On Model-Theoretic Strong Normalization for Truth-Table Natural Deduction

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Intuitionistic truth-table natural deduction (ITTND)

- Geuvers and Hurkens (2017): Turn truth tables into inference rules.
- Here only *implication* ($A \to B$), but works for any connective.
- Truth-table: **False** *True*

A	B	$A \to B$
0	**0**	1
0	1	1
1	**0**	**0**
1	1	1

$\mathbf{A}$	$\mathbf{B}$	$A \to B$
$\mathbf{A}$	B	$A \to B$
A	$\mathbf{B}$	$\mathbf{A} \to \mathbf{B}$
A	B	$A \to B$

Tables to rules

$\mathbf{A}$	$\mathbf{B}$	$A \to B$
$\mathbf{A}$	B	$A \to B$
A	$\mathbf{B}$	$\mathbf{A} \to \mathbf{B}$
A	B	$A \to B$

$$\mathsf{in}_{\to}^{00}\ \frac{\Gamma.\mathbf{A} \vdash A \to B \qquad \Gamma.\mathbf{B} \vdash A \to B}{\Gamma \vdash A \to B}$$

$$\mathsf{in}_{\to}^{01}\ \frac{\Gamma.\mathbf{A} \vdash A \to B \qquad \Gamma \vdash B}{\Gamma \vdash A \to B}$$

$$\mathsf{el}_{\to}^{10}\ \frac{\Gamma \vdash \mathbf{A} \to \mathbf{B} \qquad \Gamma \vdash A \qquad \Gamma.\mathbf{B} \vdash C}{\Gamma \vdash C}$$

$$\mathsf{in}_{\to}^{11}\ \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \to B}$$

β-Reduction (intro/elim, positive)

$$\dfrac{\dfrac{\overset{\mathbf{u}}{\Gamma.\mathbf{A} \vdash A \to B} \qquad \overset{b}{\Gamma \vdash B}}{\Gamma \vdash A \to B}\,\mathrm{in}^{01}_{\to} \qquad \overset{a}{\Gamma \vdash A} \qquad \overset{\mathbf{t}}{\Gamma.\mathbf{B} \vdash C}}{\Gamma \vdash C}\,\mathrm{el}^{10}_{\to}$$

$$\mathrm{in}^{01}_{\to}(\mathbf{u}, b) \cdot \mathrm{el}^{10}_{\to}(a, \mathbf{t})$$

$$\downarrow$$

$$\overset{\mathbf{t}[b]}{\Gamma \vdash C}$$

β-Reduction (intro/elim, negative)

$$\dfrac{\dfrac{\overset{\mathbf{u}}{\Gamma.\mathbf{A} \vdash A \to B} \qquad \overset{b}{\Gamma \vdash B}}{\Gamma \vdash A \to B}\,\mathrm{in}^{01}_{\to} \qquad \overset{a}{\Gamma \vdash A} \qquad \overset{\mathbf{t}}{\Gamma.\mathbf{B} \vdash C}}{\Gamma \vdash C}\,\mathrm{el}^{10}_{\to}$$

$$\mathrm{in}^{01}_{\to}(\mathbf{u}, b) \cdot \mathrm{el}^{10}_{\to}(a, \mathbf{t})$$

$$\downarrow$$

$$\mathbf{u}[a] \cdot \mathrm{el}^{10}_{\to}(a, \mathbf{t})$$

$$\dfrac{\overset{\mathbf{u}[a]}{\Gamma \vdash A \to B} \qquad \overset{a}{\Gamma \vdash A} \qquad \overset{\mathbf{t}}{\Gamma.\mathbf{B} \vdash C}}{\Gamma \vdash C}\,\mathrm{el}^{10}_{\to}$$

π-Reduction (elim/elim)

$$\dfrac{\dfrac{\overset{h}{\Gamma \vdash A \to B} \quad \overset{a}{\Gamma \vdash A} \quad \overset{\mathbf{t}}{\Gamma.\mathbf{B} \vdash A' \to B'}}{\Gamma \vdash A' \to B'}\,\mathrm{el}^{10}_{\to} \quad \overset{a'}{\Gamma \vdash A'} \quad \overset{\mathbf{t}'}{\Gamma.\mathbf{B}' \vdash C}}{\Gamma \vdash C}\,\mathrm{el}^{10}_{\to}$$

$$h \cdot \mathrm{el}^{10}_{\to}(a, \mathbf{t}) \cdot \mathrm{el}^{10}_{\to}(a', \mathbf{t}')$$

$$\downarrow$$

$$h \cdot \mathrm{el}^{10}_{\to}(a, \mathbf{t} \cdot \mathrm{el}^{10}_{\to}(a', \mathbf{t}'))$$

$$\dfrac{\overset{h}{\Gamma \vdash A \to B} \quad \overset{a}{\Gamma \vdash A} \quad \dfrac{\overset{\mathbf{t}}{\Gamma.\mathbf{B} \vdash A' \to B'} \quad \overset{a'}{\Gamma.\mathbf{B} \vdash A'} \quad \overset{\mathbf{t}'}{\Gamma.\mathbf{B}.\mathbf{B}' \vdash C}}{\Gamma.\mathbf{B} \vdash C}\,\mathrm{el}^{10}_{\to}}{\Gamma \vdash C}\,\mathrm{el}^{10}_{\to}$$

Normalization results

- Geuvers, Hurkens, TYPES 2017 post-proceedings
  - SN-β, saturated sets, impredicative, Tait/Girard mix
  - WN-$\beta\pi$, combinatorial, Turing
- Geuvers, van der Giessen, Hurkens (Fund. inf., 2019)
  - SN $\beta\pi$ via translation to parallel lambda-calculus
- Strong normalization in A., TYPES 2020 post-proceedings

	impredicative meta-theory	predicative meta-theory
β	Reducibility (elim-based) [Girard]	Reducibility (intro-based) [Girard, Matthes]
$\beta\pi$	(Bi)orthogonality [French school, Pitts]	Saturated sets [Tait, Joachimski/Matthes]