Semi-continuous Sized Types and Termination

Termination Checking via Type Systems

Andreas Abel

Department of Computer Science
Ludwig-Maximilians-University Munich

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Theorem Provers for Constructive Logic

Theorem Provers built on Dependent Type Theory:

- Coq (INRIA, France)
- Epigram (Nottingham, UK)
- Agda (Chalmers, Sweden)

Their soundness is based on *termination*.

Constructive Logics

The Curry-Howard Isomorphism:

Proposition	Type
A implies B	$A \to B$
Proof	Purely Functional Program
Valid Proof	Terminating Program

Non-terminating programs lead to inconsistency:

$$f : (0 = 0) \to (0 = 1)$$
$$f(p) = f(p)$$

Type-Based Termination, Informally

Recipe

Step 1 In the type system, attach sizes to data structures.

Step 2 Using type-checking, ensure that recursive calls use only arguments with decreased size.

Step 1: Sized Binary Trees

- Let $\mathsf{BTree}^{\imath}$ denote trees of height $< \imath$.
- The empty tree has height 0, hence $\mathsf{leaf} : \mathsf{BTree}^{1}$, but also $\mathsf{leaf} : \mathsf{BTree}^{2}$, $\mathsf{leaf} : \mathsf{BTree}^{3}$, ...
- In general $\mathsf{leaf} : \mathsf{BTree}^{\imath+1}$ for all $\imath$.

$$\begin{array}{lcl} \mathsf{leaf} & : & \forall \imath.\, \mathsf{BTree}^{\imath+1} \\ \mathsf{node} & : & \forall \imath.\, \mathsf{Int} \times (\mathsf{BTree}^{\imath} \times \mathsf{BTree}^{\imath}) \to \mathsf{BTree}^{\imath+1} \end{array}$$

- BTree^{∞} contains all binary trees.
- Subtyping: $\mathsf{BTree}^{\imath} \subseteq \mathsf{BTree}^{\imath+1} \subseteq \cdots \subseteq \mathsf{BTree}^{\infty}$.

Step 2: Equality Test for Sized Binary Trees

- Code annotated with sizes:

$$\mathsf{eq} : \forall\imath.\, \mathsf{BTree}^{\imath} \to \mathsf{BTree}^{\imath} \to \mathsf{Bool}$$

$$\begin{array}{l}
\mathsf{eq}\ \mathsf{leaf}\ \mathsf{leaf} = \mathsf{true} \\
\mathsf{eq}\ \mathsf{node}(i_1,(l_1,r_1))^{\imath+1}\ \mathsf{node}(i_2,(l_2,r_2))^{\imath+1} = (i_1 == i_2)\ \ \&\& \\
\qquad \mathsf{eq}\ {l_1}^{\imath}\ {l_2}^{\imath}\ \ \&\&\ \ \mathsf{eq}\ {r_1}^{\imath}\ {r_2}^{\imath} \\
\mathsf{eq}\ _\ _ = \mathsf{false}
\end{array}$$

- Input arguments assumed to be of size $\imath + 1$.
- Recursive arguments inferred to be of size $\imath$.
- Descend in size, hence, termination.

Abstracting the Branching Type

- Generalize to F-Branching Int-labelled trees $\mathsf{Tree}^{\imath}\, F$
- Constructors:

$$\begin{array}{lcl} \mathsf{leaf} & : & \forall F \forall \imath.\, \mathsf{Tree}^{\imath+1} F \\ \mathsf{node} & : & \forall F \forall \imath.\, \mathsf{Int} \times F\,(\mathsf{Tree}^{\imath} F) \to \mathsf{Tree}^{\imath+1} F \end{array}$$

- Valid instances

binary trees	$F\,T = T \times T$
lists	$F\,T = T$
finitely branching trees	$F\,T = \mathsf{List}^{\infty}\, T$
infinitely branching trees	$F\,T = \mathsf{Nat}^{\infty} \to T$

- Invalid instance (F not monotone), e.g., $F\,T = T \to \mathsf{Bool}$

Equality of F-Branching Trees

- Generalize equality test to F-branching trees:
- Termination not inferable with untyped methods.

Termination and Polymorphism

$$\mathsf{Eq}\ T = T \to T \to \mathsf{Bool}$$

$$\mathsf{eq} : (\forall T.\mathsf{Eq}\ T \to \mathsf{Eq}\,(F\ T)) \to \forall \imath.\ \mathsf{Eq}\,(\mathsf{Tree}^{\imath} F)$$

$$\mathsf{eq}\ \mathit{eqF}\ \mathsf{leaf}\ \mathsf{leaf} = \mathsf{true}$$

$$\mathsf{eq}\ \mathit{eqF}\ \mathsf{node}(i_1, \mathit{ft}_1)\ \overbrace{\mathsf{node}(i_2, \underbrace{\mathit{ft}_2}_{F\,(\mathsf{Tree}^{\imath}\ F)})}^{\mathsf{Tree}^{\imath+1}\ F} = (i_1 == i_2)\ \ \&\&$$

$$\mathit{eqF}\quad \overbrace{(\mathsf{eq}\ \mathit{eqF})}^{\mathsf{Eq}\ T = \mathsf{Eq}\,(\mathsf{Tree}^{\imath} F)}\quad \mathit{ft}_1\ \mathit{ft}_2$$

$$\mathsf{eq}\ \_\ \_\ \_ = \mathsf{false}$$

Observe the role reversal: The recursive function $(\mathsf{eq}\ \mathit{eqF})$ becomes an argument to its own argument eqF!

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Evaluation

No untyped formalism can handle this example:

- In the untyped setting, eq diverges, e.g., define

$$\mathit{eqF}\ \mathit{eqT}\ \mathit{ft}_1\ \mathit{ft}_2 = \mathit{eqT}\ \mathsf{node}(0, \mathit{ft}_1)\ \mathsf{node}(0, \mathit{ft}_2)$$

- and execute the function clause

$$\begin{aligned}\mathsf{eq}\ \mathit{eqF}\ \mathsf{node}(i_1, \mathit{ft}_1)\ \mathsf{node}(i_2, \mathit{ft}_2) &= \ldots \\ &\quad \mathit{eqF}\ (\mathsf{eq}\ \mathit{eqF})\ \mathit{ft}_1\ \mathit{ft}_2\end{aligned}$$

A typed formalism such as TBT uses the information that

$$\mathit{eqF} : \forall T. \mathsf{Eq}\ T \to \mathsf{Eq}\,(F\ T)$$

is polymorphic (hence, the above instance of eqF is ill-typed).

Type-Based Termination, Formally

Theorem

$f = s(f) : \forall\imath.\, A(\imath)$ *is well-defined if*

1. *(bottom check)* $A(0)$ *contains all programs, e.g.,* $A(\imath) = \mathsf{BTree}^{\imath} \to C$.
2. *(descent)* $f : A(\imath)$ *implies* $s(f) : A(\imath + 1)$.
3. *(admissibility)* $\bigcap_{\alpha<\lambda} A(\alpha) \subseteq A(\lambda)$ *for all limit ordinals* $\lambda \neq 0$.

Proof.

By transfinite induction on $\imath$.

1. (base) $f : A(0)$ trivial.
2. (step) ind.hyp. $f : A(\alpha)$ implies $s(f) = f : A(\alpha + 1)$.
3. (limit) $f : \bigcap_{\alpha<\lambda} A(\alpha)$ by ind.hyp., hence $f : A(\lambda)$.

□

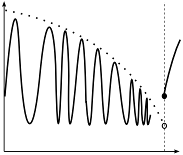
Upper Semi-Continuous Types

Definition (upper semi-continuous)

A semantical type $\mathcal{A} : \mathsf{On} \to \mathcal{P}(\mathsf{SN})$ is upper semi-continuous (*usc*) if for all limits $\lambda \neq 0$

$$\limsup_{\alpha \to \lambda} \mathcal{A}(\alpha) := \left(\bigcap_{\alpha_0 < \lambda} \bigcup_{\alpha_0 \leq \alpha < \lambda} \mathcal{A}(\alpha) \right) \subseteq \mathcal{A}(\lambda)$$

An *usc* type fulfills $\bigcap_{\alpha<\lambda} \mathcal{A}(\alpha) \subseteq \mathcal{A}(\lambda)$, hence, is admissible.

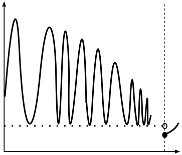


Lower Semi-Continuous Types

Definition (upper semi-continuous)

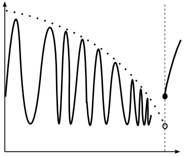
A semantical type $\mathcal{A} : \mathsf{On} \to \mathcal{P}(\mathsf{SN})$ is lower semi-continuous (*usc*) if for all limits $\lambda \neq 0$

$$\mathcal{A}(\lambda) \subseteq \liminf_{\alpha \to \lambda} \mathcal{A}(\alpha) := \bigcup_{\alpha_0 < \lambda} \bigcap_{\alpha_0 \leq \alpha < \lambda} \mathcal{A}(\alpha)$$



Continuous Types

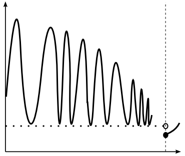
usc



$\limsup_{\alpha\to\lambda} \mathcal{A}(\alpha) \subseteq \mathcal{A}(\lambda)$

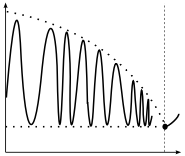
+

lsc



$\mathcal{A}(\lambda) \subseteq \liminf_{\alpha\to\lambda} \mathcal{A}(\alpha)$

continuous



$\lim_{\alpha\to\lambda} \mathcal{A}(\alpha) = \mathcal{A}(\lambda)$

Closure Properties of Semi-Continuity

usc	condition	*lsc*	condition
$\mathcal{A}$ *usc*	$\mathcal{A}$ monotone	$\mathcal{A}$ *lsc*	$\mathcal{A}$ antitone
$\mathcal{A} + \mathcal{B}$ *usc*	$\mathcal{A}$, $\mathcal{B}$ *usc*	$\mathcal{A} + \mathcal{B}$ *lsc*	$\mathcal{A}$, $\mathcal{B}$ *lsc*
$\mathcal{A} \times \mathcal{B}$ *usc*	$\mathcal{A}$, $\mathcal{B}$ *usc*	$\mathcal{A} \times \mathcal{B}$ *lsc*	$\mathcal{A}$, $\mathcal{B}$ *lsc*
$\mathcal{A} \to \mathcal{B}$ *usc*	$\mathcal{A}$ *lsc*, $\mathcal{B}$ *usc*	—	
$\nu\mathcal{F}$ *usc*	$\mathcal{F}$ *usc*	$\mu\mathcal{F}$ *lsc*	$\mathcal{F}$ *lsc*

Why Upper Semi-Continuity is Vital

Let $\mathsf{pred} : \forall\imath.\, \mathsf{Nat}^{\imath+1} \to \mathsf{Nat}^{\imath}$ such that $\mathsf{pred}\,0$ raises an exception. Define

$$f : \forall\imath.\, \overbrace{(\mathsf{Nat}^{\infty} \to \mathsf{Nat}^{\imath}) \to X}^{A(\imath)}$$

$$f(g : \mathsf{Nat}^{\infty} \to \mathsf{Nat}^{\imath+1}) = f\ ((\mathsf{pred} \circ g \circ \mathsf{succ}) : \mathsf{Nat}^{\infty} \to \mathsf{Nat}^{\imath})$$

Now $f(\mathsf{id})$ loops.

The definition passes the bottom check and the descent criterion, but $A(\imath)$ is neither *usc* nor admissible.

Related Work

Expressivity	Xi	Par	Ama	Gim	Fra	**A**	Bar	Bla	Buch
term. measures	+	−	−	−	−	−	−	+	o
dep. types	o	−	−	+	−	−	+	+	−
polymorphism	+	o	−	+	−	**+**	+	+	−
infinite branch.	−	−	−	+	+	**+**	+	−	+
semi-cont.	−	ω	−	−	−	**+**	−	−	−
productivity	−	+	+	+	+	+	+	−	+
Features									
symbolic exec.	−	−	+	+	+	+	+	+	+
soundness	V	D	SN	−	SN	SN	o	SN	D
ordinals	$< \omega$	$\leq \omega$	On	−	Ω	Ω_ω	−	$< \omega$	$\leq \omega$
equi-rec.	−	+	−	−	−	+	−	−	−
size inference	−	+	−	−	−	−	+	−	−

Conclusions

- Termination checking can be integrated into type checking
- Especially powerful in combination with polymorphism
- Type-Based Termination is a modern technology, still under active development

Future Work

- Extend to dependent types
- Investigate semi-continuity for dependent types
- Find intuitive explanations for non-admissibility of types
- Integrate into a theorem prover

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TYPES
APPSEM II

Formalizing Sized Types

- Capture the structure of data types, forget about constructor names.
- Types are build from the primitives $+$ (disjoint sum), $\times$ (cartesian product), $\to$ (function space).
- Sized types $\mu^{\imath} F$ are recursive types obtained by iterating a type transformer F:

$$
\begin{array}{lcl}
\mu^{0} F & = & \emptyset \\
\mu^{\alpha+1} F & = & F\,(\mu^{\alpha}\, F) \\
\mu^{\lambda} F & = & \bigcup_{\alpha<\lambda} \mu^{\alpha} F
\end{array}
$$

- E.g., type constructor for binary trees:
  $\mathsf{BTreeF}\ X = 1 + \mathsf{Int} \times (X \times X)$
- Sized binary trees: $\mathsf{BTree}^{\imath} = \mu^{\imath}\, \mathsf{BTreeF}$.