Part 4: Terms

An Unusual Question

What are terms for (in the context of algebras)?

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1. *to name elements of the carrier sets by only combining elements of the signature*

2. in case variables are involved: *to name several carrier elements at once*

Variables

With each signature $\Sigma = (S, \Omega)$ is associated a family $V = (V_s)_{s \in S}$ of disjoint infinite sets. The *elements of V_s* are called *variables of sort s*. It is assumed that the variables of V and the operation names of Ω are different.

A family $X \subseteq V$ is called *a set of variables for the signature Σ*, even if it formally is a family.

We deliberately confuse X with $\bigcup_{s \in S} X_s$, e.g. by stating $x \in X$ for a variable x.

Terms

Let $\Sigma = (S, \Omega)$ be a signature and X a set of variables for it.

$T_{\Sigma(X)} = (T_{\Sigma(X),s})_{s \in S}$ is the family of minimal sets satisfying:

- $X_s \subseteq T_{\Sigma(X),s}$
- if $(n :\to s) \in \Omega$, then $n \in T_{\Sigma(X),s}$
- if $(n : s_1 \times \ldots \times s_k \to s) \in \Omega$ (with $k \geq 1$), and $t_i \in T_{\Sigma(X),s_i}$, for $1 \leq i \leq k$, then $n(t_1, \ldots, t_n) \in T_{\Sigma(X),s}$

An element of $T_{\Sigma(X),s}$ is called *$\Sigma(X)$-term of sort s*, short *term*.

Occurring Variables, Ground Terms

$Var(t)$ is the *set of variables occurring in t*.

If $Var(t) = \emptyset$, then t is a *ground term*.

$T_{\Sigma,s} = T_{\Sigma(\emptyset),s}$ denotes the set of all *ground terms of sort s*.

$T_{\Sigma} = (T_{\Sigma,s})_{s \in S}$ is called *the ground terms of Σ*.

Examples

Reconsider the signature **NATBOOL**.

Be $X_{nat} = \{m, n\}$ and $X_{bool} = \{b, c\}$.

Terms of sort $bool$ are:

$False$

c

$\wedge(\wedge(True, b), False)$

$\leq(0, +(m, Succ(n)))$

Assignments

Let $\Sigma = (S, \Omega)$ be a signature and X a set of variables for Σ, and $A \in Alg(\Sigma)$.

A family $\alpha = (\alpha_s)_{s \in S}$ of total functions $\alpha_s : X_s \longrightarrow A(s)$ is called an *assignment* of X for A.

One writes $\alpha : X \longrightarrow A$.

Semantics

Let $\Sigma = (S, \Omega)$ be a signature and X a set of variables for Σ, $A \in Alg(\Sigma)$, and $\alpha : X \longrightarrow A$.

The ***value** $A(\alpha)(t)$ **of a term** t* in the algebra A, under the assignment α, is defined by:

- if $t = x$ with $x \in X_s$, then $A(\alpha)(t) = \alpha_s(x)$
- if $t = n$ and $\omega = (n :\rightarrow s) \in \Omega$, then $A(\alpha)(t) = A(\omega)$
- if $t = n(t_1, \ldots, t_k)$ and $\omega = (n : s_1 \times \ldots \times s_k \rightarrow s) \in \Omega$, $k \geq 1$,
  $t_i \in T_{\Sigma(X),s_i}$ for $1 \leq i \leq k$,
  then $A(\alpha)(t) = A(\omega)(A(\alpha)(t_1), \ldots, A(\alpha)(t_k))$

Semantics of Ground Terms

Fact 4.1.
The value of ground terms does not depend on the assignment. ⋆

Because of Fact 4.1, we can write $A(t)$ instead of $A(\alpha)(t)$, if t is a ground term.

Notational Remark

If A is an algebra, then 'A' is heavily overloaded in out meta language:

- A denotes the algebra itself
- $A(s)$ denotes the carrier set of a sort s
- $A(\omega)$, $A(n :\rightarrow s)$, or $A(n : s_1 \times \ldots \times s_k \rightarrow s)$ denote the 'meaning' function of the operations ω, $n :\rightarrow s$, or $n : s_1 \times \ldots \times s_k \rightarrow s$, resp.
  remind: – $A(n : s_1 \times \ldots \times s_k \rightarrow s) : A(s_1) \times \ldots \times A(s_k) \longrightarrow A(s)$
  – $A(n :\rightarrow s)$ denotes an element of $A(s)$
- $A(\alpha)(t)$ denotes the value of a term t in A under α.
- $A(t)$ denotes the value of a ground term t in A.
- note: $A(n :\rightarrow s) = A(n)$ (why?)

Terms and Homomorphisms

Theorem 4.2.
Let A and B be two algebras of the same signature Σ, and let $h : A \longrightarrow B$ be a Σ-homomorphism. Then:

1. $h(A(t)) = B(t)$ *for each ground term* $t \in T_\Sigma$.
2. $h(A(\alpha)(t)) = B(h \circ \alpha)(t)$ *for each term* $t \in T_\Sigma(X)$ *and assignment* $\alpha : X \longrightarrow A$

⋆

Exercises

Exercise 4.1.
Prove theorem 4.2.1., either directly (easier), or by proving the more general case 4.2.2. (slightly more technical).

Substitutions

Informally, substitutions replace variables by terms

Let $\Sigma = (S, \Omega)$ be a signature and X, Y two set of variables for Σ.

A family $\sigma = (\sigma_s)_{s \in S}$ of functions $\sigma_s : X_s \longrightarrow T_{\Sigma(Y),s}$ is called *substitution* $\sigma : X \longrightarrow T_{\Sigma(Y)}$.

$[x_1/t_1, \ldots, x_n/t_n]$ denotes a substitution σ with:

$$\sigma(x) = \begin{cases} t_i, & \text{if } x = x_i \text{ for one } i \in \{1, \ldots, n\} \\ x, & \text{otherwise} \end{cases}$$

Substitution on Terms

$t\sigma$ denotes the *application of a substitution* $\sigma : X \longrightarrow T_{\Sigma(Y)}$ *to a term* $t \in T_{\Sigma(X)}$, defined by:

- if $t = x$ with $x \in X_s$, then $t\sigma = \sigma_s(x)$
- if $t = n$ and $\omega = (n : \rightarrow s) \in \Omega$, then $t\sigma = t$
- if $t = n(t_1, \ldots, t_k)$ and $\omega = (n : s_1 \times \ldots \times s_k \rightarrow s) \in \Omega$, $k \geq 1$, $t_i \in T_{\Sigma(X),s_i}$ for $1 \leq i \leq k$, then $t\sigma = n(t_1\sigma, \ldots, t_k\sigma)$

Example

Reconsider the signature **NATBOOL**.

Let $X_{nat} = \{n, m\}$ and $X_{bool} = \{b, c\}$ and $X = Y$.

If $\sigma = [b/b \wedge c, c/True, n/m + n]$, then:

$$((True \wedge b) \wedge False)\sigma = (True \wedge (b \wedge c) \wedge False$$

$$(0 \leq m + Succ(n))\sigma = 0 \leq m + Succ(m + n)$$

note: the example is not 100% exact. Why?

Substitution Theorem

This theorem relates substitutions and algebras.

Theorem 4.3.
Let X, Y be sets of variables for a signature $\Sigma = (S, \Omega)$, α an assignment of X for A, and let $x \in X$ and $s \in T_{\Sigma(Y)}$.

Then for all $t \in T_{\Sigma(X)}$, it holds that:

$$A(\alpha)(t[x/s]) = A(\alpha_x^{A(\alpha)(s)})(t)$$

⋆

Fact 4.4.
If $t \in T_{\Sigma(\{x\})}$ and $s \in T_\Sigma$, then:

$$A(t[x/s]) = A(\alpha_x^{A(s)})(t)$$

⋆

Part 5: Quotient Algebras

Congruence Relations

Let $\Sigma = (S, \Omega)$ be a signature and $A \in Alg(\Sigma)$.

A *congruence relation on A* is a family $Q = (Q_s)_{s \in S}$ of equivalence relations Q_s on $A(s)$, such that:

for any $(n : s_1 \times \ldots \times s_k \to s) \in \Omega$, $k \geq 1$:
for any $a_i, a'_i \in A(s_i)$ with $a_i \; Q_{s_i} \; a'_i$, $1 \leq i \leq k$,

$$A(\omega)(a_1, \ldots, a_k) \; Q_s \; A(\omega)(a'_1, \ldots, a'_k)$$

Informally: *Equivalent arguments lead to equivalent function values.*

Example and Exercise

Reconsider the **BOOL**-Algebra D:

Define: $m \; Q_{bool} \; n$ iff $n + m$ is even

Then $Q = (Q_{bool})$ is congruence relation on D.

Exercise 5.1.
Show this (if not done in the lecture).

Quotient Algebras

Let $\Sigma = (S, \Omega)$ be a signature, $A \in Alg(\Sigma)$, and $Q = (Q_s)_{s \in S}$ a congruence relation on A.

The *quotient algebra* (or *quotient*) of A by Q is the Σ-Algebra $A/_Q$, defined by:

$$A/_Q(s) = \{[a]_{Q_s} \mid a \in A(s)\}, \text{ for all } s \in S$$

$$A/_Q(\omega)([a_1]_{Q_{s_1}}, \ldots, [a_k]_{Q_{s_k}}) = [A(\omega)(a_1, \ldots, a_k)]_{Q_s}$$
$$\text{for all } n : s_1 \times \ldots \times s_k \to s \in \Omega,\ k \geq 0$$

Fact 5.1.
The above definition is consistent. ⋆

Exercise

Exercise 5.2.
Reflect:

- *Why is fact 5.1 worth mentioning?*
- *Why does it hold?*

Quotient Algebra Example

Reconsider the **BOOL**-Algebra D, and $Q = (Q_{bool})$ with:
$m\ Q_{bool}\ n$ iff $n + m$ is even

Construct $D/_Q$.

Compare $D/_Q$ with the **BOOL**-Algebra C.

Exercise 5.3.
If not done in the lecture, do this as an exercise.
(recommended.)

Congruence Relation induced by Homomorphisms

Let $\Sigma = (S, \Omega)$ be a signature, $A, B \in Alg(\Sigma)$ and let $h : A \longrightarrow B$ be a $homomorphism$.

Now, let $\equiv_h = (\equiv_{h_s})_{s \in S}$ be the family of relations on A, defined by:

$$a \equiv_{h_s} a' \text{ iff } h_s(a) = h_s(a').$$

Then $\equiv_h$ is called the *congruence relation induced by the homomorphism* h.

Fact 5.2.
$\equiv_h$ *is indeed a congruence relation.* ⋆

Quotient Homomorphisms

Let $\Sigma = (S, \Omega)$ be a signature, $A \in Alg(\Sigma)$, and Q a congruence relation on A.

Then the family $hom_Q : A \longrightarrow A/_Q$ of functions, defined by

$$hom_{Q_s}(a) = [a]_Q$$

is called the *quotient homomorphism of the congruence relation* Q.

Fact 5.3.
hom_Q *is indeed a homomorphism.* ⋆

The Theorem of Homomorphisms

Theorem 5.4.
Let $\Sigma = (S, \Omega)$ *be a signature,* $A, B \in Alg(\Sigma)$ *and let* $h : A \longrightarrow B$ *be a* $homomorphism$.

If h *is surjective, then:* $A_{\equiv_h} \simeq B$ ⋆

Exercises

Exercise 5.4.
Prove some the facts 5.2, 5.3, or theorem 5.4. (if not proved during the lecture).

Quotients and Terms

Informally:
The value of a term in the quotient algebra is in the equivalence class of its value in the original algebra.

Theorem 5.5.
Let $\Sigma = (S, \Omega)$ *be a signature,* $A \in Alg(\Sigma)$*, and* Q *a congruence relation on* A*. Then:*

- $A/_Q(t) = [A(t)]$ *for each ground term* $t \in T_\Sigma$*.*
- $A_Q(\beta)(t) = [A(\alpha)(t)]$ *for each term* $t \in T_\Sigma(X)$ *and assignment* $\alpha : X \longrightarrow A$*, where* $\beta : X \longrightarrow A/_Q$ *is defined by:*
  $\beta(x) = [\alpha(x)]$ *f.a.* $x \in X$*.*

⋆