A Note on Formal Iterated Function Systems

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Introduction

Edalat has introduced the notion of weakly hyperbolic iterated function systems [3] and showed that it allows for natural domain theoretic proofs (also for the probabilistic case). This note tries to analyse some of these arguments from the point of view of formal (or point-free) topology [4]. It can be also be seen as a first attempt towards a formal approach to iterated function systems (see [6] for a related work). It also generalises and simplifies some results in [2].

We first limit ourselves to the case of compact totally disconnected spaces. We then have a purely algebraic notion of formal iterated function systems, as a boolean algebra B with a finite number of endomaps. We associate to any such system a particular downward closed subset Z of the boolean algebra, called the *zero subset* of the function system. We give next in this context a definition of weak hyperbolicity and show that in this case the zero subset is decidable and is prime. The proof of decidability of the zero subset can be read as an algorithm that computes the attractor of the iterated function systems. We show also in this case that there exists a unique invariant measure on the boolean algebra B. We then consider two possible extensions. The first generalisation concerns the case where the space is compact totally disconnected, but the system is not necessarily weakly hyperbolic. We use the zero subset to characterise a geometric theory describing a generic attractor. The second extension is the case of a weakly iterated function system on a compact Hausdorff space, not necessarily totally disconnected. We sketch in this case how to define an invariant measure.

1 Point-Free Topology

In this section, we fix notations and recall some basic facts about point-free, or *formal*, topology. We intend to use a purely constructive framework, so that our proofs can be directly read as algorithms. Though it is still work in progress how to adapt the definitions of [4] to a purely constructive and predicative framework, some simple cases are understood. The simplest case is the one of compact, Hausdorff totally disconnected. Such a space can be defined formally as a boolean algebra. Notice that the notion of boolean algebra, as a set B with an equality relation and constants $0, 1$ and binary operations $\cdot, \vee$ satisfying some equational axioms is purely elementary. We can define as usual $x \leq y$ by $x \cdot y = x$ or $x \vee y = y$. If we work in a constructive framework, it is implicitely understood that the operations $\cdot, \vee$ are computable. In all examples we have analysed so far, we even have further that the equality $=$ on B is decidable[1], and hence $\leq$ is decidable as well. To simplify the treatment of this work, we will hence assume that all boolean algebras we consider have a *decidable* equality, and that they are *not trivial*, in the sense that we have $0 \neq 1$.

[1] The decidability of $=$ cannot be expressed in a classical framework.

Classically a boolean algebra B corresponds by Stone duality [4] to a compact totally disconnected space X; the elements of B may be thought of as the closed open subset of X. Furthermore any continuous map $f : X \to X$ corresponds to a endomap $g : B \to B$, which is a function preserving $0, 1, \cdot, \vee$. We write simply by juxtaposition gx the result of appyling a map g to an element $x \in B$. In a formal approach to topology, we would have only the boolean algebra B with an endomap $g : B \to B$ and consider that this defines a space with a continuous map over it. Classically this provides the same information as the pair X, f, but constructively this has some direct advantages; take for instance Cantor space $X = 2^\omega$. Classically this is a set of infinite objects (infinite sequence of $0, 1$) while the elements of its corresponding boolean algebra B are finite and concrete objects[2]. It may seem surprising that the spaces we consider in this way, though constructively given, are not necessarily metric spaces. In most examples however, B will be classically countable, and hence the space X, compact Hausdorff and with a countable basis of open sets, will be metrizable. We emphasize again that, in a constructive framework, it is implicit that any map $g : B \to B$ is computable.

We use the letters $w, w', \ldots$ to denotes finite sequence of natural numbers. If we have a set A together with a family of functions $f_i : A \to A$ for $i < n$ we let f_w be the composition $f_{i_1} \ldots f_{i_k}$ for any sequence $w = i_1 \ldots i_k$ of numbers $i_l < n$.

We define inductively when a finite set of finite sequences $i_0 \ldots i_{m-1}$, $i_l < n$ is a *bar*: first the empty sequence alone $\{\epsilon\}$ is a bar, second if $L \cup \{w\}$ is a bar then so is $L \cup \{w0, \ldots, w(n-1)\}$. If L is a bar for any infinite sequence α there is m such that $\alpha(0) \ldots \alpha(m-1) \in L$.

An important class of iterated function systems is given by looking at the spaces $X_n = \{0, \ldots, n-1\}^\omega$ with the maps $f_i(i_0 i_1 i_2 \ldots) = i i_0 i_1 i_2 \ldots$. Formally, we consider the initial boolean algebra B_n initial among the algebra B with an operation $B^n \to B$. Intuitively an element of B_n may be described inductively as being either 0 or 1 or a list $(x_0, \ldots, x_{n-1})$ of elements of B_n. Furthermore, the map $B_n^n \to B_n$ is an isomorphism and the composition of the inverse map with projection defines n endomaps $\kappa_i : B_n \to B_n$ such that $x = (\kappa_0 x, \ldots, \kappa_{n-1} x)$ and $x_i = \kappa_i(x_0, \ldots, x_{n-1})$. Each element $x \in B_n$ describes a closed open subset of X_n : the element 1 represents X_n, the element 0 the empty set and if $x = (x_0, \ldots, x_n)$ the element x represents inductively the set of infinite sequences $i_0 i_1 i_2 \ldots$ such that $i_1 i_2 \ldots$ is in the set represented by x_{i_0}.

Given two iterated functions systems $(X, u_i)_{i<n}$ and $(Y, v_i)_{i<n}$ a *morphism* between these two function systems is a continuous map $f : X \to Y$ such that $v_i \circ f = f \circ u_i$ for all $i < n$.

Similarly, given two formal iterated functions systems $(A, h_i)_{i<n}$ and $(B, g_i)_{i<n}$ a *morphism* between these two function systems will be a map $f : B \to A$ such that $h_i \circ f = f \circ g_i$ for all $i < n$.

We recall that a *filter* of B is a subset $F \subseteq B$ which is *upward closed*, if $x \in F$ and $x \leq y$ then $y \in F$, contains 1 and is closed by product, if $x, y \in F$ then $x \cdot y \in F$. Dually an *ideal* of B is a subset $I \subseteq B$ which is *downard closed*, if $x \in I$ and $y \leq x$ then $y \in I$, contains 0 and is closed by disjunction, if $x, y \in I$ then $x \vee y \in I$.

2 Formal Iterated Function Systems

A *formal iterated function system* is a boolean algebra B with a finite set $g_0, \ldots, g_{n-1}$ of endomaps of B.

If X is the stone space (compact, totally disconnected) associated to B [4] the maps $g_i,\ i < n$ correspond to continuous functions $f_i : X \to X,\ i < n$

[2] In this case B can be described as the Lindenbaum algebra of propositional logic, and the elements of B may be thought of as propositional formulae.

If $x \in B$ we let F_x to be the least filter containing x and closed by the maps g_i, $i < n$. Intuitively F_x is the set of elements that are $\geq$ a product of elements of the form $g_w x$.

Lemma 2.1 *If $y \leq x$ then $F_x \subseteq F_y$.*

Proof. F_y is a filter closed by g_i, $i < n$ which contains y and hence x. Since F_x is the *least* filter containing x and closed by g_i we have $F_x \subseteq F_y$. $\square$

Lemma 2.2 *For any $x, y \in B$ and $i < n$ we have $F_{y.g_i x} \subseteq F_{y.x}$.*

Proof. It is enough to show that $y \cdot g_i x \in F_{y \cdot x}$. But this follows from $y \cdot x \cdot g_i(y \cdot x) \leq y \cdot g_i x$. $\square$

Let $Z \subseteq B$ be the set of all $x \in B$ such that $0 \in F_x$. We call Z the *zero subset* of the system $(B, g_i)_{i<n}$. It satisfies the following important properties.

Proposition 2.3 *We have $0 \in Z$ and Z is downward closed. Furthermore $y \cdot g_i x \in Z$ implies $y \cdot x \in Z$.*

Proof. Direct by lemmas 2.1 and 2.2. $\square$

Corollary 2.4 *If $g_i x \in Z$ then $x \in Z$.*

Proof. Take $y = 1$ in proposition 2.3. $\square$

Define gx to be $g_0 x \cdot \ldots \cdot g_{n-1} x$. The application g is not a boolean map in general but it respects product: we have $g(x \cdot y) = gx \cdot gy$. Furthermore, F_x can be described as the least filter containing x which is closed by g.

Proposition 2.5 *$y \cdot x \in Z$ implies $y \cdot gx \in Z$.*

Proof. We show that $u \in F_{y \cdot x}$ implies $g(u \cdot gu) \in F_{y \cdot gx}$. This will imply the proposition in the special case $u = 0$. It is enough to show that the set X of u such that $g(u \cdot gu) \in F_{y \cdot gx}$ is a filter containing $x \cdot y$ and closed by g. It is clear that X is upward closed. Since $y \cdot gx \cdot g(y \cdot gx) g^2(y \cdot gx) \leq g(x \cdot y \cdot g(x \cdot y))$ we have $x \cdot y \in X$. Finally $g(u \cdot gu) \in F_{y \cdot gx}$ implies $g(gu \cdot g^2 u) = g^2(u \cdot gu) \in F_{y \cdot gx}$ and hence $u \in X$ implies $gu \in X$. $\square$

3 Weakly Hyperbolic System

We recall [3] that a (pointwise) system $(X, f_i)_{i<n}$ is *weakly hyperbolic* iff for each infinite sequence α of numbers $< n$ the intersection $\cap_m f_{\alpha(0)\ldots\alpha(m-1)} X$ is a singleton.

We say that the system $(B, g_i)_{i<n}$ is *weakly hyperbolic* iff for any $x \in B$ there exists N such that any element $g_w x$ is 0 or 1 if w is a sequence of length $\geq N$.

Let $(B, g_i)_{i<n}$ be a formal system and $(X, f_i)_{i<n}$ the corresponding pointwise system by Stone duality.

Proposition 3.1 *The system $(X, f_i)_{i<n}$ is weakly hyperbolic iff the formal system $(B, g_i)_{i<n}$ is weakly hyperbolic.*

Proof. If $X, f_i,\ i < n$ is weakly hyperbolic, let a be an arbitrary closed open subset of X. For any infinite sequence α, we have that $a \cap \cap_m f_{\alpha(0)\ldots\alpha(m-1)}X = \emptyset$ or $(X - a) \cap \cap_m f_{\alpha(0)\ldots\alpha(m-1)}X = \emptyset$ since $\cap_n f_{\alpha(0)\ldots\alpha(n-1)}X$ is a singleton. By compactness for each α there exists N such that $a \cap f_{\alpha(0)\ldots\alpha(N-1)}X = \emptyset$ or $(X - a) \cap f_{\alpha(0)\ldots\alpha(N-1)}X = \emptyset$ and by König lemma, we can take the same N for all α. This proves that $(B, g_i)_{i<n}$ is weakly hyperbolic.

Conversely if B, g_i is weakly hyperbolic, that is we have N such that $g_w a = 0$ or 1 for any $a \in B$ and any sequence w of length N, then we claim that $\cap_m f_{\alpha(0)} \ldots f_{\alpha(m-1)}X$, which is non empty by compactness, is necessarily a singleton. Otherwise, thinking of B as the set of closed open subset of X, there exists an element $a \in B$ such that both $a \cap \cap_m f_{\alpha(0)} \ldots f_{\alpha(m-1)}X$ and $(X - a) \cap \cap_m f_{\alpha(0)} \ldots f_{\alpha(m-1)}X$ are non empty. In particular both $a \cap f_{\alpha(0)} \ldots f_{\alpha(N-1)}X$ and $(X - a) \cap f_{\alpha(0)} \ldots f_{\alpha(N-1)}X$ are non empty. This contradicts the fact that we have $g_w a = 0$ or 1 for any sequence w of length N. □

We can rephrase this definition using the notion of inductive definitions [1]. Let $(B, g_i)_{i<n}$ be any formal iterated function systems, and define $a \in B$ to be *accessible* iff a is 0 or 1 or, inductively, all elements $g_i a,\ i < n$ are accessible. Similarly say that a is *secured* iff a is 1 or, inductively, all elements $g_i a,\ i < n$ are secured. It is clear that B is weakly hyperbolic iff all its elements are accessible.

We still have a third possible formulation, which will be useful for the generalisation to the case of spaces compact Hausdorff not necessarily totally disconnected. We can then say that B is weakly hyperbolic iff for any $x \in B$ there exists a bar L such that for any $w \in L$ we have $g_w x = 0$ or $g_w x = 1$. Notice that, with this formulation, x will be secured iff we have $g_w x = 1$ for all $w \in L$.

Lemma 3.2 *For any formal iterated function system $(B, g_i)_{i<n}$, the set S of secured elements is a filter which is closed by each map g_i.*

Proof. It is clear that $1 \in S$ and that $y \in S$ if $x \leq y$ and $y \in S$. We show next that $x \cdot y \in S$ if $x \in S$ and $y \in S$ by induction on the proof that x and y are secured: if $x = 1$ or $y = 1$ it is direct that $x \cdot y \in S$; otherwise we have by induction hypothesis $g_i(x \cdot y) = g_i x \cdot g_i y \in S$ for all $i < n$ and hence $x \cdot y \in S$. Finally, if a is accessible then $a = 0$ or 1 and then $g_i a = a$ is accessible or each $g_i a$ is accessible by definition. □

Theorem 3.3 *If $(B, g_i)_{i<n}$ is weakly hyperbolic then its zero subset Z is decidable and prime, that is, $x \cdot y \in Z$ implies $x \in Z$ or $y \in Z$.*

Proof. We show that S is a decidable subset and that Z is the complement of S.

First, it is clear that $x \in S$ is decidable by induction on the proof that x is accessible: if $x = 0$ then x does not belong to S because we have assumed $0 \neq 1$. If $x = 1$ then $x \in S$. In the other case, we have by induction that each statement $g_i x \in S$ is decidable and $x \in S$ is equivalent to the finite conjunction of these statements.

Second, we show that $x \in S$ iff x is not in Z. If $x \in S$ then $F_x \subseteq S$ because S is a filter containing x and closed by each g_i. Since $0 \neq 1$ we have that 0 is not in S and hence not in F_x so that x is not in Z. Finally, if x is not in S, we show by induction on the proof that x is accessible that $x \in Z$. Since x is accessible we have $x = 0$ or one $g_i x$ is not in S. In the second case we have by induction $g_i x \in Z$, and then $x \in Z$ by corollary 2.4.

Since Z is the complement of a filter, we have that $x \cdot y \in Z$ implies $x \in Z$ or $y \in Z$. □

In term of points, the attractor of $(X, f_i)_{i<n}$ is a compact subset K and S is the set of all closed open subsets x such that $K \subseteq x$. So we see that $K \subseteq x$ is a decidable property of x for

weakly hyperbolic function system. Since $K \cap x = \emptyset$ iff $K \subseteq (X - x)$ it follows that $K \cap x = \emptyset$ is decidable as well, and so is the assertion that $K \cap x$ is non empty. From this follows the following algorithm to represent approximations of the attractor K. To "approximate" the attractor K, we can then take an arbitrary disjoint covering $X = x_1 \cup \ldots \cup x_p$ and we know that K is a subset of the union $\cup x_i$ over x_i such that $K \cap x_i$ is non empty.

Theorem 3.4 *The system $(B_n, \kappa_i)_{i<n}$ is* initial *among the weakly hyperbolic systems.*

Proof. Let $(B, g_i)_{i<n}$ be a weakly hyperbolic system. We build a map $f : B \to B_n$ by defining fa by induction on the proof of accessibility of a. We take $fa = 0$ if $a = 0$, and $fa = 1$ if $a = 1$. By induction we take $fa = (fg_0a, \ldots, fg_{n-1}a)$ in the other case. By construction we have $f \circ g_i = \kappa_i \circ f$. Furthermore we can show $f(x \cdot y) = fx \cdot fy$ and $f(x \vee y) = fx \vee fy$ by induction on the proofs that x and y are accessible.

Such a map f is furthermore uniquely determined since it has to satisfy $f(g_ia) = \kappa_i(fa)$ and hence $fa = (fg_0a, \ldots, fg_{n-1}a)$. □

4 Existence of an Invariant Measure

We define a *measure* on B [8] to be an increasing map $m : B \to [0, 1]$ satisfying the following conditions:

- $m0 = 0$,
- $m1 = 1$,
- $m(x \vee y) + m(x \cdot y) = mx + my$.

Let $(B, g_i)_{i<n}$ be a formal function system and let $p_i \in [0, 1]$ such that $\Sigma_{i<n} p_i = 1$. The map

$$m \longmapsto T\ m = \Sigma_{i<n} p_i(m \circ g_i)$$

defines an operation on measures on B.

Theorem 4.1 *If $(B, g_i)_{i<n}$ is weakly hyperbolic, there exists one and only one measure m such that $m = T\ m$.*

Proof. We define mx by induction on the proof that x is accessible. We have $mx = 1$ if $x = 1$ and $mx = 0$ if $x = 0$ and by induction we can define $mx = \Sigma p_i m(g_i x)$. We can then show by induction on the proof that x, y are accessible that we have

$$m(x \vee y) + m(x \cdot y) = mx + my;$$

indeed this is direct if x or y are 0 or 1, and in the other case, it follows from the fact that $g_i(x \cdot y) = g_i x \cdot g_i y$ and $g_i(x \vee y) = g_i x \vee g_i y$. □

Another way to look at this construction is the following: let $x \in B$ and let L be a bar such that if $w \in L$ then $g_w x = 0$ or $g_w x = 1$. Define $p_w = p_{i_0} \ldots p_{i_{m-1}}$ if $w = i_0 \ldots i_{m-1}$ and notice that $\Sigma_{w \in L} p_w = 1$ and that if m is invariant then we have $mx = \Sigma_{w \in L} p_w m(g_w x)$. We can then define mx to be Σp_w where we take the sum of all $w \in L$ such that $g_w x = 1$.

5 Examples

On each iterated function system $B_n,\ \kappa_i,\ i < n$ it is direct how to define mx by induction on x : we take $mx = 0$ if $x = 0$ and $mx = 1$ if $x = 1$ and finally

$$m(x_0, \ldots, x_{n-1}) = p_0 m x_0 + \ldots + p_{n-1} m x_{n-1}.$$

It is clear that if $(B, g_i)_{i<n}$ is another formal function systems and $f : B \to B_n$ is the initial morphism then the unique invariant measure on B can be defined by $x \longmapsto m(fx)$: this map defines indeed a measure on B and it is invariant since

$$m(fx) = m((\kappa_0(fx), \ldots, \kappa_{n-1}(fx))) = \Sigma p_i m(\kappa_i(fx)) = \Sigma p_i m(f(g_i x)).$$

The algebra $B = B_4$ can be described alternatively as the algebra of "continuous" matrices of 0 and 1: its elements may be 0 or 1 or recursively a 2×2 matrix

$$\begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix}$$

of elements of B; 1 (resp. 0) is identified with a matrix with only 1 (resp. 0). We have then 4 "access" functions $a_{ij}x\ i,j = 0$ or 1 for $x \in B$ which associates to the matrix x its four corners:

- $a_{ij}0 = 0$,
- $a_{ij}1 = 1$,
- $a_{ij} \begin{pmatrix} x_{00} & x_{01} \\ x_{10} & x_{11} \end{pmatrix} = x_{ij}$.

Proposition 5.1 *$B, a_{00}, a_{10}, a_{11}$ is a weakly hyperbolic function system.*

Indeed, it is direct by induction on $x \in B$ to show that x is accessible. In this case the attractor of the system is a Sierpinski triangle.

6 First Extension: General Formal Iterated Function System

We know consider the case of a formal iterated function system, not necessarily weakly hyperbolic. The zero subset Z is then not decidable in general. The existence of a proper filter F such that $x \in F$ iff $gx \in F$ can however be proved using Zorn's lemma: it is enough for F to be maximal among filters disjoint from Z. In this section, we give an infinitary propositional theory, the standard models of which are maximal filters F disjoint from Z. We build a boolean (non standard) model of this theory using the previous results (in particular, this shows in an elementary way its consistency). In general, this theory cannot be given a standard model in a constructive way.

6.1 Phase Semantics

We first recall some results about phase semantics [7]. Let $(M, \cdot, 1)$ be a meet-semi lattice, and $Z \subseteq M$ a downward closed subset of M. We define the following "polarity" operation on subsets $A \subseteq M : y \in -A$ iff $yx \in Z$ for all $x \in A$. We say that $A \subseteq M$ is a *fact* iff $- - A = A$. The following result is proved in detail in [7].

Theorem 6.1 *The facts of M form a complete boolean algebra for the operation $\bigvee_i A_i = --(\cup_i A_i)$ and $\bigwedge_i A_i = \cap_i A_i$.*

Let $\mathcal{B}$ be this complete boolean algebra.

We give now an infinitary propositional theory T whose models are exactly the maximal filter F such that $F \cap Z = \emptyset$. We introduce an atomic proposition $S(x)$ for each $x \in M$ with the axioms

- $S(z) \vdash \perp$ if $z \in Z$,
- $\vdash S(1)$,
- $S(y) \vdash S(x)$, if $y \leq x$,
- $S(x), S(y) \vdash S(xy)$,
- $\vdash S(x) \vee \bigvee_{x \cdot y \in Z} S(y)$.

The last axiom is infinitary. If we read intuitively $S(x)$ as meaning $x \in F$ this axiom expresses the fact that F is maximal among the filters disjoint from Z : it says that for any x either $x \in F$ or the filter generated by F and x will have a non empty intersection with Z.

Theorem 6.2 *The boolean algebra $\mathcal{B}$ is a model of the theory T interpreting $S(x)$ as the fact $\phi(x) = --x \in \mathcal{B}$.*

Proof. We have $-x \subseteq -y$ and hence $S(y) \vdash S(x)$ is satisfied by this interpretation if $x \leq y$. Similarly $S(z) \vdash \perp$ if $z \in Z$ and $\vdash S(1)$ are checked directly.

Let us prove that $\phi(x) \cap \phi(y) \subseteq \phi(x \cdot y)$. We take $z \in \phi(x) \cap \phi(y)$ and prove $z \in \phi(x \cdot y)$. If $t \in -x \cdot y$ we have $x \cdot y \cdot t \in Z$ and hence since $z \in \phi(x)$ we have $yzt \in Z$. Since $z \in \phi(y)$ this implies $z \cdot t = z^2 \cdot t \in Z$. Hence $z \in \phi(x \cdot y)$.

Finally let us prove the $1 = \phi(x) \vee \bigvee_{x \cdot y \in Z} \phi(y)$. We take $z \in -x$ and $z \in -y$ whenever $x \cdot y \in Z$ and prove $z \in Z$. Since $x \cdot z \in Z$ we have $z \in -z$ and hence $z = z^2 \in Z$. □

Corollary 6.3 *We have $S(x) \vdash S(y)$ iff $-y \subseteq -x$.*

Proof. The theorem shows that $S(x) \vdash S(y)$ implies $\phi(x) \subseteq \phi(y)$ and hence $-y \subseteq -x$. Conversely, if $-y \subseteq -x$ we show $S(x) \vdash S(y)$. We know $\vdash S(y) \vee \bigvee_{t \cdot y \in Z} S(t)$. So it is enough to show $S(x), S(t) \vdash \perp$ whenever $t \cdot y \in Z$. But in this case $t \in -y$ and hence $t \in -x$ that is $t \cdot x \in Z$. We have then $S(x), S(t) \vdash S(x \cdot t)$ and $S(x \cdot t) \vdash \perp$. □

We notice also the following decidability result.

Proposition 6.4 $\vdash S(x) \vee \neg S(x)$ *is a formal consequence of the theory T.*

Proof. By the last condition on G, it is enough to show $\neg S(x)$ if $x \cdot y \in Z$ and $S(y)$ but this direct because $S(x), S(y) \vdash S(x \cdot y)$. □

6.2 A Generic Attractor

Let $(B, g_i)_{i<n}$ be *any* formal iterated formal system. We apply the previous results on $M = B$ seen as a meet-semi lattice with its zero subset $Z \subseteq B$.

Proposition 6.5 *In the theory T we have equivalence between $S(x)$ and $S(gx)$.*

Proof. We use corollary 6.3.

The entailment $S(x) \vdash S(g_i x)$ follows from proposition 2.3. Hence $S(x) \vdash S(gx)$. The converse follows from proposition 2.5. $\square$

It follows that in this case, the models of the theory T are exactly the proper maximal filter of B closed by $g_i,\ i < n$. In term of points they describe minimal proper compact subsets K such that $K = \cup_i f_i(K)$. (We give a concrete example of this situation in [2].)

Proposition 6.6 *If $(B, g_i)_{i<n}$ is weakly hyperbolic $\vdash S(x)$ iff x is secured.*

Proof. It is direct by induction on the proof that x is secured that this implies $\vdash S(x)$. Indeed we have $\vdash S(1)$ and if $\vdash S(g_i x)$ for all $i < n$ then $\vdash S(gx)$ and hence $\vdash S(x)$ by proposition 6.5.

Conversely, it follows from the corollary 6.3 that if $\vdash S(x)$ holds then $-x \subseteq -1 = Z$. This implies that 1 does not belong to $-x$ and hence that we do not have $x \in Z$. Hence x belongs to the complement of Z, that is x is secured. $\square$

In the case of a waekly hyperbolic system, we have not only a boolean model of the theory T but we can also directly model $S(x)$ by the fact that x is secured. The proposition 6.6 expresses the completeness property of this interpretation.

7 Second Extension: Compact Hausdorff Spaces

We limit ourselves to some remarks concering the extension of the definition of weakly hyperbolic in the case of compact Hausdorff's cases. This would be needed if we can to capture the case $X = [0, 1]$ or $X = [0, 1]^2$. We prove now the existence of an invariant measure. But we should first precise what we mean by a measure on a compact space, since, in a constructive framework, it is not always possible to assign a measure to any open sets [5]. Even if we consider only a basis of open sets of a compact space, it does not seem possible to assign constructively a measure to them. Instead, the right notion of measure seems to be of an *integral* that is a continuous linear functional If defined for $f \in C(X)$, set of continuous function over X, such that $I1 = 1$ and $0 \leq If$ if $0 \leq f$.[3]

A continuous function f_i will be in a point-free way defined as a relation on open sets, expressing $U \subseteq f_i^{-1}V$. We say that $(X, f_i)_{i<n}$ is *weakly hyperbolic* iff for any finite covering $U_0, \ldots, U_{m-1}$ of X there exists a bar L such that for all $w \in L$ there exists j such that $X = f_w^{-1}U_j$. (In term of points, this means $f_w x \in U_j$ for any $x \in X$.) Like in the case of spaces that are totally disconnected we can show that this is equivalent to the definition in [3].

If we have a system $f_i : X \to X,\ i < n$ of continuous functions over X and $p_i,\ i < n$ such that $0 \leq p_i$ and $\Sigma p_i = 1$, we say that an integral I is *invariant* iff $If = \Sigma p_i I(f \circ f_i)$ for any $f \in C(X)$.

Theorem 7.1 *If $(X, f_i)_{i<n}$ is weakly hyperbolic there exists one and only one invariant integral I.*

[3]This generalises the previous definition of measure for spaces compact totally disconnected, since in this case, the characteristic function ϕ_x of a closed open subset is continuous and we can define $mx = I\phi_x$.

Proof. (sketch) Indeed, let f be an element of $C(X)$ and $\epsilon > 0$. We can find a finite covering $U_0, \ldots, U_{m-1}$ of X and a finite sequence of rationals $r_0, \ldots, r_{m-1}$ such that $|f - r_j| < \epsilon$ on U_j. There exists then a bar L such that for all $w \in L$ there exists j_w such that $f_w(X) \subseteq U_{j_w}$. We define then $p_w = p_{i_0} \ldots p_{i_{N-1}}$ if $w = i_0 \ldots i_{N-1}$. Notice that we should have $If = \Sigma_{w \in L} p_w I(f \circ f_w)$. Since we have $|f \circ f_w - r_{j_w}| < \epsilon$, we should have also $|I(f \circ f_w) - r_{j_w}| < \epsilon$. Hence we can find an ϵ approximation of If by taking $\Sigma_{w \in L} p_w r_{j_w}$ This shows the existence and uniqueness of the integral If. $\square$

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