{Pos}\ a$, denoting that the element a is "positive", as meaning that, for any such of an experiment $b \in B\ a$, it is possible to find a $c \in C\ a\ b$ such that $\mathsf{Pos}\ (d\ a\ b\ c)$ is positive. In term of "interaction" or "game" of an observer against nature, it means that there exists a winning strategy for nature (who wins because the game goes on forever). Having this intuition in mind, we define $S_n\ a$ the set of strategy at "level n" inductively

$$S_0\ a \equiv N_1$$

$$S_{n+1}\ a \equiv (\Sigma f \in B\ a \to C\ a\ b)(\Pi b \in B\ a)\ S_n\ (d\ a\ b\ (f\ b))$$

It is then natural to define a strategy as a infinite sequence (σ_n) in $\Pi_n\ S_n\ a$ such that σ_{n+1} "extends" σ_n.

The problem now lies in the definition of "extends". This is where the relation $\sqsubseteq$ is needed. We define by induction on n the relation $\sigma' \sqsubseteq_n \sigma$ for $\sigma' \in S_{n+1}\ a'$ and $\sigma \in S_n\ a$:

$$\sigma' \sqsubseteq_0 \sigma \equiv a' \sqsubseteq a$$

$$(f', g') \sqsubseteq_{n+1} (f, g) \equiv (\forall b \in B\ a)\ (\exists b' \in B\ a')\ g'\ b' \sqsubseteq_n g\ b$$

We define then $\mathsf{Pos}\ a$ as the set of sequences (σ_n) in $\Pi_n\ S_n\ a$ such that $\sigma_{n+1} \sqsubseteq_n \sigma_n$ for all n.

1 On the definition of Formal Spaces

If A is a subset of X we let A' be the subset $\{w \in X \mid \exists u \in A\ w \sqsubseteq u\}$. If A, B are subsets of X, we write $A \sqsubseteq B$ iff for all $a \in A$, there exists $b \in B$ such that $a \sqsubseteq b$, that is $A \subseteq B'$. If A is a subset of X, A is *monotone* iff $u \in A$ and $v \sqsubseteq u$ implies $v \in A$; this is equivalent to $A' \subseteq A$. We let $\mathrm{M}(X)$ be the collection of all monotone subsets of X. We define a *formal space* is a structure $(X, \sqsubseteq, \mathsf{Cov})$ which is a complete lattice such that

- if $S \in \mathsf{Cov}_x$ and $y \in S$ then $y \sqsubseteq x$.

The elements of X are the *basic open* of the space X. The condition that $\sqsubseteq$ is a simulation for (X, Cov) is called the *localisation condition.*

Given such a structure, and U a subset of X, we define inductively $u \lhd U$ by the clauses:

- (C1) $u \lhd U$ if $u \in U$,
- (C2) $u \lhd U$ if $u \sqsubseteq v$ and $v \lhd U$,
- (C3) $u \lhd U$ if $S \lhd U$ for one $S \in \mathsf{Cov}_u$,

where $S \lhd U$ means $s \lhd U$ for all $s \in S$.

In term of spaces as set of points, the relation $u \lhd U$ means that basic open u is covered by the collection U of basic open. We write $u \lhd v$ for $u \lhd \{v\}$ if $u, v \in X$.

TO BE ADDED: if U is monotone, $U = U'$, we don't need the clause (C2) in the definition of $u \lhd U$. This was proved essentially by Brouwer.

Furthermore, this can be proved in a strict predicative way.

WARNING: it may well be that there is a better description of formal spaces as special case of tree set constructors.

One important thing here is that we have a natural example of a generalised inductive definition. If we represent a derivation of $u \lhd U$ as a tree, then the branching of the tree is determined by the "size" of the sets in Cov_x for $x \in X$.

We discuss shortly the representation of this in Type Theory. We must have for each $x \in X$, a set C_x and for each $x_1 \in C_x$ a set D_{xx_1} together with a map $f : (\Sigma_{x \in X} \Sigma_{x_1 \in C_x} D_{xx_1}) \to X$. We then express that $f(x, x_1, x_2) \sqsubseteq x$ for all $x_1 \in C_x,\ x_2 \in D_{xx_1}$.

It may be interesting that Kent Petersson and Dan Synek have introduced a similar structure, the *tree type*, in Type Theory [6], motivated by considerations from language theory. What seems to me important here is that formal spaces are closely connected to the theory of generalised inductive definitions, since, as noted by Per Martin-Löf, tree types are the natural expressions of Post systems in Type Theory.

Furthermore, we have to express that if $y \sqsubseteq x$ and $x_1 \in C_x$, then there exists $y_1 \in C_y$ such that for all $y_2 \in D_{yy_1}$, there exists $x_2 \in D_{xx_1}$ such that $f(y, y_1, y_2) \sqsubseteq f(x, x_1, x_2)$.

If we think of the elements of X as stage of knowledge as before, the order $x \sqsubseteq y$ can be read as "x represents a possible future stage of y" or "x is more informative than y." The condition $f(x, y, z) \sqsubseteq x$ is then natural: one has more knowledge after doing this experiment. The localisation condition expresses that if $y \sqsubseteq x$ one can simulate any experiment of the stage x at the stage y.

Lemma 1: if $u \sqsubseteq v$, then $u \lhd v$.

Proof: By (C1), we have $v \lhd v$, and by (C2), we get $u \lhd v$. $\square$

In general, $u \lhd v$ may hold without having $u \sqsubseteq v$.

Lemma 2: If $U \subseteq V$, then $U \lhd V$.

Proof: If $u \in U$, then $u \in V$ and hence $u \lhd V$ by the rule (C1).

Lemma 3: If $u \lhd U$ and $v \lhd V$, then $w \lhd U' \cap V'$ whenever $w \sqsubseteq u$ and $w \sqsubseteq v$.

Proof: We prove this by induction on the proof that $u \lhd U$ and on the proof that $v \lhd V$. A priori there are 9 cases to analyse. However, the cases where one proof ends with $(C2)$ is direct.

Furthermore, if $u \lhd U$ because of the rule $(C3)$, then there exists $R \in \mathsf{Cov}_u$ such that $r \lhd U$ for all $r \in R$. By definition of a formal space, since $w \sqsubseteq u$, we can find $T \in \mathsf{Cov}_w$ such that $T \sqsubseteq R$. Also, all $t \in T$ satisfy $t \sqsubseteq w$ and hence $t \sqsubseteq u$. By induction hypothesis, we get then $t \lhd U' \cap V'$ for all $t \in T$ and hence $w \lhd U' \cap V'$, because $T \in \mathsf{Cov}_w$.

The case where $v \lhd V$ because of the rule $(C3)$ can be handled similarly. The only left case is that both proofs are instances of the rule $(C1)$: in this case $w \in U' \cap V'$ and hence $w \lhd U' \cap V'$. $\square$

Corollary: If $w \lhd u$ and $w \lhd v$, then $w \lhd \{t \in X \mid t \sqsubseteq u \wedge t \sqsubseteq v\}$.

If $U \subseteq X$, we define $\nabla U = \{u \in X \mid u \lhd U\}$. We have then that $U \lhd V$ iff $U \subseteq \nabla V$. We say that U is a *formal open* of the space X iff $U = \nabla U$. We let $\mathrm{O}(X)$ be the collection of all formal open of the space X.

Lemma 4: $U \subseteq X$ is a formal open iff

- U is monotone,
- if $S \in \mathsf{Cov}_u$ and $S \subseteq U$ then $u \in U$.

Proof: If U is a formal open, and $u \in U'$ then $u \lhd U$ by lemma 1, and hence $u \in U$, so that U is monotone. Furthermore, by (C1) and (C3), if $S \in \mathsf{Cov}_u$ and $S \subseteq U$ then $u \lhd U$, and hence $u \in U$ if $U \in \mathrm{O}(X)$.

Conversely, if U satisfies these two properties, we show that $u \lhd U$ implies $u \in U$ by induction on the proof that $u \lhd U$. There are three cases

1. (C1) we have directly $u \in U$,
2. (C2) we have v such that $v \lhd U$ and $u \sqsubseteq v$; we deduce $v \in U$ by induction hypothesis and hence $u \in U$ because U is monotone,
3. (C3) we have $S \in \mathsf{Cov}_u$ such that $S \lhd U$; we deduce $S \subseteq U$ by induction hypothesis and hence $u \in U$ by hypothesis on U.

Lemma 5: If $U, V \subseteq X$, $u \in X$ and $u \lhd U$, $U \lhd V$ then $u \lhd V$.

Proof: By induction on the proof that $u \lhd U$. If this proof is an instance of the rule $(C1)$, then $u \in U$ and $U \lhd V$ can be instantiated to $u \lhd V$. If it ends with the rule $(C2)$, then $u \sqsubseteq v$ with $v \lhd U$. By induction, $v \lhd V$, and hence $u \lhd V$. Finally, if the rule $(C3)$ has been applied last, we have $S \in \mathsf{Cov}_u$ such that $s \lhd U$ for all $s \in S$. By induction hypothesis $s \lhd V$ for all $s \in S$ and hence $u \lhd V$. $\square$

Corollary: If $U \subseteq X$, then ∇U is a formal open.

Lemma 6: we have

1. $U \subseteq \nabla U$,
2. $U \subseteq V$ implies $\nabla U \subseteq \nabla V$,
3. $\nabla\nabla U = \nabla U$.

Proof: The first property is a reformulation of the rule (C1) in the definition of covering. The second property follows from lemmas 2 and 5, while the third property is a reformulation of the corollary of lemma 5. $\square$

Corollary: If $\nabla U \subseteq U$ then U is a formal open.

Lemma 7: If $U \subseteq X$ then $U' \subseteq \nabla U$.

Proof: This is a reformulation of the rule (C2). $\square$

Corollary: A formal open is monotone.

When we define $a \lhd U$ over a tree set (A, B, C, d), it may be that $\lhd U$ is not monotone in the sense that $a \lhd U$ may not imply $d\ a\ b\ c \lhd U$ in general. For instance, if a has two basic covering $\{a_1\}$ and $\{a_2\}$ and $U = \{a_1\}$. Then $a \lhd U$ but we do not have $a_2 \lhd U$.

Lemma 8: If $U, V \subseteq X$ are formal open, then so is $U \cap V$.

Proof: By lemma 6, we have $\nabla(U \cap V) \subseteq \nabla U$ and $\nabla(U \cap V) \subseteq \nabla V$. But $\nabla U \subseteq U$ and $\nabla V \subseteq V$ because U and V are formal open. Hence $\nabla(U \cap V) \subseteq U \cap V$ and $U \cap V$ is a formal open. $\square$

Lemma 9: If U and V are monotone, then $\nabla(U \cap V) = \nabla U \cap \nabla V$.

Proof: By lemma 6, we have $\nabla(U \cap V) \subseteq \nabla U \cap \nabla V$ for any subsets $U, V \subseteq X$. Conversely, lemma 3 shows that $\nabla U \cap \nabla V \subseteq \nabla(U' \cap V')$, and hence $\nabla U \cap \nabla V \subseteq \nabla(U \cap V)$, if U and V are monotone. $\square$

Notice the monotonicity assumption in lemma 8. In general, it may be that $U \cap V$ is empty while $\nabla V \subseteq \nabla U$, for instance if $U = \{u\}$ and $V = \{v\}$ with $v \sqsubseteq u$ and $v \neq u$, and we have a counter-example to the statement of lemma 9 without the monotonicity assumption as soon as $\nabla\emptyset \neq \nabla\{v\}$.

We consider the collection of all formal open, with the operations of binary intersection $U \cap V$, which is well-defined by lemma 8, and of arbitrary union $\bigvee U_i = \nabla(\cup U_i)$.

Lemma 10: If $U, U_i \subseteq X$ are formal open, we have $U \cap \bigvee U_i = \bigvee(U \cap U_i)$.

Proof: We have $U \cap U_{i0} \subseteq U \cap \bigvee U_i$ by lemma 6, and hence $\bigvee(U \cap U_i) \subseteq U \cap \bigvee U_i$ by lemma 5 again. Notice next that $\cup U_i$ is monotone, hence by lemma 9, we have

$$U \cap \bigvee U_i = \nabla U \cap \nabla(\cup U_i) \subseteq \nabla(\cup(U \cap U_i))$$

which can be rewritten as $U \cap \bigvee U_i \subseteq \bigvee(U \cap U_i)$. $\square$

Proposition 1: The formal open of a space X form a locale.

We denote by $\mathrm{O}(X)$ this locale. Notice that this is a type (large set) and not a set (small set) when expressed in Martin-Löf type theory.

2 Special kind of Spaces in Type Theory

For representing formal spaces in Type Theory, we need an extra structure on top of the tree set (A, B, C, d), namely a poset relation on A that is a simulation. In some cases however, it is possible to define directly a formal space from (A, B, C, d) by taking intuitively the poset relation to be $a' \sqsubseteq a$ iff there exists $(a_0, b_0, c_0), (a_1, b_1, c_1), \dots, (a_n, b_n, c_n)$ such that $a_0 = a$, $a_{i+1} = d\ a_i\ b_i\ c_i$ and $a' = d\ a_n\ b_n\ c_n$. In practice, one does not need to consider this relation, but one associates directly a complete Heyting algebra to the tree set (A, B, C, d).

2.1 Triples

Such an example is provided by the notion of "triples"[2] that are special case of tree set (A, B, C, d) where $B\ a$ is always a singleton. Thus a triple is a structure (A, C, d) with C dependent set over A and $d\ a\ c \in A$ for $a \in A$ and $c \in C\ a$.

This is a suitable type theoretic representation of transition systems.

2.2 Exploding Models

The next example is provided by the case where $B\ a$ is a type of the form $1 + B'\ a$, whose elements are 0 or elements of $B'a$, and $C\ a\ b$ is empty for all $b \in B'a$. Intuitively, the elements of a such that $B'\ a$ holds are "exploding" elements.

2.3 Products

Finally, another way to get formal spaces in Type Theory without the introduction of a poset relation is by taking the product of two formal spaces. For instance, if we have two structures (A_1, C_1, d_1, E_1) and (A_2, C_2, d_2, E_2) then if we define $A = A_1 \times A_2$, $C(a_1, a_2) = C_1\ a_1 + C_2\ a_2$ and $d\ (a_1, a_2)\ (i\ c_1) = (d_1\ a_1\ c_1, a_2)$, $d\ (a_1, a_2)\ (j\ c_1) = (a_1, d_2\ a_2\ c_2)$, and $E\ (a_1, a_2) = E_1\ a_1 \wedge\ E_2\ a_2$ then we get a formal space in Type Theory.

2.4 Large Complete Heyting Algebra

There are natural examples of complete Heyting algebras in Type Theory, actually complete Boolean algebras, for which it is not known yet if they can be described "inductively" with a tree set structure. For instance, starting with a inf semi lattice X and a downward closed subset $Z \subseteq X$, we can define $U^{\perp} = \{y \in X \mid yU \subseteq Z\}$. Then the type of all U such that $U^{\perp\perp} = U$ forms a complete Boolean algebra.

3 Points of a Formal Space

3.1 Definition

Let X be a formal space. It is convenient to adopt the following notation: if α is a predicate on X and U a subset of X, we write $\alpha\ U$ for expressing that there exists $u \in U$ such that $\alpha\ u$.

A *point* of X is a predicate α on X satisfying the following conditions:

- $\alpha\ X$,
- if $u \sqsubseteq v$ and $\alpha\ u$ then $\alpha\ v$,
- if $\alpha\ u$ and $U \in \mathsf{Cov}_u$ then $\alpha\ U$,
- if $\alpha\ u_1$ and $\alpha\ u_2$ then there exists u such that $u \sqsubseteq u_1,\ u \sqsubseteq u_2$ and $\alpha\ u$.

Proposition 2: A predicate α on X is a point iff

- $\alpha\ X$,
- if $\alpha\ u$ and $u \lhd U$ then $\alpha\ U$,
- if $\alpha\ u_1$ and $\alpha\ u_2$ then there exists u such that $u \sqsubseteq u_1,\ u \sqsubseteq u_2$ and $\alpha\ u$.[3]

Proof: If α is a point, we show the second property by induction on the proof that $u \lhd U$. There are three cases

[2]This terminology is due to Peter Hancock [2].

[3]This is implied by the seemingly weaker condition that requires only $u \lhd u_1,\ u \lhd u_2$ and $\alpha\ u$. Indeed, u is then covered by the set $\{w \in X \mid w \sqsubseteq u_1 \wedge w \sqsubseteq u_2\}$ and we can apply the second condition.

1. (C1) we have $u \in U$, and hence $\alpha\ U$,

2. (C2) we have $u \sqsubseteq v$ and $v \lhd U$. Since α is a point $\alpha\ u$ implies $\alpha\ v$, and hence $\alpha\ U$ by induction hypothesis,

3. (C3) we have $S \in C_u$ such that $S \lhd U$; since α is a point, we have $\alpha\ S$ and hence $\alpha\ U$ by induction hypothesis.

Conversely, if α is a predicate satisfying these three conditions, we show that α is a point. We have to show that if $\alpha\ u$ and $u \sqsubseteq v$ then $\alpha\ v$. But this follows from the second condition and the fact that $u \sqsubseteq v$ implies $u \lhd v$. $\square$

3.2 When do Points form a set

In some cases, for instance formal spaces defined over a triple, the points of a space can be enumerated by a set. More precisely, the points of a space X, as predicates over X, have a natural notion of equality. A priori, the collection of all points form a type (a "large" set) $\mathsf{Pt}(X)$ and not a set. In some cases, there exists however a *set* I with an enumeration $f : I \rightarrow \mathsf{Pt}(X)$, such that for any $\alpha \in \mathsf{Pt}(X)$ there exists $i \in I$ such that $f\ i = \alpha$.

For instance, in the case of the Baire space, the large class of points is enumerated by the set N^{N}, and in the case of Cantor space, the class of points is enumerated by 2^{N}.

3.3 Formal Spaces without Points

Here is an example, essentially due to A. Joyal, of a non trivial formal space without points. Intuitively, the points of this space should be onto maps from N to $\mathsf{C} = 2^{\mathsf{N}}$. The basic neighborhood are finite sequences $\sigma = c_1 \ldots c_n$ of elements of C, the ordering $\sigma' \sqsubseteq \sigma$ is the extension ordering and the basic neighborhood are of two kind:

1. σ is covered by the set of all its direct extensions $\sigma.c$,

2. for any c_0 in C, the neighborhood σ is covered by the set of all its extensions $\sigma' \sqsubseteq \sigma$ that contain c_0.

It can be checked that this defines a non trivial formal space (all neighborhood are positive) that have no points, because a point would be a surjective map from N to C.

4 Continuous Maps

A *continuous map* $f : Y \rightarrow X$ can be defined as a point in the sense of the topological model defined by the space Y. Thus while we write $f : Y \rightarrow X$, because in term of points, f represents a map from the points of Y to the points of X, the map f in formal terms is really a map $f^* : X \rightarrow \mathrm{O}(Y)$. If $U \subseteq X$, we write f^*U for $\bigvee_{u \in U} f^*u$. Such a map is a continuous map iff the following conditions hold:

- $Y \lhd f^*X$,
- if $u \sqsubseteq v$ then $f^*u \lhd f^*v$,
- if $U \in \mathsf{Cov}_u$ then $f^*u \lhd f^*U$,
- $f^*u_1 \cap f^*u_2 \lhd f^*\{u \mid u \sqsubseteq u_1 \wedge u \sqsubseteq u_2\}$.

Proposition 3: A map $f^* : X \rightarrow \mathrm{O}(Y)$ is continuous iff the following conditions hold for any $U, V \subseteq X$:

- $Y \lhd f^*X$,
- if $U \lhd V$ then $f^*U \lhd f^*V$,

- $f^*U \cap f^*V = f^*(U' \cap V')$.

Proof: If $f^* : X \to \mathrm{O}(Y)$ satisfies these properties, and $u \sqsubseteq v$, then $u \lhd v$ and hence $f^*u \lhd f^*v$. If $U \in \mathsf{Cov}_u$, we have $u \lhd U$ and hence $f^*u \lhd f^*U$. Finally, $f^*u_1 \cap f^*u_2 = f^*(u_1' \cap u_2')$ and $u_1' \cap u_2' = \{u \mid u \sqsubseteq u_1 \wedge u \sqsubseteq u_2\}$.

Conversely, let $f^* : X \to \mathrm{O}(Y)$ be a continuous map. We prove first that $f^*u \lhd f^*U$ if $u \lhd U$ by induction on the proof that $u \lhd U$. There are three cases:

1. (C1) we have $u \in U$, and hence $f^*u \subseteq f^*U$ and hence $f^*u \lhd f^*U$,
2. (C2) we have $u \sqsubseteq v$ and $v \lhd U$; by induction we get $f^*v \lhd f^*U$ while the second property entails $f^*u \lhd f^*v$, and hence $f^*u \lhd f^*U$,
3. (C3) we have $S \in \mathsf{Cov}_u$ such that $S \lhd U$; by induction we get $f^*S \lhd f^*U$ while the third property shows that $f^*u \lhd f^*S$, and hence $f^*u \lhd f^*U$.

It follows that if $U \lhd V$ then $f^*U \lhd f^*V$.

Finally, we always have $f^*(U' \cap V') \lhd f^*U' \cap f^*V'$ by the second property, while the second property shows also $f^*U' \lhd f^*U$ and $f^*V' \lhd f^*V$ because $U' \lhd U$ and $V' \lhd V$. This shows $f^*(U' \cap V') \lhd f^*U \cap f^*V$. Conversely, by using the distributivity rule (proposition 1), we have that $f^*U \cap f^*V = \bigvee_{u \in U, v \in V} f^*u \cap f^*v$. Since $f^*u \cap f^*v \lhd f^*(U' \cap V')$ by the fourth property, we get that $f^*U \cap f^*V \lhd f^*(U' \cap V')$. $\square$

5 Product of Two Spaces

The product of two spaces X and Y is defined as follow:

- the basic open of $X \times Y$ are the pairs (u, v) where $u \in X$ and $v \in Y$,
- $(u_1, v_1) \sqsubseteq (u_2, v_2)$ iff $u_1 \sqsubseteq u_2$ and $v_1 \sqsubseteq v_2$,
- $\mathsf{Cov}_{(u,v)}$ is the disjoint sum $\mathsf{Cov}_u + \mathsf{Cov}_v$; an element $S \in \mathsf{Cov}_u$ defines the covering $(s, v),\ s \in S$ of (u, v), while an element $T \in \mathsf{Cov}_v$ defines the covering $(u, t),\ t \in T$ of (u, v).

Let us give a formulation in Type Theory. Given (X, A_x, B_{xx_1}, f) with $f(x, x_1, x_2) \in X$ for $x \in X,\ x_1 \in A_x,\ x_2 \in B_{xx_1}$ and (Y, C_y, D_{yy_1}, g) with $g(y, y_1, y_2) \in Y$ for $y \in Y,\ y_1 \in C_y,\ y_2 \in D_{yy_1}$. We define then the structure $E_{(x,y)} = A_x + C_y$ and $F_{(x,y)x_1} = A_{xx_1},\ F_{(x,y)y_1} = B_{yy_1}$, while $h(z, z_1, z_2)$ for $z \in X \times Y,\ z_1 \in E_z,\ z_2 \in F_{zz_1}$ is defined by the equations $h((x, y), x_1, x_2) = (f(x, x_1, x_2), y)$ and $h((x, y), y_1, y_2) = (x, g(y, y_1, y_2))$. It is then direct that the condition $h(z, z_1, z_2) \sqsubseteq z$ holds if this condition is satisfied both by f and g.

Proposition 4: The structure $X \times Y$ is a formal space; furthermore the map $\pi_1 : X \times Y \to X$ defined by $\pi_1\ x = \{(u, v) \in X \times Y \mid u \lhd x\}$ and the map $\pi_2 : X \times Y \to Y$ defined by $\pi_2\ y = \{(u, v) \in X \times Y \mid v \lhd y\}$ are continuous maps.

6 Positive and Connected Spaces

A space $X, \sqsubseteq, \mathsf{Cov}$ is *positive* iff each element of Cov_u are inhabited, for any $u \in X$.

Lemma 10: If X is positive U is inhabited whenever $u \lhd U$.[4]

Proof: By induction on the proof that $u \lhd U$. There are three cases:

1. (C1) $u \in U$ and hence U is inhabited,
2. (C2) there exists v such that $u \sqsubseteq v$ and $v \lhd U$; by induction hypothesis U is inhabited,

[4] Brouwer had a similar condition in his definition of a spread.

3. (C3) there exists $S \in \mathsf{Cov}_u$ such that $S \lhd U$; since X is positive, there exists $s \in S$ and then $s \lhd U$, so that U is ingabited by induction hypothesis.

Given a positive space X, we say that two basic open u and v of X are *overlapping* iff there exists a basic open w which is included in both u and v. This definition extends to a finite sequence $u_1, \ldots, u_n$ of elements of X : they are overlapping iff there exists $w \in X$ such that $w \sqsubseteq u_i$ for $i = 1, \ldots, n$. More generally, two subsets U and V of X are overlapping iff there exists $u \in U$ and $v \in V$ such that u and v are overlapping. This is equivalent to: there exists $w \in X$ such that $w \lhd U$ and $w \lhd V$.

A *chain* in X is a non empty sequence $u_1 \ldots u_k$ of basic open of X such that u_i and u_{i+1} are overlapping for all $i < k$.

We say that X is *connected* iff X is positive and whenever $X \lhd U$ and $u, v \in U$, there exists a chain $u = u_1, \ldots, u_n = v$ of elements of U joining u and v.

We say that X is *locally connected* iff X is positive and whenever $u \lhd U$ and v, w are overlapping with u, there exists a chain $u_1 = v, \ldots, u_n = w$ of elements of U all overlapping with u.

Proposition: Let X be a positive space such that whenever $U \in \mathsf{Cov}_u$ and $v, w \in U$ there exists a chain of elements of U joining v and w, then X is locally connected.

7 Open Surjective Maps

A continuous map $f : Y \to X$ is *surjective* iff $f^*U_1 \lhd f^*U_2$ implies $U_1 \lhd U_2$ for any $U_1, U_2 \subseteq X$. By proposition 3, we have that $f^*U_1 \lhd f^*U_2$ iff $U_1 \lhd U_2$.

An important class of continuous surjective maps is obtained by using the following proposition.

Proposition 5: If $\phi : Y \to X$ is a monotone map of posets such that

- ϕ is onto, and if $u \sqsubseteq \phi\ v$, then there exists $w \in Y$ such that $w \sqsubseteq v$ and $\phi\ w = u$,
- if $\phi(v) \sqsubseteq u$ and $S \in \mathsf{Cov}_u$ then there exists $T \in \mathsf{Cov}_v$ such that $\phi(T) \sqsubseteq S$,
- if $V \in \mathsf{Cov}_v$ then $\phi\ v \lhd \phi\ V$,

then the map $f^*\ U = \phi^{-1}\ (\nabla U)$ defines a continuous surjective map $Y \to X$.

Proof: We notice that $\phi(\phi^{-1}(U)) = U$ because ϕ is onto, and $\phi(V)$ is monotone if V is monotone, because of the first condition on ϕ.

We prove that $v \lhd V$ implies $\phi(v) \lhd \phi(V)$ by induction on the proof of $v \lhd V$. There are three cases:

- (C1) we have $v \in V$, hence $\phi(v) \in \phi(V)$ and $\phi(v) \lhd \phi(V)$,
- (C2) we have $v \sqsubseteq v'$, and $v' \lhd V$; by induction hypothesis $\phi(v') \lhd \phi(V)$ and since ϕ is monotone, $\phi(v) \sqsubseteq \phi(v')$; hence $\phi(v) \lhd \phi(V)$,
- (C3) we have $S \in \mathsf{Cov}_v$ and $S \lhd V$; by induction hypothesis $\phi(S) \lhd \phi(V)$, and by hypothesis, $\phi(v) \lhd \phi(S)$; hence $\phi(v) \lhd \phi(V)$.

We deduce from this that if $U \in \mathrm{O}(X)$ then $\phi^{-1}(U) \in \mathrm{O}(Y)$. Indeed, if $U \in \mathrm{O}(X)$ and $v \lhd \phi^{-1}(U)$, then $\phi(v) \lhd U$ because $U = \phi(\phi^{-1}(U))$ and hence $\phi(v) \in U$, that is $v \in \phi^{-1}(U)$. If $U_1 \lhd U_2$ we have $\nabla U_1 \subseteq \nabla U_2$ and hence $f^*U_1 \subseteq f^*U_2$.

Next, we show that if $\phi(v) \sqsubseteq u$ and $u \lhd U$, with U monotone, then $v \lhd \phi^{-1}(U)$.

It follows from this that $\nabla(\phi^{-1}U) = \phi^{-1}(\nabla U)$ if U is monotone. Hence $f^*(\bigvee U_i) = \bigvee(\phi^{-1}U_i)$.

Furthermore, $f^*U_1 \cap f^*U_2 = \phi^{-1}(\nabla U_1 \cap \nabla U_2) = \phi^{-1}(\nabla(U_1' \cap U_2') = f^*(U_1' \cap U_2')$. Since $f^*X = Y$, we have, by proposition 3, that f^* defines a continuous map $Y \to X$.

$\square$

If ϕ is a map satisfying the conditions of proposition 1, we say that the continuous map f defined by $f^*\ U = \phi^{-1}\ (\nabla U)$ is *open surjective.*

8 Simulation

One can define a relation R between two spaces Y and X to be a *simulation* iff whenever $R(y, x)$ and $S \in \mathsf{Cov}_x$ there exists $T \in \mathsf{Cov}_y$ such that for all $t \in T$ there exists $s \in S$ such that $R(t, s)$.

Furthermore it is natural to ask that if $y' \sqsubseteq y$, $x \sqsubseteq x'$ and $R(y, x)$ then $R(y', x')$.

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