On the measure problem

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Abstract

Using the notion of *entailment relation* [Sco74] we give a simple proof of a basic result of Tarksi (1929) on the existence of an additive function over a monoid.

1 Introduction

There are several examples of topological spaces naturally associated to a mathematical structure: the spectrum (with Zariski topology) associated to a ring, the space of valuations associated to a field, and the space $\mathsf{Fn}\ E$ of linear functionals of norm not exceding 1 associated to a normed vector space E. In most of these examples, the associated space is compact (and sometimes compact Haussdorff), and one has an extension theorem of the form: if A is a substructure of B then any point of the space associated to A extend to a point of the space associated to B. It is very natural to analyse such situation from a "topology without points" or localic [Joh82] framework: instead of associating a space, one associates a distributive lattice. This has been done in the case of the spectrum of a ring, by a construction due to A. Joyal [Esp86]. It is shown in [Esp86] how some theorems of commutative algebra become constructive and elementary if understood in such a "pointfree" way. Another suggestive view is to see this distributive lattice as the (Tarski-)Lindenbaum algebra of a propositional theory. The extension theorem mentionned above becomes then a conservativity theorem between two theories.

In this note we associate to any commutative group, and then commutative monoid, an entailment relation. In this way, we can prove a basic result of Tarski on the existence of an additive measure on a monoid. An application is a necessary and sufficient condition for the existence of an invariant finitely additive measure function on a Boolean algebra with a group of automorphisms.

2 Entailment Relation

First we give a version for commutative group. Let A be a commuative group. We denote by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ arbitrary finite subsets of A. We define the following relation $\mathcal{A} \vdash \mathcal{B}$ iff there exists a non empty sum of elements of $\mathcal{A}$ equal to a sum of elements in $\mathcal{B}$.

Theorem 2.1 *The relation $\vdash$ satisfies the following conditions of reflexivity, monotonicity and transivity*

$$\mathcal{A} \vdash \mathcal{B} \ if \ \mathcal{A} \cap \mathcal{B} \neq \emptyset \qquad (R)$$

$$\frac{\mathcal{A} \vdash \mathcal{B}}{\mathcal{A}, \mathcal{A}' \vdash \mathcal{B}, \mathcal{B}'} \tag{M}$$

$$\frac{\mathcal{A} \vdash x, \mathcal{C} \quad \mathcal{A}, x \vdash \mathcal{C}}{\mathcal{A} \vdash \mathcal{C}} \tag{T}$$

Theorem 2.2 *The relation $\vdash$ is the least entailment relation such that*

$$x + y \vdash x, y$$

$$x, -x \vdash$$

3 Monoid

Let now be M a commutative monoid with a 0 element. It is standard that we can associate to M a commutative group A: the elements are formal differences $x - y \in M^2$ and $x_1 - y_1 = x_2 - y_2$ iff there exists $x \in M$ such that $x_1 + y_2 + x = x_2 + y_1 + x$. Furthermore $x_1 - y_1 + x_2 - y_2$ is $x_1 + x_2 - y_1 + y_2$ which is a congruence with respect to the equality of A. If $i : M \to A$ is defined by $i(x) = x - 0$ then i is a monoid morphism. Furthermore it has the following universal property: if B is commutative group and $f : M \to B$ a monoid map, then there exists one and only one map $\bar{f} : A \to B$ such that $\bar{f} \circ i = f$.

We can use this construction to a characterisation of the least entailment relation on M^2 such that

- $a < b, b < a \vdash$
- $a + b < c + d \vdash a < c, b < d$
- $a < b \vdash a + x < b + x$

Theorem 3.1 *$X \vdash Y$ iff there exist $x_i < y_i \in X$, $i \leq n$, and $z_j < t_j \in Y$, $n > 0$ and x such that*

$$\Sigma y_i + \Sigma z_j + x = \Sigma x_i + \Sigma t_j + x$$

Let $<$ be a model of this theory. We suppose $0 < a_0$ and that for any $a \in M$ there exists x and n such that $a + x = n.a_0$. We can define $f : M \to R$ by defining $r < f(a)$ iff there exists $p/q > 0$ such that $r < p/q$ and $p.a < q.a_0$. Similarly we define $f(a) < s$ iff there exists $p/q \geq 0$ such that $p/q < s$ and $p.a_0 < q.a$. It can be checked that $f(a) \in R$ is uniquely defined by these conditions and that f is finitely additive, and that $f(a_0) = 1$.

Conversely, any such function defines a model of the theory above by taking $a < b$ to mean $f(a) < f(b)$. And we have $0 < a_0$ if $f(a_0) = 1$.

Hence there exists such a function iff the previous theory is consistent that is iff we don't have $a_0 > 0 \vdash$, which is equivalent to the existence of n and x such that

$$(n + 1).a_0 + x = n.a_0$$

4 Application: invariant measure

Let B be a boolean algebra. We associate to B a monoid $N(B)$: the elements are finite sequences (x_i, n_i) where x_i is a partition of unity in B and $n_i \in N$. We write this also $\Sigma n_i x_i$. We define $\Sigma m_i x_i = \Sigma n_j y_j$ to mean that $m_i \neq n_j$ implies $x_i y_j = 0$. We then have a map $i : B \to N(B)$, such that

$$i(xy) + i(x \vee y) = i(x) + i(y)$$

(that is i is a *valuation*) and this map is universal with this property. If $f : B \to M$ is another valuation in a commutative monoid M, there exists only and only one monoid morphism $\bar{f} : N(B) \to M$ such that $\bar{f} \circ i = f$.

If G is a group of automorphisms of B then G acts on $M(B)$. We can define an equaivalence relation on $M(B)$ by $x \equiv y$ iff there exists $x_i, y_i \in M$ and $g_i \in G$ such that $x = \Sigma x_i,\ y = \Sigma y_i$ and $y_i = g_i x_i$. It is clear that if $x \equiv y$ and $x' \equiv y'$ then $x + x' \equiv y + y'$. Let M be the quotient monoid.

To find an invariant measure on B is then equivalent to find a positive additive funtion $f : M \to R$ such that $f(1) = 1$.

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