A Constructive Analysis of Stone-Weierstrass Theorem

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Two fundamental papers

We present an analysis of the following papers

F. Riesz "Sur la décomposition des opérations fonctionelles linéaires" 1928

M.H. Stone "A General Theory of Spectra II" 1940 Proc. N.A.S.

“Algebraization” of functional analysis

- *formal* description of compact Hausdorff spaces X
- constructive proof of Stone-Weierstrass theorem
- constructive treatment of Gelfand duality (in the real case)
- formal description of the space of probability measure on X
- formal description of Baire subsets of X

Lattice ordered abelian group

Riesz introduced the following structure of l-*group*

an abelian group E with an order

$$x \leq y \rightarrow x + z \leq y + z$$

such that any two elements x, y have a l.u.b. $x \vee y$

Typical examples

- $C(X)$ continuous functions over a space X
- space of continuous linear functionals over $C(X)$ (Riesz)
- positive non zero rationals $a_1/b_1 \mid a_2/b_2$ iff a_1b_2 divides a_2b_1

$P \subseteq E$ will be the set of *positive* elements

Relatively prime elements

We have $x + (y \vee z) = (x + y) \vee (x + z)$

Any two elements x, y have a g.l.b. $x \wedge y$

$$x + y = x \vee y + x \wedge y$$

If $a, b \in P$ we define $a \perp b$ as $a \wedge b = 0$

If $a \perp b$ then $a \vee b = a + b$

We define $x^+ = x \vee 0,\ x^- = (-x) \vee 0$

We have $x = x^+ - x^-$ and $x^+ \perp x^-$

Euclides lemma

Lemma: If $a \leq b + c$ and $a \perp c$ then $a \leq b$

"If $a \mid bc$ and a is relatively prime to c then $a \mid b$"

Corollary: if $a \perp b$ and $a \perp c$ then $a \perp b + c$

Corollary: If $x = a - b$ and $a \perp b$ then $a = x^+$ and $b = x^-$

Corollary: We have $nx^+ = (nx)^+$ and $nx^- = (nx)^-$. Hence also $n(x \vee y) = nx \vee ny$ and $n(x \wedge y) = nx \wedge ny$, and if $nx \leq ny$ then $x \leq y$

It follows that we can *embed* E in a Q vector space

Distributivity

Lemma: We have $(x_1 \wedge x_2)^+ = x_1^+ \wedge x_2^+$

Corollary: The lattice $E, \wedge, \vee$ is *distributive*

More generally, if y_i has a g.l.b. $\wedge y_i$ then y_i^+ has a g.l.b. and

$$\wedge y_i^+ = (\wedge y_i)^+$$

Spectrum of an l-group

We define $a \preceq b$ iff $(\exists n) a \leq nb$

$a \sim b$ iff $a \preceq b$ and $b \preceq a$

Proposition: $(P/\sim, \vee, \wedge)$ is a distributive lattice L

This is a pointfree description of the *spectrum* of E

Spectrum as a theory

Propositional theory: atoms $D(x),\ x \in E$

1. $D(x), D(-x) \vdash$
2. $D(x) \vdash$ if $x \leq 0$
3. $D(x+y) \vdash D(x), D(y)$
4. $D(x \vee y) \vdash D(x), D(y)$

Think of $D(x)$ as the proposition $x > 0$

The models are the total order on E

Formal consequences

Proposition: The map

$$x \longmapsto x^{+}, \qquad E \to L$$

is an interpretation of this theory

Proposition: If $x \leq y$ then $D(x) \leq D(y)$. Also $D(x \vee y) = D(x) \vee D(y)$ and $D(x \wedge y) = D(x) \wedge D(y)$ and $D(x) = D(x^{+})$.

Completness theorem

Proposition: If

$$x_1^+ \wedge \cdots \wedge x_n^+ \preceq y_1^+ \vee \cdots \vee y_m^+$$

then

$$x_1, \ldots, x_n \vdash y_1, \ldots, y_m$$

Theorem: The interpretation

$$x \longmapsto x^+, \qquad E \to L$$

is *complete*

Corollary: We have $x \vdash y$ iff $x^+ \preceq y^+$

Strong unit

We assume that E has a strong unit

$1 \in P$ and $x \preceq 1$ for all $x \in E$

$$\Gamma(E) = [0, 1] = \{x \in E \mid 0 \leq x \leq 1\}$$

Define $\neg a = 1 - a$ and $a \oplus b = 1 \wedge (a + b)$

Theorem: (Munidici) $\Gamma(E), \oplus, \neg$ is an *MV-algebra*, and Γ is an equivalence of category between MV-algebras and l-group

MV-algebra can be equationally described, while l-groups with strong unit cannot be described in a first-order way.

Representation of compact spaces

Let E be a l-group with a strong unit 1

We consider the following presentation $S(E)$ of a frame (locale)

1. $D(x), D(-x) \vdash$

2. $D(x) \vdash$ if $x \leq 0$

3. $D(x+y) \vdash D(x), D(y)$

4. $D(x \vee y) \vdash D(x), D(y)$

5. $\vdash D(1)$

6. $D(x) \vdash \bigvee_n D(nx-1)$

Representation of compact spaces (continued)

Theorem: $S(E)$ describes a compact regular space X. The *points* of X (models of $S(E)$) can be identified to positive linear mapping $E \longmapsto R$ that preserve sups.

$D(x)$ can be thought of as the open

$$D(x) = \{\alpha \in X \mid x(\alpha) > 0\}$$

Gelfand duality

Theorem: We have $X = D(x)$ iff $1 \preceq x$

In term of points: if $x(\alpha) > 0$ for all $\alpha \in X$ then we have n such that $1/n \leq x(\alpha)$ for all $\alpha \in X$

In term of points this follows from the *compactness* of X

Theorem: We have $D(x) = 0$ iff $x \leq 0$. And $D(a) \wedge D(b) = 0$ iff $a \perp b$.

It follows that $X - \overline{D(a)}$ is $\cup\{D(b) \mid b \perp a\}$

Stone-Weierstrass

Proposition: If U_i is a covering of X we can find a *finite* family $p_1, \ldots, p_n \in [0, 1]$ such that $\Sigma p_i = 1$ and each $D(p_i)$ is a formal subset of some U_j.

Theorem: If E is divisible, E can be identified to a *dense* subgroup of $C(X)$.

Baire subsets of X

Let $S_\sigma(E)$ be the Boolean σ-algebra generated by the theory $S(E)$

$S_\sigma(E)$ in term of points, is the σ-algebra of *Baire subsets* of X i.e. the σ-algebra generated by the basic open $D(a),\ a \in P$

Components

$x \in E$ is a *component* of E iff

$$x \in \Gamma(E) \qquad x \perp (1 - x)$$

By distributivity, the set $G(E)$ of all components is a Boolean algebra. It is σ-complete if E is σ-complete.

$e \in [0, 1]$ is a component iff

$$e \oplus e = e$$

σ-complete l-group

We assume now E to be divisible, with a strong unit, and to be σ-*complete*

We define (Riesz, 1928)

$$e(a) = sup_n(1 \wedge na) \in \Gamma(E)$$

Theorem: $e(a)$ is a *component* of E, so that

$$D(e(a)), D(1 - e(a))$$

partition of X. Furthermore, if $b \in [0, 1]$ we have $b \perp a$ iff $b \leq 1 - e(a)$, hence

$$D(e(a)) = \overline{D(a)}$$

In term of points, we have

$$a(\alpha) > 0 \to \alpha \in D(e(a)) \to a(\alpha) \geq 0$$

Components of E

By distributivity, the components of E form a σ-complete Boolean algebra $G(E)$

Theorem: The map $E \to G(E)$

$$x \longmapsto e(x^+)$$

gives an *interpretation* of $S(E)$ in $G(E)$

Spectral Decomposition

If $x \in E$ let $\phi : Q \to G(E)$

$$\phi(r) = e(x - r)^+$$

This is the *spectral decomposition* of x.

If $-n \leq x \leq n$

- $\phi(r) = 0$ for $n \leq r$
- $\phi(r) = 1$ for $r \leq -n - 1$
- $\phi(r) = \vee_{s>r} \phi(s)$

We have

$$x = \int r \; d\phi(r)$$

Applications

If B is σ-complete Boolean algebra let $F(B)$ be the divisible σ-complete l group of distribution functions over B

Theorem: E is isomorphic to $F(G(E))$, if E divisible σ-complete; $G(F(B))$ is isomorphic to B

Theorem: In general, the σ-completion of E is $F(S_\sigma(E))$.

Hence $S_\sigma(E)$ is isomorphic to the algebra of Baire subsets of X.

Cut-elimination

Theorem: We have $D(x) \vdash D(y)$ in $S_\sigma(E)$ iff $D(x) \vdash D(y)$ in $S(E)$ iff $(nx-1)^+ \preceq y^+$ for all n.

This means that we can think of $D(x)$, seen as a Baire subset, extensionally as a set of points.

Space of measures

We consider the following presentation $M(E)$ of a frame (or locale)

1. $D(x), D(-x) \vdash$
2. $D(x) \vdash$ if $x \leq 0$
3. $D(x+y) \vdash D(x), D(y)$
4. $\vdash D(1)$
5. $D(x) \vdash \bigvee_n D(nx-1)$

We simply removed the clause

$$D(x \vee y) \vdash D(x), D(y)$$

from $S(E)$

Space of measures (continued)

Theorem: $M(E)$ describes a compact regular space $M(X)$. The *points* of $M(X)$ (models of $M(E)$) can be identified to positive linear mapping $I : E \longmapsto R$ such that $I(1) = 1$. Hence the points of $M(X)$ can be identified to *probability measure* over X.

This follows from Stone-Weierstrass theorem

Generic measure

Let $M_\sigma(E)$ be the σ-complete Boolean algebra generated by the theory $M(E)$.

Theorem: The map $E \to F(M_\sigma(E))$

$$x \longmapsto (r \longmapsto D(x-r))$$

is a positive linear mapping such that $I(1) = 1$