Severi-Brauer variety

Introduction

We assume to know that $\mathbb{P}^n = \mathbb{P}^n$ is $PGL_{n+1}(R)$ in SAG. Using this, we prove the following statement, for X scheme

$$X \to \|X = \mathbb{P}^n\|_{et} \to \|X = \mathbb{P}^n\|$$

Here $\|T\|_{et}$ represents the propositional truncation w.r.t. the étale topology.

A scheme X such that $\|X = \mathbb{P}^n\|_{et}$ for some n is called a *Severi-Brauer* variety. The statement says that if a Severi-Brauer variety has a rational point, it is locally for Zariski topology the projective space. In the case where the base ring is a field, this was the content of the thesis of F. Châtelet.

This uses the fact that $PGL_{n+1}(R)$ is also $M_{n+1}(R) = M_{n+1}(R)$ as equality of R-algebra. Furthermore the proof follows closely the original reasoning of Châtelet.

1 Delooping

Let G be a group and B_0 and B_1 be two delooping of G, i.e. B_i is a connected groupoid with a point a_i such that G is $a_i = a_i$.

Let T_G the type of G-torsors and $*_G$ the trivial G-torsor.

If x is in B_i then $x = a_i$ is a G-torsor.

So we have a map (which is an equivalence) $\alpha_i : B_i \to T_G,\ x \mapsto (x = a_i)$. Note that $\alpha_i(a_i)$ is the trivial G-torsor $*_G$.

We define then a correspondance R between B_0 and B_1 by taking $R\ x\ y$ to be $\alpha_0(x) =_{T_G} \alpha_1(y)$.

For x in B_0 let $P(x)$ be the type $\Sigma_{y:B_1} R\ x\ y$. The type $P(a_0)$ is equivalent to the singleton type $\Sigma_{y:B_1} y = a_1$ and hence is contractible. Since B_0 is connected we have that $P(x)$ is contractible for all x.

Similarly the type $\Sigma_{x:B_0} R\ x\ y$ is contractible for all y in B_1.

Hence the relation R defines an equivalence between B_0 and B_1.

2 The étale modality

A type X is modal for the étale modality (we shall simply write that X is étale) iff for any unramifiable monic polynomial P the diagonal map $X \to X^E$ is an equivalence, where $E = \|Sp(R[X]/(P))\|$ is the proposition expressing that P has a root.

Let M be a R-module which is quasi-coherent, i.e. $M \otimes_R A \to M^{Sp(A)}$ is an equivalence for any f.p. R-algebra A.

Lemma 2.1. *The proposition that M is flat is étale.*

The proof follows the usual external argument.

Proof. Let $LC = 0$ be a linear relation with L line vector in R and C column vector in M. We assume $\|\Sigma_P \Sigma_{C_1} C = PC_1 \wedge LP = 0\|^{Sp(A)}$ for some $A = R[X]/(P)$, P unramifiable, and we prove

$$\|\Sigma_P \Sigma_{C_1} C = PC_1 \wedge LP = 0\|$$

Using Zariski local choice, we get $(u_1, \dots, u_m) = 1$ in A and using the fact that M is quasi-coherent, we get P_1, C_1 in $A[1/u_1]$, $\dots$, P_m, C_m in $A[1/u_m]$ such that $C = P_1C_1 = \cdots = P_mC_m$ and $LP_1 = \cdots =$

$LP_m = 0$. We then get P_0, C_0 in A such that $C = P_0 C_0$ and $LP_0 = 0$. If P is of degree m, we can write any element of A as a polynomial of degree $< m$. Looking at the constant term, we can assume P_0 and C_0 have coefficients in R, and this shows that $\|\Sigma_{P_0}\Sigma_{C_0} C = P_0 C_0 \wedge LP_0 = 0\|$ as required. □

We can prove similarly.

Lemma 2.2. *The proposition that M is f.p. is étale.*

Since to be finite projective is equivalent to be flat and f.p., we get the following result.

Corollary 2.1. *The proposition that M is finite projective is étale.*

3 The argument

We work now in SAG. We let G be the group $PGL_{n+1}(R)$.

If A is a R-algebra free of dimension $(n+1)^2$ with an explicit basis, we can consider the scheme $SB(A)$ of all left ideals of A of dimension $n+1$. We can check that we have $SB(M_{n+1}(R)) = \mathbb{P}^n$.

We know (Hugo) that any scheme is étale. So in particular $\mathbb{P}^n$ is étale.

We consider in the étale topos the connected component of $\mathbb{P}^n$ in the universe. This is a delooping of $PGL_{n+1}(R)$.

Another delooping is the type of algebras A that are modal and quasi-coherent and such that $\|A = M_{n+1}(R)\|_{et}$. This is the type of Azumaya algebra of degree $n+1$.

If we look at A as a R-module this is a free R-module of dimension $(n+1)^2$ in the étale topos.

It should follow that it is finite projective as a R-module and hence free of dimension $(n+1)^2$ since R is local and the notion of being finite projective is étale.

If X is a Severi-Brauer variety, we write $\alpha_0(X)$ the G-torsor $X = \mathbb{P}^n$.

If A is a Azumaya algebra of degree n, we write $\alpha_1(A)$ the G-torsor $A = M_{n+1}(R)$.

Using Section 1, we see that we have an equivalence between the type of Severi-Brauer variety X of dimension n and the type of Azumaya algebra A of degree $n+1$. This is given by the relation

$$\alpha_0(X) = \alpha_1(A)$$

where the equality is equality in the type of G-torsors.

Given X Severi-Brauer variety of dimension n, we thus find explicitely an Azumaya algebra A of degree $n+1$ such that

$$\alpha_0(X) = \alpha_1(A)$$

Since we want to prove $\|X = \mathbb{P}^n\|$ which is a proposition, we can assume that A is explicitely free of dimension $(n+1)^2$ as a R-module.

We can check that, in this case, $SB(A)$ is a scheme together with an equality of G-torsor

$$\alpha_1(A) = \alpha_0(SB(A))$$

It follows from that that $\alpha_0(SB(A)) = \alpha_0(X)$ and hence $SB(A) = X$.

If X has a point, so has $SB(A)$ and we get a left ideal V of A of dimension $n+1$. Then $A = End_R(V)$ and we have $\alpha_1(A)$ is the trivial G-torsor.

So $\alpha_0(X)$ is the trivial G-torsor as well and we have an element of $X = \mathbb{P}^n$.