How to Measure Borel Sets

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Borel's Measure Function

"Leçons sur la Théorie des Fonctions", 1898

Domain: *generalised inductive* definition

Function: *generalised recursive* definition

Borel's Measure Function

Subsets of $(0, 1)$

- (r, s) is well-defined and $\mu(r, s)$ is $s - r$
- If A_n *disjoint* family of well-defined sets $A = \bigcup A_n$ is well-defined, and $\mu(A) = \Sigma\mu(A_n)$.
- If $A \subseteq B$ are well-defined, $B - A$ is well-defined, and $\mu(B - A) = \mu(B) - \mu(A)$.

Examples

The measure of a singleton is 0

Hence also the measure of any countable subset

We have

$$\mu((0, 1/2]) = 1/2,\ \mu((1/2, 3/4]) = 1/4,$$

$$\mu((3/4, 7/8]) = 1/8,\ \mu((7/8, 15/16]) = 1/16, \ldots$$

An open set is the countable disjoint union of its component

$$\mu((0, 1/2) \cup (1/2, 3/4) \cup (3/4, 7/8) \cup \ldots)) = 1$$

Borel's definition

Notice the difference with the usual definition of Borel subsets: the union has to be *disjoint*

In this way, we get a clearly motivated definition

With the usual definition (any countable union), this clear motivation is lost

Borel's definition

Compare with Jordan-Peano's definition of measure

First measure of finite union of intervals

Then outer and inner measure, but with finite union!

inner measure $\mu_*(A) = 1 - \mu^*((0,1) - A)$

A set A is measurable if

$$\mu_*(A) = \mu^*(A)$$

Problem: the set of rationals in $(0,1)$ is not measurable

Coherence problem

Three problems (Luzin)

- Does the sum $\Sigma\mu(A_n)$ converge??
- Can $\mu(B) - \mu(A)$ be negative??
- Is the definition coherent??

For instance

$$\mu((0,1/2] \cup (1/2,3/4] \cup (3/4,7/8] \cup \dots)) = 1$$

and

$$(0,1/2] \cup (1/2,3/4] \cup (3/4,7/8] \cup \cdots = (0,1)$$

Borel's definition

We write "axiomatically" the essential properties that the measure should have

This defines a theory of a new object

In order to justify the introduction of this new object, it has to be shown that this theory is not inconsistent

Borel cites Drach's exposition of Galois theory as a motivation of such an approach

Quite similar (but in 1898!) to Hilbert's notion of ideal elements in proof theory

Bourbaki: Borel's definition "opens a new era in Analysis"

Coherence Problem

Borel limits himself to a proof of Heine-Borel covering theorem (a complete proof would be "long and tedious")

Solved indirectly by Lebesgue 1902

Outer measure: g.l.b. of open supersets

Lebesgue measurable sets/Borel measurable sets

$$B_1 \subseteq A \subseteq B_2$$

$$\mu(B_1) = \mu(B_2)$$

"measurable" became "B measurable" (Lebesgue) and then "well-defined"

Regular measures

In this approach if A is measurable

$$\mu(A) = \bigwedge_{U\ open,\ A \subseteq U} \mu(U)$$

Such a measure is called *regular*

Young (1911), Daniell, Stone, Bourbaki

One replaces open subsets by *lower semi-continuous* functions

But the essential idea stays the same

Borel's measure problem

Lusin (1928): there is a difference between Borel's purely inductive definition, and Lebesgue's solution

Cannot we have a direct inductive justification of an inductive definition of measure of Borel sets??

This is Borel's measure problem

We present a solution which is inductive and use only constructive logic

Well-defined sets/any subset

Lebesgue (1905) had an article with the result that Borel subsets are closed by projection

Uses that, for A_n decreasing

$$f(\bigcap_n A_n) = \bigcap_n f(A_n)!!!$$

Suslin (1917) found a counter-example

Introduction of co-analytical sets (projection of Borel sets) and analytical sets (complement)

Beginning of descriptive set theory $\Sigma^1_1,\ \Pi^1_1$

Borel = analytical $\cap$ co-analytical (Suslin, Lusin)

Any analytical set is measurable

Not known for coanalytical sets...

Proof Theory

A similar hierarchy can be seen in proof theory

$$ID_1, \ , ID_2, \ldots, ID_{<\omega}, ID_\omega = \Pi^1_1, \ldots$$

Here we use only intuitionistic theory of inductive definitions

Constructive Probability Theory

We work with Cantor space Ω instead of $(0,1)$

Basic open sets (closed and open) instead of intervals

Form a Boolean algebra B with usual measure $\mu : B \to [0,1]$

$$\mu(x \wedge y) + \mu(x \vee y) = x + y$$

B, μ can be described purely syntactically without references to Ω

Lindenbaum-Tarski algebra of propositional logic!!

Formal definition of Borel sets

A Borel subset of Ω is a symbolic infinitary expression built from simple sets by repeated formal union and intersection

Inclusion can be defined via an infinitary sequent calculus (Novikov,Lorenzen,Schutte)

Lindenbaum-Tarski algebra of propositional ω-logic (Scott-Tarski)

Martin-Löf "Notes on Constructive Mathematics"

Set of Normal Sequences

Define $r_i(\omega) = 2\omega_i - 1$ and $s_n = \Sigma_{i \leq n} r_i$

$$b_{n,k} = \{\omega \in \Omega \mid \quad |\frac{s_n(\omega)}{n}| \leq \frac{1}{k} \quad \}$$

is a simple set $b_{n,k} \in B$

The Borel subset

$$N = \bigwedge_k \bigvee_m \bigwedge_{n \geq m} b_{n,k}$$

is the set of *normal sequences*

"well-defined" set

If k_n strictly increasing

$$N' \subseteq \bigvee_{n \geq m} b'_{n,k_n} \qquad \bigwedge_{n \geq m} b_{n,k_n} \subseteq N$$

Initiality Property of Borel Sets

Let B_1 be the σ-algebra of Borel subsets of Ω

Theorem: B_1 is the *free* σ-algebra on B

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "i"'] & A \\ B_1 \arrow[ur, "\exists!\bar{f}"'] & \end{tikzcd}$$

We can *define* define the algebra of Borel sets as the free σ-algebra on B

Axiomatic approach

Intuitively, we introduce infinitary symbolic expressions and use freely the law of σ-complete Boolean algebras

We have to show that this does not introduce inconsistency

In "Notes on Constructive Mathematics" this is justified via a cut-elimination theorem, similar to Gentzen's cut-elimination theorem

Well-defined $=$ represented by a symbolic expression

Solution of the measure problem

Define the measure of Borel sets by using initiality

This would solve the coherence problem

Measures on Boolean algebras

Already Tarski (1929) showed that it is convenient to "linearize" the problem of measure

Replace the Boolean algebra of basic event by the space of basic random variables $V(B)$

The elements of $V(B)$ can be seen as finite formal sums $\Sigma q_i b_i$

$B \to V(B)$ is the universal valuation!

The measure μ on B can be seen as a positive linear functional $E : V(B) \to Q$ (expectation)

Riesz space

$V(B)$ is an example of a Riesz space

$C(X)$ is another example

Ordered vector space

Any two elements have a sup

One can consider also commutative ordered monoid that are lattices

Riesz space

Very basic structure, due to Frederik Riesz (1928)

Rich properties: for instance, any Riesz space is a distributive lattice

Cover very different class of examples: monoid of natural numbers for multiplication and divibility as ordering, and $C(X)$

The basic property

$$x \vee y + x \wedge y = x + y$$

naturally connects with the definition of measure on Boolean algebras

$$\mu(x \vee y) + \mu(x \wedge y) = \mu(x) + \mu(y)$$

Riesz space

On a monoid, we define $x \perp y$ iff $x \wedge y = 0$

Euclides' lemma: if $x \leq y + z$ and $x \perp z$ then $x \leq y$

This holds for numbers and for continuous functions!

Bounded Baire functions

Strong unit: element 1 such that for any x

$$-n \cdot 1 \leq x \leq n \cdot 1$$

for some n

Dedekind σ-complete: any bounded increasing sequence has a sup

Theorem: the space $B(\Omega)$ of bounded Baire functions on Ω is the σ-completion of $V(B)$.

Baire functions: first continuous functions, then we close by (bounded) pointwise limits

The theorem is quite close to Rasiowa-Sikorski lemma; also very close to completness of propositional ω-logic, and close to Loomis-Sikorski representation of σ-complete algebras

How to define measure inductively

We let M_I be the space of functionals l on $V(B)$

$$-nI(f) \leq l(f) \leq nI(f)$$

for $f \geq 0$

We define $I_f \in M_I$

$$I_f(g) = I(fg)$$

Main remark: $I_{f_1 \vee f_2}$ is $I_{f_1} \vee I_{f_2}$

How to define measure inductively

By initiality $f \longmapsto I_f$ extends to $B(\Omega)$

So if f Baire functions and $g \in B(V)$ we can consider $I_f(g)$

In particular $I_f(1)$ is the integral of f

Notice that the initiality states exactly the monotone convergence theorem!

Constructive Probability Theory

$$b_{n,k} = \{\omega \in \Omega \mid \quad |\frac{s_n(\omega)}{n}| \leq \frac{1}{k} \quad \}$$

$$N = \bigwedge_k \bigvee_m \bigwedge_{n \geq m} b_{n,k}$$

If k_n strictly increasing

$$N' \subseteq \bigvee_{n \geq m} b'_{n,k_n}$$

Lemma: We can find k_n such that $\Sigma\mu(b'_{n,k_n})$ converges

Theorem: (Borel) $\mu_1(N) = 1$