Wellfounded Recursion with Copatterns

A Unified Approach to Termination and Productivity

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Abstract

In this paper, we study strong normalization of a core language based on System F_ω which supports programming with finite and infinite structures. Building on our prior work, finite data such as finite lists and trees are defined via constructors and manipulated via pattern matching, while infinite data such as streams and infinite trees is defined by observations and synthesized via copattern matching. In this work, we take a type-based approach to strong normalization by tracking size information about finite and infinite data in the type. This guarantees compositionality. More importantly, the duality of pattern and copatterns provide a unifying semantic concept which allows us for the first time to elegantly and uniformly support both well-founded induction and coinduction by mere rewriting. The strong normalization proof is structured around Girard's reducibility candidates. As such our system allows for non-determinism and does not rely on coverage. Since System F_ω is general enough that it can be the target of compilation for the Calculus of Constructions, this work is a significant step towards representing observation-centric infinite data in proof assistants such as Coq and Agda.

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General Terms Languages, Theory

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1. Introduction

Integrating infinite data and coinduction with dependent types is tricky. For example, in the Calculus of (Co)Inductive Constructions, the core theory underlying Coq (INRIA 2012), coinduction is broken, since computation does not preserve types (Giménez 1996; Oury 2008). In Agda (Norell 2007), a dependently typed proof and programming environment based on Martin-Löf Type Theory, inductive and coinductive types cannot be mixed in a compositional way.[1] In previous work (Abel et al. 2013) we have introduced *copatterns* as a novel perspective on defining infinite structures that might serve as a new foundation for coinduction in dependently-typed languages, overcoming the problems in the present solutions.

In the copattern approach, finite data such as finite lists and trees are defined as usual via constructors and manipulated via pattern matching, while infinite data such as streams and infinite trees are defined by observations and synthesized via copattern matching. For example, instead of conceiving streams as built by the constructor `cons`, we consider the observations `head` and `tail` about streams as primitive. Programs about streams are defined in terms of the observations `head` and `tail`.

Our previous work left the question of termination of recursive function and the productivity of infinite objects open. Both issues are crucial since we want to program inductive proofs as recursive functions and coinductive proofs as infinite objects or corecursive functions producing infinite objects. In this article, we adapt type-based termination (Hughes et al. 1996; Amadio and Coupet-Grimal 1998; Barthe et al. 2004; Blanqui 2004; Abel 2006; Sacchini 2011, 2013) to definitions by copatterns.

A syntactic termination check would ensure that recursive calls occur only with arguments smaller than the ones of the original call. In type-based termination, inductive types are tagged with a size expression that denotes the (ordinal) maximal height of the trees inhabiting it, i. e., an upper bound on the number of constructors in the longest path of the tree. To prove termination of a recursive function means to show that it can safely handle arguments of arbitrary size. This can be established by well-founded induction: to show that a function can handle arguments up to a fixed size a, we may assume it already safely processes arguments of any smaller size $b < a$. This induction principle can be turned into a typing rule for recursive functions, using sized types and size quantification. How can this be dualized to coinduction? A stream is *productive* if we can make arbitrarily deep observations, i. e., if we can take its tail arbitrarily many times. To show that a stream definition is productive, we also proceed by well-founded induction. To show that it can safely handle a observations, we may assume that b observations are fine for any $b < a$. The number of observations we can safely make is called the *depth* of the stream, or more general, of the coinductive structure. One should not be mislead and think of the depth as "size"; streams do not have a size since they are not tree-structures in memory—they only exist as processes that con-

[1] In Agda, one can encode the property "infinitely often" from temporal logic, but not its dual "eventually forever" (Altenkirch and Danielsson 2010).

tinuously yield elements on demand. But it is fruitful to transfer the concept of *depth* to (co)recursive functions. The depth of a function is the maximal size of arguments it can safely handle. As we are only interested in streams of infinite depth in the end, we care only about functions of infinite depth. Yet to establish productivity and termination, we need to induct on depth.

The type-based termination approach is in contrast to common approaches taken in systems such as Coq (Bertot and Castéran 2004) and Agda (Norell 2007) which employ a syntactic guardedness check to ensure corecursive programs are productive: all corecursive calls must occur under a constructor. This ensures that the next unit of information can be computed in a finite amount of time (Sijtsma 1989). However, this approach has also known limitations: it is difficult to handle higher-order programs such as $g\,f = \mathsf{cons}\,0\,(f\,(g\,f))$ where the productivity of g depends on the behavior of the function f. It is also not compositional, i. e., we cannot easily abstract over a constructor cons in a productive program and replace it with a function f. Both limitations are due to the lack of information we have about f in the syntactic guardedness check. Types on the other hand already track information about each argument to a definition and its output. Type-based termination piggybacks on the typing analysis and avoids a separate formal system to traverse the definitions. By indexing types with sizes, we are able to carry more precise information about input and output arguments and their relation which is then verified simultaneously while type checking the definitions.

The contributions of our work are:

- We present $\mathsf{F}^{\mathsf{cop}}_\omega$, an extension of System F_ω by inductive and coinductive types, sizes and bounded size quantification, pattern and copattern matching and lexicographic termination measures.
- In contrast to previous approaches on type-based termination, we use well-founded induction on ordinals instead of conventional induction that distinguishes between zero, successor and limit ordinals. Disposing of this case distinction, we operate within constructive foundations of mathematics (Taylor 1996).
- Well-founded induction leads to a construction of inductive types by inflationary iteration, which has been utilized to justify cyclic proofs in the sequent calculus (Sprenger and Dam 2003). We are the first to utilize inflationary iteration in a type system.
- Well-founded induction alleviates the need for a semi-continuity check for sized types of recursive functions (Hughes et al. 1996; Abel 2008b) which sometimes disguises itself as a monotonicity check (Barthe et al. 2004; Blanqui 2004; Barthe et al. 2008; Sacchini 2013). Thus, we put type-based termination on leaner and better understandable foundations.
- Since we construct infinite objects by copattern matching, standard rewriting becomes strongly normalizing even for corecursive definitions, and productivity becomes an instance of termination. Thus, we add the last brick to a unified treatment of recursion and corecursion that is central to type-based termination.
- Our typing rules are formulated as a bidirectional type-checking algorithm that can be implemented as such. A prototype, which additionally features dependent types, is MiniAgda (Abel 2012).
- We prove soundness of $\mathsf{F}^{\mathsf{cop}}_\omega$ by an untyped term model based on Girard's reducibility candidates. The proof exhibits semantic counterparts of pattern and copattern typing and accounts for incomplete and overlapping rewrite rules.

2. Copatterns and Termination

Let us illustrate how to program with copatterns using a simple example of generating a stream of zeros. A streams s over an element type A is given by the two *observations* head and tail: We can inspect the head of s by applying the projection s .head and obtain an element of A. To obtain the tail of s, we use the projection s .tail. We can then define the stream of zeros recursively by the following two clauses:

$$\begin{aligned}\mathsf{zeros}\ .\mathsf{head} &= 0\\ \mathsf{zeros}\ .\mathsf{tail} &= \mathsf{zeros}\end{aligned}$$

More generally, zeros can be coded as repeat 0 with

$$\begin{aligned}\mathsf{repeat}\ a\ .\mathsf{head} &= a\\ \mathsf{repeat}\ a\ .\mathsf{tail} &= \mathsf{repeat}\ a\end{aligned}$$

The left hand side of each clause is considering the definiendum, here repeat, in a *copattern*, here $\cdot\ a$ .head and $\cdot\ a$ .tail, resp. A copattern consists of a hole, $\cdot$, applied to a sequence of patterns and/or projections. The hole is filled, e. g., by the definiendum. In this case, we have first a variable pattern, a, and then a projection head/tail.

The definition of repeat is *complete* because the given copatterns are covering all possible cases (Abel et al. 2013). In this article, we investigate the *termination* of definitions by copatterns if read as rewrite rules, regardless of their completeness. In systems without the copattern facility, repeat would be defined using a stream constructor cons as follows:

$$\mathsf{repeat}\ a = \mathsf{cons}\ a\ (\mathsf{repeat}\ a)$$

Read as rewrite rule, this equation leads immediately to nontermination; this is why in the absence of copatterns one speaks about *productivity* instead of termination (Coquand 1993). A definition is productive if it unfolds to an infinite stream in all cases—which certainly holds for repeat a. In the presence of copatterns, productivity is subsumed under plain termination.

Coming back to our copattern-based definition we see that repeat a *terminates* in all contexts since it does not unfold by itself and consumes one projection in each unfolding. For example, projecting the $(n+1)$st element (counting from 0) of repeat a, i. e., repeat a $.\mathsf{tail}^{n+1}$ .head reduces in one step to repeat a $.\mathsf{tail}^{n}$ .head and after n more steps to repeat a .head.

There are many formalisms that ensure termination or productivity of recursive definitions. In this article, we adapt type-based termination (Hughes et al. 1996; Barthe et al. 2004; Abel 2006) to copatterns, i. e., we will present a type system that only accepts terminating definitions. There are good reasons to integrate termination checking into the type system, the foremost one is *compositionality*. Good type systems are defined in a compositional way, i. e., one can replace any expression with a different one of the same type without destroying well-typedness, in particular, one can replace a complex expression by a variable, abstracting from the concrete behavior or the expression. In contrast, syntactic termination checks often lack similarly powerful means of abstraction. For instance, if we abstract the constructor

$$f\ a = \mathsf{cons}\ a$$

in the second, non-copattern definition of repeat, obtaining

$$\mathsf{repeat}\ a = f\ a\ (\mathsf{repeat}\ a),$$

then syntactic productivity checks such as constructor-counting will fail unless they have access to the definition of f. Put f into a different module and per-module termination checking will fail.

Type-based termination restores compositionality by giving function f a refined type that not only expresses that it takes an element an a stream and produces a stream, but also that the generated stream is extended by one element in the front. In this way, productivity of repeat is guaranteed by the typing of f, without need to reveal its definition. One could say that type-based termination facilitates *termination checking under assumptions*.

2.1 Example: Fibonacci

Let us look at programming with copatterns and type-based termination for a more interesting example, the stream of Fibonacci numbers. It can be elegantly implemented in terms of zipWith $f\ s\ t$ which pointwise applies the binary function f to the elements of streams s and t.

$$\begin{array}{ll} \mathsf{zipWith}\ f\ s\ t\ .\mathsf{head} & = f\ (s\ .\mathsf{head})\ (t\ .\mathsf{head}) \\ \mathsf{zipWith}\ f\ s\ t\ .\mathsf{tail} & = \mathsf{zipWith}\ f\ (s\ .\mathsf{tail})\ (t\ .\mathsf{tail}) \\ \\ \mathsf{fib}\ .\mathsf{head} & = 0 \\ \mathsf{fib}\ .\mathsf{tail}\ .\mathsf{head} & = 1 \\ \mathsf{fib}\ .\mathsf{tail}\ .\mathsf{tail} & = \mathsf{zipWith}\ (+)\ \mathsf{fib}\ (\mathsf{fib}\ .\mathsf{tail}) \end{array}$$

The last equation states in terms of streams that the $(n+2)$nd element of the Fibonacci stream is the sum of the nth and the $(n+1)$st. It looks like fib is a terminating definition since fib .tail .tail only refers to fib and fib .tail, thus, one projection is removed in each recursive call. However, termination of fib is also dependent on good properties of zipWith. For instance, the following faulty clause for zipWith would make fib .tail .tail .head loop:

$$\mathsf{zipWith}\ f\ s\ t\ .\mathsf{head} = f\ (s\ .\mathsf{tail}\ .\mathsf{head})\ (t\ .\mathsf{tail}\ .\mathsf{head})$$

$$\begin{array}{l} \mathsf{fib}\ .\mathsf{tail}\ .\mathsf{tail}\ .\mathsf{head} \\ \ = \mathsf{zipWith}\ (+)\ \mathsf{fib}\ (\mathsf{fib}\ .\mathsf{tail})\ .\mathsf{head} \\ \ = (\mathsf{fib}\ .\mathsf{tail}\ .\mathsf{head}) + (\mathsf{fib}\ .\mathsf{tail}\ .\mathsf{tail}\ .\mathsf{head}) \\ \ = (\mathsf{fib}\ .\mathsf{tail}\ .\mathsf{head}) + (\mathsf{fib}\ .\mathsf{tail}\ .\mathsf{head}) + (\mathsf{fib}\ .\mathsf{tail}\ .\mathsf{tail}\ .\mathsf{head}) \\ \ = \dots \end{array}$$

The problem is that the faulty zipWith adds again one tail projection that has been removed in going from the original call fib .tail .tail to the recursive call fib .tail, thus, we are left with the same number of projections, leading to an infinite call cycle.

What we learn from this counterexample is that in order to reason about termination of stream expressions, we need to trade the naive image of streams as infinite sequences for a notion of streams that can safely be subjected to α many projections, where $\alpha \le \omega$ can be a natural number or (the smallest) infinity ω. We refer to such streams as *sized streams*, or streams having *depth* α. Clearly, if a stream of depth α is required, we can safely supply a stream of depth $\beta \ge \alpha$, thus, sized streams are subject to contravariant subtyping.

The original zipWith delivers, if called with input streams of depth α, an output stream of the same depth. This allows us to reason about the termination of fib as follows. We show that fib is a stream of arbitrary depth α by induction on $\alpha \le \omega$. Cases $\alpha < 2$ are easy. The interesting case is $\alpha = n+2$ when we take two tail projections and then another n projections, thus, $n+2$ projections in total. Then we may assume (by induction hypothesis) that on the rhs taking up to $n+1$ projections of fib is fine, thus, fib and fib .tail behave well under another n projections—they both can be assigned depth n using subtyping. Passing them to zipWith $(+)$ returns in turn a stream of the same depth n, hence the lhs fib .tail .tail can be assigned depth n and, consequently, fib depth $n+2$, which was our goal.

The faulty zipWith, however, needs streams of depth $n+1$ to deliver a stream of depth n. Since fib .tail can only safely be assumed to have depth n, not depth $n+1$, the termination proof attempt fails, and rightfully so.

In this model proof we assumed that taking a projection will decrease the depth by exactly one. In the following, we will loosen this assumption and let projections take us to any strictly smaller depth.

2.2 Type-based termination for copatterns

In this section, we present the key ideas behind $\mathsf{F}_\omega^{\mathsf{cop}}$, our polymorphic core language for type-based termination checking of recursive definitions involving inductive and coinductive types. We illustrate how the integration of size expressions into the type system captures and mechanizes the informal reasoning about termination employed in the previous section.

Size quantification for inductive and coinductive types. Besides quantification over types $\forall A{:}{*}.\ B$ we have quantification over sizes $\forall i{<}a.\ B$. To unify these two forms of quantification we add to the base kind $*$ of types the base kinds $<a$ denoting sets of ordinals below a and conceive $\forall i{<}a.\ B$ as shorthand for $\forall i{:}({<}a).\ B$. Thus, size expressions fall in the same syntactic class as type expressions. We introduce a special ordinal ∞, the closure ordinal for all (co)inductive types we consider. As far as streams are concerned, ∞ can be thought of as ω. In general, valid size expressions are of the form $a ::= i + n \mid \infty + n$ where i is a size variable and n a concrete number (we drop $+0$).

The type of streams of depth a over element type A will be denoted by $\mathsf{Stream}^a A$, and we consider the following typing rules for the projections:

$$\frac{s : \mathsf{Stream}^a A}{s\ .\mathsf{head} : \forall i{<}a^\uparrow.\ A} \qquad \frac{s : \mathsf{Stream}^a A}{s\ .\mathsf{tail} : \forall i{<}a^\uparrow.\ \mathsf{Stream}^i A} \tag{1}$$

These rules state that if you want to project a stream of depth a, you will need to provide a *witness* that you are able to do so, i. e., an ordinal $i < a^\uparrow$. In case of tail, this witness serves also as the depth of the projected stream. For instance, if $s : \mathsf{Stream}^{i+2} A$, then $s\ .\mathsf{tail}\ (i+1)\ .\mathsf{head}\ i\ :\ A$. *Bound normalization* $a^\uparrow$, defined by $(i+n)^\uparrow = i+n$ and $(\infty+n)^\uparrow = \infty+1$, allows us to turn bounds $a \ge \infty$ into $\infty+1$ and project from the fixpoint $\mathsf{Stream}^\infty A$ without information loss. For $s : \mathsf{Stream}^\infty A$ we have $s\ .\mathsf{tail}\ \infty : \mathsf{Stream}^\infty A$ since $\infty < \infty^\uparrow = \infty + 1$, reflecting that the tail of a fully defined stream has infinite depth as well.

In practice, we often use the following derived rule which eliminates the universal quantifier and directly compares sizes.

$$\frac{s : \mathsf{Stream}^a A}{s\ .\mathsf{head}\, b : A}\ b < a^\uparrow \qquad \frac{s : \mathsf{Stream}^a A}{s\ .\mathsf{tail}\, b : \mathsf{Stream}^b A}\ b < a^\uparrow$$

More generally, following previous work (Abel et al. 2013), we represent coinductive types as recursive records νR, with $R = \{d_1 : F_1; \dots; d_n : F_n\}$ giving (sized) types to the projections $d_{1..n}$ as follows:

$$\frac{r : \nu^a R}{r\ .d_k : \forall i{<}a^\uparrow.\ F_k(\nu^i R)}$$

For instance, with $\mathsf{Stream}^i\, A = \nu^i\{\mathsf{head} : \lambda X.\, A; \mathsf{tail} : \lambda X.\, X\}$ we obtain the typing of head and tail presented above (1). Considering R as a finite map from projections to type constructors, we write R_{d_k} for F_k.

Dually, inductive types are recursive variants μS with $S = \langle c_1 : F_1; \dots; c_n : F_n\rangle$ and constructor typing

$$\frac{t : \exists i{<}a^\uparrow.\ F_k(\mu^i S)}{c_k\, t : \mu^a S}.$$

For instance, finite lists can be defined as follows: $\mathsf{List}^i A = \mu^i\langle \mathsf{nil}\ :\ \lambda X.\, 1;\ \mathsf{cons}\ :\ \lambda X.\, A \times X\rangle$. Integrating the quantifier rules, we derive the following inferences for constructors and destructors:

$$\frac{s : S_c(\mu^b S)}{c^b s : \mu^a S}\ b < a^\uparrow \qquad \frac{r : \nu^a R}{r\ .d\ b : R_d(\nu^b R)}\ b < a^\uparrow.$$

Specifying termination measures. The polymorphically typed version of zipWith officially looks as follows, where we write $\forall i{\le}a$ as abbreviation for $\forall i{<}(a+1)$:

$$\begin{aligned}\mathsf{zipWith} : {} & \forall i{\le}\infty.\ |i| \Rightarrow \forall A{:}{*}.\ \forall B{:}{*}.\ \forall C{:}{*}.\\ & (A \to B \to C) \to\\ & \mathsf{Stream}^i A \to \mathsf{Stream}^i B \to \mathsf{Stream}^i C\end{aligned}$$

$$\begin{aligned}\mathsf{zipWith}\ i\ A\ B\ C\ f\ s\ t\ .\mathsf{head}\ j &= f\ (s\ .\mathsf{head}\ j)\ (t\ .\mathsf{head}\ j)\\ \mathsf{zipWith}\ i\ A\ B\ C\ f\ s\ t\ .\mathsf{tail}\ j &= \mathsf{zipWith}\ j\ A\ B\ C\ f\\ & \quad\ (s\ .\mathsf{tail}\ j)\ (t\ .\mathsf{tail}\ j)\end{aligned}$$

The first equation has type C and the second one type $\mathsf{Stream}^j C$. The kind of j is $<i$ due to the typing of head and tail, thus, zipWith is well-defined (and terminating) by induction on its first argument, the size argument. The associated termination measure is located after the size variable(s) and, in general, a tuple $|a, b, c|$ of size expressions under the lexicographic order.[2] In this case, it is just the unary tuple $|i|$, meaning that the termination measure is just the value of size variable i. The measure is not officially part of the type; it is rather an annotation that allows us to termination check the clauses without having to infer a termination order.

High-level idea of size-based termination checking. When we check a corecursive definition such as the second clause of zipWith we start with traversing the the left hand side (lhs). We first introduce assumption $i{\le}\infty$ into the context and now hit the measure annotation $|i|$ in the type. At this point we introduce the assumption $\mathsf{zipWith} : \forall j{\le}\infty.\ |j|{<}|i| \Rightarrow \forall A{:}{*}.\ \forall B{:}{*}.\ \forall C{:}{*}.\ (A \to B \to C) \to \mathsf{Stream}^j A \to \mathsf{Stream}^j B \to \mathsf{Stream}^j C$ which will be used to check the recursive call on the right hand side (rhs). It has a constraint $|j| < |i|$, a lexicographic comparison of size expression tuples (which here just means $j < i$), that is checked before applying zipWith j to A. Continued checking of the lhs introduces further assumptions $A, B, C : *$, $f : A \to B \to C$, $s : \mathsf{Stream}^i A$, $t : \mathsf{Stream}^i B$, and $j < i$. Checking the rhs succeeds since the constraint $|j| < |i|$ is satisfied and $s\ .\mathsf{tail}\ j : \mathsf{Stream}^j A$ and $t\ .\mathsf{tail}\ j : \mathsf{Stream}^j B$.

In the following, we abbreviate $\forall A{:}{*}$ to just $\forall A$ and $\forall i{\le}\infty$ to just $\forall i$. With all size and type-arguments, the definition of the Fibonacci stream becomes:

$$\begin{aligned}&\mathsf{fib}\ :\ \forall i.\ |i| \Rightarrow \mathsf{Stream}^i \mathbb{N}\\ &\mathsf{fib}\ i\ .\mathsf{head}\ j \qquad\quad = 0\\ &\mathsf{fib}\ i\ .\mathsf{tail}\ j\ .\mathsf{head}\ k = 1\\ &\mathsf{fib}\ i\ .\mathsf{tail}\ j\ .\mathsf{tail}\ k \quad = \mathsf{zipWith}\ k\ \mathbb{N}\ \mathbb{N}\ \mathbb{N}\ (+)\ (\mathsf{fib}\ k)\ (\mathsf{fib}\ j\ .\mathsf{tail}\ k)\end{aligned}$$

In the last line, the lhs introduces size variables i and $j < i$ and $k < j$ and an assumption $\mathsf{fib} : \forall i'.\ |i'| < |i| \Rightarrow \mathsf{Stream}^{i'} \mathbb{N}$ and expects a rhs of type $\mathsf{Stream}^k \mathbb{N}$. Since $k < j < i$, both recursive calls are valid, and the expressions $\mathsf{fib}\ k$ and $\mathsf{fib}\ j\ .\mathsf{tail}\ k$ both have type $\mathsf{Stream}^k \mathbb{N}$. With $\mathsf{zipWith}\ k\ \mathbb{N}\ \mathbb{N}\ \mathbb{N} : \mathsf{Stream}^k \mathbb{N} \to \mathsf{Stream}^k \mathbb{N} \to \mathsf{Stream}^k \mathbb{N}$, the rhs is well-typed, and fib is terminating.

2.3 Example: Stream processor

Ghani et al. (2009) describe programs for continuous stream functions $\mathsf{Stream}\ A \to \mathsf{Stream}\ B$ in terms of a mixed coinductive-inductive data type SP with two constructors $\mathsf{get} : (A \to \mathsf{SP}) \to \mathsf{SP}$ and $\mathsf{put} : (B \times \mathsf{SP}) \to \mathsf{SP}$. We use this example to illustrate how our foundation supports size-based reasoning on such mixed datatypes and lexicographic termination measures for mutually recursive functions. A stream processor can either get an element $v : A$ from the input stream and enter a new state, depending on the read value, or it can put an element $w : B$ on the output stream and enter a new state. To be productive, it can only read finitely many values from the input stream before writing a value on the output stream, thus, SP is actually a nesting of a least fixed-point into a greatest one: $\mathsf{SP} = \nu X.\ \mu Y.\ (A \to Y) + (B \times X)$. We express this nesting by the definition of two data types, an inductive variant SP_μ and a coinductive record type SP_ν.

$$\begin{aligned}\mathsf{SP}^i_\mu\ X &= \mu^i \langle \mathsf{get} : \lambda Y.\ A \to Y;\ \mathsf{put} : \lambda Y.\ B \times X\rangle\\ \mathsf{SP}^i_\nu &= \nu^i \{\mathsf{out} : \lambda X.\ \mathsf{SP}^\infty_\mu X\}\end{aligned}$$

Inside the coinductive type, we use the inductive type SP_μ at size ∞ since we want to allow an arbitrary (finite) number of gets between two puts. We get the following derived rules for typing constructors and destructors:

$$\frac{f : A \to \mathsf{SP}^b_\mu X}{\mathsf{get}^b f : \mathsf{SP}^a_\mu X}\ b < a^\uparrow \qquad \frac{w : B \qquad sp : X}{\mathsf{put}^b(w, sp) : \mathsf{SP}^a_\mu X}\ b < a^\uparrow$$

$$\frac{sp : \mathsf{SP}^a_\nu}{sp\ .\mathsf{out}\ b : \mathsf{SP}^\infty_\mu\ \mathsf{SP}^b_\nu}\ b < a^\uparrow$$

In the context of stream processors it is convenient to consider streams as given by a single destructor force which returns head and tail in a pair, thus, $\mathsf{Str}^i A = \nu^i\{\mathsf{force} : \lambda X.\ A \times X\}$. Dedicated projections hd and tl can be defined by

$$\begin{aligned}\mathsf{hd} &: \forall i.\ \mathsf{Str}^{i+1} A \to A\\ \mathsf{hd}\ i\ s &= \mathsf{fst}\ (s\ .\mathsf{force}\ i)\end{aligned}$$

$$\begin{aligned}\mathsf{tl} &: \forall i.\ \mathsf{Str}^{i+1} A \to \mathsf{Str}^i A\\ \mathsf{tl}\ i\ s &= \mathsf{snd}\ (s\ .\mathsf{force}\ i)\end{aligned}$$

with fst and snd the obvious first and second projections from pairs. Via bound normalization, facilitating $\mathsf{Str}^\infty = \mathsf{Str}^{\infty+1}$, we obtain instances $\mathsf{hd}\ \infty : \mathsf{Str}^\infty A \to A$ and $\mathsf{tl}\ \infty : \mathsf{Str}^\infty A \to \mathsf{Str}^\infty A$.

Running a stream processor on an input stream produces an output stream as follows (informally coded in a Haskell-like language):

$$\begin{aligned}\mathsf{run}\ (\mathsf{get}\ f)\ (v, vs) &= \mathsf{run}\ (f\ v)\ vs\\ \mathsf{run}\ (\mathsf{put}(w, sp))\ vs &= (w,\ \mathsf{run}\ sp\ vs)\end{aligned}$$

We represent this function via two mutually recursive functions, one handling SP_μ and one SP_ν:

$$\begin{aligned}&\mathsf{run}_\mu : \forall i \forall j.\ |i, j+1| \Rightarrow \mathsf{SP}^j_\mu(\mathsf{SP}^i_\nu) \to \mathsf{Str}^\infty A \to B \times \mathsf{Str}^i B\\ &\mathsf{run}_\mu\ i\ j\ (\mathsf{get}^{j'} f) \qquad vs = \mathsf{run}_\mu\ i\ j'\ (f\ (\mathsf{hd}\ \infty\ vs))\ (\mathsf{tl}\ \infty\ vs)\\ &\mathsf{run}_\mu\ i\ j\ (\mathsf{put}^{j'}(w, sp))\ vs = (w,\ \mathsf{run}_\nu\ i\ sp\ vs)\end{aligned}$$

$$\begin{aligned}&\mathsf{run}_\nu : \forall i.\ |i, 0| \Rightarrow \mathsf{SP}^i_\nu \to \mathsf{Str}^\infty A \to \mathsf{Str}^i B\\ &\mathsf{run}_\nu\ i\ sp\ vs\ .\mathsf{force}\ i' \qquad = \mathsf{run}_\mu\ i'\ \infty\ (sp\ .\mathsf{out}\ i')\ vs\end{aligned}$$

The recursive run_μ handles a sequence of gets terminated by put and emits the head of a forced stream $B \times \mathsf{Str}^i B$. The tail is produced by the corecursive run_ν which, upon forcing, calls run_μ again. The termination is guaranteed by the lexicographic measures, which decrease in each recursive call:

$$\begin{aligned}\mathsf{run}_\mu \to \mathsf{run}_\mu &: |i, j+1| > |i, j'+1| &&\text{since } j > j'\\ \mathsf{run}_\mu \to \mathsf{run}_\nu &: |i, j+1| > |i, 0|\\ \mathsf{run}_\nu \to \mathsf{run}_\mu &: |i, 0| \quad\ \ > |i', \infty+1| &&\text{since } i > i'\end{aligned}$$

Note that since we are not doing induction on SP^i_ν, but coinduction into Str^i, we could use SP^∞_ν instead of SP^i_ν in the types of run_μ and run_ν. However, the given types are more precise: instead of a stream processor of infinite depth, they only require a stream processor of depth i to produce a stream of depth i.

[2] The notation for termination measures is taken from Xi (2002)

2.4 Example: breadth-first labelled infinite trees

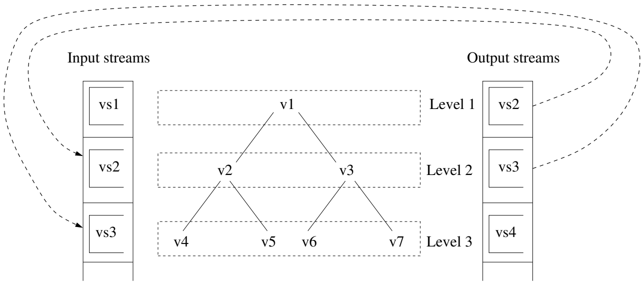

Figure 1. Breadth-first labeled infinite tree

Jones and Gibbons (1993) present tree labeling as a cyclic program. We will now describe a modified version for infinite trees and apply type-based termination to it. Figure 1 explains the core idea of this algorithm. Given a stream $vs_1 = \mathsf{cons}\, v_1\, vs_2$ of labels, we construct an infinite tree with root v_1 (at level 1) and use vs_2 to construct the left and right subtree (both at level 2). To provide labels for all levels, a stream of streams $vs_1, vs_2, vs_3, \ldots$ is used as input and a stream of streams of the remaining labels $vs_2, vs_3, \ldots$ is output. In a Haskell-like language, we would code this as follows:

$$
\begin{array}{rcl}
\mathsf{bfs}\,(\mathsf{cons}\,(\mathsf{cons}\,v\,vs)\,vss) & = & (\mathsf{node}\,v\,l\,r,\ \mathsf{cons}\,vs\,vss'') \\
\mathsf{where}\quad (l, vss') & = & \mathsf{bfs}\,vss \\
(r, vss'') & = & \mathsf{bfs}\,vss'
\end{array}
$$

The stream vss of streams is created from a single label stream vs by *tying the knot*:

$$\mathsf{bf}\ vs = t \quad \mathsf{where}\ (t, vss) = \mathsf{bfs}\,(\mathsf{cons}\,vs\,vss)$$

Is this cyclic program productive, or will the creation of tree t get stuck in an infinite loop? Danielsson has shown productivity by coding an interpreter for stream expressions in Agda (Danielsson 2010); we shall succeed by appropriate size assignment. At this point, it is worth mentioning that bfs does not fall into the usual scheme of a *corecursive definition* such as supported by the Coq proof assistant (INRIA 2012), since its target is not a coinductive type, but a tuple type. Our approach, however, breaks out of this restriction since it unifies recursion and corecursion under measure-based termination on ordinals (sizes and depths).

Fixing a type V of labels, we define a coinductive type of infinite binary trees, a type of streams of streams, and a type of results of function bfs.

$$
\begin{array}{lcl}
\mathsf{SS}^i & = & \mathsf{Stream}^i(\mathsf{Stream}^\infty V) \\
\mathsf{Tree}^i & = & \nu^i\ \{\mathsf{label} : \lambda X.V; \mathsf{left} : \lambda X.X; \mathsf{right} : \lambda X.X\} \\
\mathsf{Result}^i & = & \nu^\infty\{\mathsf{tree} : \lambda X.\mathsf{Tree}^i; \mathsf{rest} : \lambda X.\mathsf{SS}^i\}.
\end{array}
$$

Since Result is not recursive (X is not used), Result^i is just a lazy product of Tree^i and SS^i. We need a record here instead of a tuple because we want to define bfs by copattern matching, the copatterns being $.\mathsf{tree}\,\infty$ and $.\mathsf{rest}\,\infty$.

In the following definition of bfs, each of the five components v, l, r, vs, and vss'' of its result $(\mathsf{node}\,v\,l\,r,\ \mathsf{cons}\,vs\,vss'')$ is given by one equation:

$$
\begin{array}{l}
\mathsf{bfs}\ :\ \forall i.\ |i| \Rightarrow \mathsf{SS}^i \to \mathsf{Result}^i \\
\mathsf{bfs}\ i\ ss\ .\mathsf{tree}\ \infty\ .\mathsf{label}\ j\ =\ v \\
\mathsf{bfs}\ i\ ss\ .\mathsf{tree}\ \infty\ .\mathsf{left}\ \ j\ =\ p_1\ .\mathsf{tree}\ \infty \\
\mathsf{bfs}\ i\ ss\ .\mathsf{tree}\ \infty\ .\mathsf{right}\ j\ =\ p_2\ .\mathsf{tree}\ \infty \\
\mathsf{bfs}\ i\ ss\ .\mathsf{rest}\ \infty\ .\mathsf{head}\ j\ =\ vs \\
\mathsf{bfs}\ i\ ss\ .\mathsf{rest}\ \infty\ .\mathsf{tail}\ \ j\ =\ p_2\ .\mathsf{rest}\ \infty
\end{array}
$$

$$
\begin{array}{rrcll}
\mathsf{where} & v & : & V & = ss\ .\mathsf{head}\ j\ .\mathsf{head}\ \infty \\
 & vs & : & \mathsf{Stream}^\infty V & = ss\ .\mathsf{head}\ j\ .\mathsf{tail}\ \infty \\
 & vss & : & \mathsf{SS}^j & = ss\ .\mathsf{tail}\ j \\
 & p_1 & : & \mathsf{Result}^j & = \mathsf{bfs}\ j\ vss \\
 & p_2 & : & \mathsf{Result}^j & = \mathsf{bfs}\ j\ (p_1\ .\mathsf{rest}\ \infty)
\end{array}
$$

For the sake of readability, and to make the connection to the original program obvious, we have taken the liberty to name and type the intermediate results v, vs, vss (decomposition of ss) and p_1, and p_2 (the pairs created by the recursive calls). Well-definedness of bfs is apparent since recursive calls are restricted to depth $j < i$. For well-typedness it is crucial that the SS of input and output and the output Tree are all considered at the same depth i.

The final step is tying the knot, $(t, vss) = \mathsf{bfs}\,(\mathsf{cons}\,vs\,vss)$. We define the pair (t, vss) by recursion, informally by $\mathsf{bfp}\ vs = \mathsf{bf}\ (\mathsf{cons}\ vs\ (\mathsf{bfp}\ vs\ .\mathsf{rest}))$. How to assign sizes?

$$
\begin{array}{l}
\mathsf{bfp} : \forall i.\ |i| \Rightarrow \mathsf{Stream}^\infty V \to \mathsf{Result}^i \\
\mathsf{bfp}\ i\ vs\ =\ \mathsf{bf}\ i\ (\mathsf{cons}\ vs\ (\mathsf{bfp}\ ?\ vs\ .\mathsf{rest}\ \infty))
\end{array}
$$

For the recursive call, we need a depth ? smaller than i, but we only get one by pattern matching if we analyse the result further:

$$
\begin{array}{l}
\mathsf{bfp}\ :\ \forall i.\ |i| \Rightarrow \mathsf{Stream}^\infty V \to \mathsf{Result}^i \\
\mathsf{bfp}\ i\ vs\ .\mathsf{tree}\ \infty\ .\mathsf{label}\ j\ =\ p\ .\mathsf{tree}\ \infty\ .\mathsf{label}\ j \\
\mathsf{bfp}\ i\ vs\ .\mathsf{tree}\ \infty\ .\mathsf{left}\ \ j\ =\ p\ .\mathsf{tree}\ \infty\ .\mathsf{left}\ \ j \\
\mathsf{bfp}\ i\ vs\ .\mathsf{tree}\ \infty\ .\mathsf{right}\ j\ =\ p\ .\mathsf{tree}\ \infty\ .\mathsf{right}\ j \\
\mathsf{bfp}\ i\ vs\ .\mathsf{rest}\ \infty\ .\mathsf{head}\ j\ =\ p\ .\mathsf{rest}\ \infty\ .\mathsf{head}\ j \\
\mathsf{bfp}\ i\ vs\ .\mathsf{rest}\ \infty\ .\mathsf{tail}\ \ j\ =\ p\ .\mathsf{rest}\ \infty\ .\mathsf{tail}\ \ j \\
\quad \mathsf{where}\ \ p\ :\ \mathsf{Result}^{j+1} \\
\qquad\qquad\ p\ =\ \mathsf{bf}\ (j+1)\ (\mathsf{cons}\ vs\ (\mathsf{bfp}\ j\ vs\ .\mathsf{rest}\ \infty))
\end{array}
$$

This works, but is a lot of boilerplate code. In previously studied type systems for productivity (Pareto 2000; Abel 2006) one assumes size $i+1$ on the lhs, which in our notation would simply become

$$
\begin{array}{l}
\mathsf{bfp} : \forall i.\ \mathsf{Stream}^\infty V \to \mathsf{Result}^i \\
\mathsf{bfp}\ (i+1)\ vs\ =\ \mathsf{bf}\ (i+1)\ (\mathsf{cons}\ vs\ (\mathsf{bfp}\ i\ vs\ .\mathsf{rest}\ \infty)).
\end{array}
$$

Our present system disallows such matching on sizes, which has some consistency issues (Abel 2010, Sec. 5.2) and also requires the result type to be upper semi-continuous in i (which it is in this case) (Hughes et al. 1996; Abel 2008b). However, we can first code a fixpoint combinator for Result and then use it to define bfp, hiding the unpleasant boilerplate.

$$
\begin{array}{l}
\mathsf{fixR}\ :\ \forall i.\ |i| \Rightarrow (\forall j.\ \mathsf{Result}^j \to \mathsf{Result}^{j+1}) \to \mathsf{Result}^i \\
\mathsf{fixR}\ i\ f\ .\mathsf{tree}\ \infty\ .\mathsf{label}\ j\ =\ r\ .\mathsf{tree}\ \infty\ .\mathsf{label}\ j \\
\mathsf{fixR}\ i\ f\ .\mathsf{tree}\ \infty\ .\mathsf{left}\ \ j\ =\ r\ .\mathsf{tree}\ \infty\ .\mathsf{left}\ \ j \\
\mathsf{fixR}\ i\ f\ .\mathsf{tree}\ \infty\ .\mathsf{right}\ j\ =\ r\ .\mathsf{tree}\ \infty\ .\mathsf{right}\ j \\
\mathsf{fixR}\ i\ f\ .\mathsf{rest}\ \infty\ .\mathsf{head}\ j\ =\ r\ .\mathsf{rest}\ \infty\ .\mathsf{head}\ j \\
\mathsf{fixR}\ i\ f\ .\mathsf{rest}\ \infty\ .\mathsf{tail}\ \ j\ =\ r\ .\mathsf{rest}\ \infty\ .\mathsf{tail}\ \ j \\
\quad \mathsf{where}\ \ r\ :\ \mathsf{Result}^{j+1} \\
\qquad\qquad\ r\ =\ f\ j\ (\mathsf{fixR}\ j\ f)
\end{array}
$$

$$
\begin{array}{lcl}
\mathsf{bfp} & : & \forall i.\ \mathsf{Stream}^\infty V \to \mathsf{Result}^i \\
\mathsf{bfp}\ i\ vs & = & \mathsf{fixR}\ i\ f \\
\quad \mathsf{where}\ \ f\ j\ r & = & \mathsf{bfs}\ (j+1)\ (\mathsf{cons}\ vs\ (r\ .\mathsf{rest}\ \infty)).
\end{array}
$$

Digression. For which types A^i can we define a fixpoint combinator of type $\forall i.\ (\forall j.\ A^j \to A^{j+1}) \to A^i$? We conjecture those are at least the admissible types of Pareto (2000) and Abel (2008b). While in these works admissible types are determined by inference rules derived from by semantic criteria, in our present types system we can "prove" admissibility by programming the fixed-point principle ourselves! This gives greater flexibility (and we could employ generic programming to derive fixpoint combinators in the standard cases).

3. Syntax

In this section, we formally define $\mathsf{F}_\omega^{\mathsf{cop}}$, our higher-order polymorphic lambda-calculus with sized inductive and coinductive types, polarized higher-order subtyping, and definitions by pattern and copattern matching. As in previous work (Abel 2006) we choose System F_ω rather than System F as basis since the notion of a *type constructor* is required (at least, semantically) if one wants to talk its fixed-points, i. e., about (co)inductive types.

$$\begin{array}{lll}
\mathsf{SizeVar} & \ni i, j & \\
\mathsf{SizeExp} & \ni a, b & ::= i + n \mid \infty + n \quad (n \geq 0) \\
\mathsf{SizeExp}^+ & \ni a^+, b^+ & ::= a \mid n \\
\mathsf{Measure} & \ni \mathfrak{m} & ::= \cdot \mid a^+, \mathfrak{m} \\
\\
\mathsf{Pol} & \ni \pi & ::= \circ \mid + \mid - \mid \top \\
\mathsf{SizeCxt} & \ni \Psi & ::= \cdot \mid \Psi, i{:}\pi({<}a)
\end{array}$$

Figure 2. Sizes and measures.

3.1 Sizes

Fig. 2 gives a grammar for sizes, measures, and size contexts. A *size expression* a consists of a base, which is either a size variable i or ∞, and an offset, a natural number n.

$$a ::= i + n \mid \infty + n$$

We omit the offset when 0. Each size variable i comes with a bound $i < a$, which is recorded in a *size context*

$$\Psi ::= \cdot \mid \Psi, i{:}\pi({<}a).$$

A size context is considered as finite map from size variables i to their *polarity* π (see below) and their *kind* ${<}a$. We write ${\leq}a$ for ${<}(a+1)$ and size for ${\leq}\infty$. *Extended size expressions* a^+ allow as a third base, n, i.e. just a natural number. *Measures* $\mathfrak{m}$ are tuples of extended size expressions. There are a number of trivial judgements concerning well-formedness and partial ordering of (extended) size expressions and measures (see Table 1). These judgements may use the bounds stored in size context Ψ and are all defined as expected; their inference rules can be found in Fig. 12.

$\Psi \vdash a$	size a is well-formed
$\Psi \vdash a < b$	strict size comparison
$\Psi \vdash a \leq b$	size comparison
$\Psi \vdash a^+$	extended size a^+ is well-formed
$\Psi \vdash a^+ < b^+$	strict comparison
$\Psi \vdash a^+ \leq b^+$	comparison
$\Psi \vdash_n \mathfrak{m}$	measure $\mathfrak{m}$ is a well-formed n-tuple
$\Psi \vdash \mathfrak{m} < \mathfrak{m}'$	strict lexicographic measure comparison
$\Psi \vdash \mathfrak{m} \leq \mathfrak{m}'$	lexicographic measure comparison
$\Psi \vdash \exists\Psi'$	Ψ' is consistent for each valuation of Ψ

Table 1. Size-related judgements.

In constraint-based systems, strong normalization is usually lost in inconsistent contexts.[3] While our size contexts Ψ are always consistent, i. e., enjoy a valuation[4] η of the declared size variables (by natural numbers even), we need sometimes a stronger property that a size context extension Ψ' is consistent with a fixed valuation η of Ψ, i. e., Ψ' must be consistent even when we apply η to its declared bounds. For instance, $i{\leq}\infty, j{<}i$ is consistent, but $j{<}i$ is not a consistent extension of $i{\leq}\infty$ under valuation $\eta(i) = 0$, since there is no solution for j. We write $\boxed{\Psi \vdash \exists\Psi'}$ if Ψ' consistently extends Ψ in this sense. This judgement is inspired by Blanqui and Riba (2006).

Proposition $\Psi \vdash \exists\Psi'$ can be tested by computing a minimal valuation η of Ψ and then checking whether Ψ' has a (minimal) valuation under η. In the following, let η be a finite map from size variables to natural numbers. Then $\eta(a)$ is an extended size expression. We say $\eta \models \Psi$ if $\eta(i) < \eta(a)$ for all $(i < a) \in \Psi$. A minimal valuation $\mathsf{val}_\eta(\Psi)$ for Ψ above η can be defined as follows:

$$\begin{array}{lll}
\mathsf{val}_\eta(\cdot) & = \eta & \\
\mathsf{val}_\eta(\Psi, j < a) & = \mathsf{val}_\eta(\Psi) & \text{if } \eta(j) < \eta(a) \\
\multicolumn{3}{l}{\text{otherwise, if } \eta(j) \not< \eta(a):} \\
\mathsf{val}_\eta(\Psi, j < i + n) & = \mathsf{val}_{\eta[i \mapsto \eta(j)-n+1]}(\Psi) & \text{if } i \in \mathsf{dom}(\Psi) \\
\mathsf{val}_\eta(\Psi, j < i + n) & = \text{undefined} & \text{if } i \notin \mathsf{dom}(\Psi)
\end{array}$$

Note that if $\eta' = \mathsf{val}_\eta(\Psi)$ is defined, then $\eta' \geq \eta$ (pointwise), and $\eta' \models \Psi$. If $\mathsf{val}_\eta(\Psi)$ is undefined and $\eta' \geq \eta$ then $\eta' \not\models \Psi$. In particular, if $\eta(i) = 0$ for all $i \in \mathsf{dom}(\Psi)$ and $\mathsf{val}_\eta(\Psi)$ is undefined, then Ψ is inconsistent. To check $\Psi \vdash \exists\Psi'$ we let $\eta_0(i) = 0$ the null-valuation and $\eta = \mathsf{val}_{\eta_0}(\Psi)$. Then we check whether $\mathsf{val}_\eta(\Psi')$ is defined.

$$\begin{array}{lll}
\mathsf{SKind} & \ni \iota & ::= * \mid o \mid \iota \to \iota' \\
\mathsf{Kind} & \ni \kappa & ::= * \mid {<}a \mid \pi\kappa \to \kappa' \\
\mathsf{TyCxt} & \ni \Delta & ::= \cdot \mid \Delta, X{:}\pi\kappa \\
\mathsf{Cxt} & \ni \Gamma & ::= \cdot \mid \Gamma, x{:}A \mid \Gamma, x{:}{}^?\!A \\
\\
\mathsf{TyVar} & \ni X, Y, Z, i, j & \\
\mathsf{TyAtom} & \ni K & ::= a \mid X \mid 1 \mid \times \mid \to \mid \forall_\kappa \mid \exists_\kappa \\
\mathsf{Type} & \ni F, G, A, B, C & ::= K \mid \lambda X{:}\iota.\, F \mid F\, G \\
& & \quad \mid \mu^a S \mid \nu^a R \\
\\
\mathsf{Var} & \ni x, y, z & \\
\mathsf{Cons} & \ni c & \\
\mathsf{Proj} & \ni d & \\
\mathsf{Variant} & \ni S & ::= \langle c_1{:}F_1; \ldots; c_n{:}F_n \rangle \qquad n \geq 0 \\
\mathsf{Record} & \ni R & ::= \{d_1{:}F_1; \ldots; d_n{:}F_n\} \qquad n \geq 0 \\
\\
\mathsf{MType} & \ni {}'\!A, {}'\!B & ::= \forall\Psi.\, \mathfrak{m} \Rightarrow A \\
\mathsf{CType} & \ni {}^?\!A, {}^?\!B & ::= \forall\Psi.\, \mathfrak{c} \Rightarrow A \\
\mathsf{Cond} & \ni \mathfrak{c} & ::= \mathfrak{m} {<} \mathfrak{m}'
\end{array}$$

Figure 3. Kinds and type constructors.

3.2 Kinds and type constructors

The type constructors of F_ω are assigned kinds $\iota ::= * \mid \iota \to \iota'$, with base kind $*$ classifying all proper types and function kinds

[3] For instance, in extensional type theory, $X : \mathsf{Type},\ p : X = (X \to X) \vdash (\lambda x{:}X.\, x\, x)(\lambda x{:}X.\, x\, x) : X$. The blame is on the false equality assumption $X = X \to X$ which is used for type conversion.

[4] A valuation η of size context Ψ is a map from size variables i to sizes $\eta(i)$ that fulfills the constraints for the size variables given by Ψ. Formally, $\eta(i) < [\![a]\!]_\eta$ must hold in case $i{:}\pi({<}a) \in \Psi$, where $[\![a]\!]_\eta$ is the value of size expression a in environment η.

$\iota \to \iota'$ the (higher-order) type operators. We add a second base kind $\iota ::= \cdots \mid o$ that classifies size expressions, which we locate at the type level, since they are computationally irrelevant and can be erased during compilation, just as the types are.

These *simple kinds* ι form with the type constructor a simply-"typed" type-level lambda calculus. We *refine* these kinds into $\mathsf{F}^{\mathsf{cop}}_{\omega}$-kinds

$$\kappa ::= * \mid {<}a \mid \kappa \xrightarrow{\pi} \kappa'$$

where ${<}a$ refines o into the kind of size expressions $b < a$. The *polarized function kind* $\kappa \xrightarrow{\pi} \kappa'$, also written $\pi\kappa \to \kappa'$, allows us to express that the classified type constructor is co-variant ($\pi = +$), contravariant ($\pi = -$), constant ($\pi = \top$) or mixed-variant or of unknown variance ($\pi = \circ$). The polarities π are partially ordered $\circ \leq +, - \leq \top$ according to their information content. This and the order on size expressions induce a subkinding relation $\Psi \vdash \kappa \leq \kappa'$ on kinds of the same structure, i. e., the same underlying simple kind $|\kappa| = |\kappa'|$. Here, when comparing two o-kinds $({<}a) \leq ({<}b)$, we resort to size comparison $a \leq b$. The default variance is $\circ$ (no information) and we may omit it, writing simply $\kappa \to \kappa'$ or $\Psi, i{:}({<}a)$, which is further abbreviated by $\Psi, i{<}a$.

Kinding or *type variable contexts* $\Delta ::= \cdot \mid \Delta, X{:}\pi\kappa$, which provide scoping and kinding information for type constructors, generalize size contexts from bounds $({<}a)$ to arbitrary kinds κ. We may use a Δ where a Ψ is formally required, silently erasing all non-size variables from Δ. More generally, context *restriction* $\Delta \upharpoonright \vec{X}$ of context Δ to a set of variables $\vec{X}$ deletes the bindings for all $Y \notin \vec{X}$ from Δ.

$\Psi \vdash \kappa$	kind κ is well-formed in Ψ
$\Psi \vdash \kappa \leq \kappa'$	κ is a subkind of κ'
$\Delta \vdash \Delta'$	kinding context Δ' is well-formed in Δ
$\Delta \vdash \exists\Delta'$	Δ' is consistent for each valuation of Δ

Table 2. Kind-related judgements.

The judgement $\Delta \vdash \exists\Delta'$ (see Table 2) states that Δ' is consistent for each valuation of Δ. Only the size declarations matter here, so it is a straightforward extension of $\Psi \vdash \exists\Psi'$.

Figure 3 contains a grammar for the type constructors of $\mathsf{F}^{\mathsf{cop}}_{\omega}$. Its core is a simply-kinded lambda-calculus $X \mid \lambda X{:}\iota.\, F \mid F\, G$ with constants 1, $\times$, $\to$, $\forall_\kappa$, and $\exists_\kappa$ to form unit, product, function, universal, and existential types. Size expressions a are considered type constructors so that sizes can be abstracted over and applied. We use the following short-hands:

$\lambda X F$	for	$\lambda X{:}\iota.\, F$	if ι inferable
$A \times B$	for	$(\times)\, A\, B$	product type
$A \to B$	for	$(\to)\, A\, B$	function type
$\forall X{:}\kappa.\, A$	for	$\forall_\kappa(\lambda X{:}|\kappa|.\, A)$	universal type
$\exists X{:}\kappa.\, A$	for	$\exists_\kappa(\lambda X{:}|\kappa|.\, A)$	existential type
$\forall i{<}a.\, A$	for	$\forall_{<a}\,(\lambda i{:}o.\, A)$	bounded universal
$\exists i{<}a.\, A$	for	$\exists_{<a}\,(\lambda i{:}o.\, A)$	bounded existential
$\forall i.\, A$	for	$\forall i{:}\mathsf{size}.\, A$	"unbounded" universal
$\exists i.\, A$	for	$\forall i{:}\mathsf{size}.\, A$	"unbounded" existential.

We also write $\forall\Delta.\, A$ for the universal abstraction of all type variables of Δ in type A.

The simple kind annotation ι in $\lambda X{:}\iota.\, F$ allows us to infer a unique simple kind for closed type constructors. The simple kind of an open type constructor depends only on the simple kinds of its free type variables. This property simplifies the interpretation $[\![F]\!]$ of type constructors as set-theoretic functions on semantic types we will give later.

For the purpose of type checking, we are only interested in β-normal type constructors. We write $F \;@^\iota\; G$ for the normalizing application of F to an argument G of simple kind ι. We may write $@^\kappa$ instead of $@^{|\kappa|}$, or even just $@$.

Sized inductive $\mu^a S$ and coinductive types $\nu^a R$ are given in terms of *variant rows* S and *record rows* R. A variant row $S = \langle c_1{:}F_1; \ldots; c_n{:}F_n\rangle$ is a finite map from variant labels c_i, called *constructors*, to type constructors $S_{c_i} = F_i$. Dually, a record row R maps record labels d, called *destructors* or *projections*, to type constructors R_d. Instead of presenting, for instance, streams as $\nu^a X.\, \{\mathsf{head} : A;\ \mathsf{tail} : X\}$, we move the abstraction over X into the record row as $\nu^a\{\mathsf{head} : \lambda X.A;\ \mathsf{tail} : \lambda X.X\}$, in order to formulate the typing rules more conveniently.

Finally, we have *constrained types* $\forall\Psi.\, \mathfrak{m}{<}\mathfrak{m}' \Rightarrow A$ that allow its inhabitants to be used only if the condition $\mathfrak{m} < \mathfrak{m}'$ is fulfilled. We use them to restrict recursive calls to situations where the termination measure has decreased. Recursive function definitions come with *measured types* $'A ::= \forall\Delta.\ \mathfrak{m} \Rightarrow A$. These are not proper types but rather blueprints for constrained types. The idea is that kinding context Δ declares some size variables that are used in measure $\mathfrak{m}$ (and type A). When we analyze the body of a recursive function of measure type $'A$ and the variables of Δ are in scope (thus, the measure $\mathfrak{m}$ is well-formed), we make a copy $'B = \forall\Delta'.\, \mathfrak{m}' \Rightarrow A'$ of $'A$ by renaming the variables of Δ to Δ'. Then, by *measure replacement* $'B^{<\mathfrak{m}}$ we create the constrained type $\forall\Delta'.\ \mathfrak{m}'{<}\mathfrak{m} \Rightarrow A'$ which is used to type the recursive occurrences of the function in its body.

$\Delta \vdash A$	type A is well-formed
$\Delta \vdash F \rightrightarrows \kappa$	F has kind κ (inference)
$\Delta \vdash F \leftleftarrows \kappa$	F has kind κ (checking)
$\Delta \vdash \Gamma$	typing context Γ is well-formed
$\Delta \vdash A \leq A'$	A is subtype of A'
$\Delta \vdash F \leq^\pi F' \rightrightarrows \kappa$	F is higher-ord. subtype of F' (κ inferred)
$\Delta \vdash F \leq^\pi F' \leftleftarrows \kappa$	F is higher-ord. subtype of F' (κ given)

Table 3. Type-related judgements.

Table 3 lists judgements for well-kindedness and partial ordering of types and type constructors. The judgements for types A only invoke the judgments for type constructors F in checking mode at base kind ($\leftleftarrows *$). The judgements for constructors are *bidirectional* with inference mode that computes the kind κ and checking mode that starts with a given κ. Bidirectional checking is complete since we are only interested in normal type constructors.

The rules for these judgements are given in figures 13 and 14 . A thorough discussion of polarized higher-order subtyping, i. e., subtyping for type constructors that take variance into account, is available in Abel (2008a) and Steffen (1998), we just recapitulate the basic principle here: A constructor F with $X_1{:}\pi_1\kappa_1, \ldots, X_n{:}\pi_n\kappa_n \vdash F \leftleftarrows \kappa$ is interpreted as an operator

$$\lambda X_1 \ldots \lambda X_n.F \;:\; \kappa_1 \xrightarrow{\pi_1} \ldots \kappa_n \xrightarrow{\pi_n} \kappa$$

with variance given as noted in its kinding context. This induces the kinding rules, for instance $X{:}{-}*, Y{:}{+}* \vdash X \to Y : *$ is valid since function space is contravariant in its domain and covariant in its codomain. In particular, the hypothesis rule $X{:}\pi\kappa \vdash X : \kappa$ is only valid if $\pi \leq +$, i. e., $\pi = \circ$ which just states that $\lambda X.X : \kappa \to \kappa$ is a well-formed operator, or $\pi = +$ which additionally states that $\lambda X.X$ is monotone. Using the hypothesis rule on $\pi = -$ or $\pi = \top$ is invalid since $\lambda X.X$ is neither an antitone nor a constant operator.

Given a partial order $G \leq G'$, its π-parameterized version $G \leq^{\pi} G'$ can be defined as follows:

$$\begin{array}{lcl} G \leq^{+} G' & = & G \leq G' \\ G \leq^{-} G' & = & G' \leq G \\ G \leq^{\circ} G' & = & G \leq G' \text{ and } G' \leq G \\ G \leq^{\top} G' & = & \text{true} \end{array}$$

π-variance of a constructor $F \rightrightarrows \pi\kappa \to \kappa'$ means that $F\,G \leq F\,G' \rightrightarrows \kappa$ whenever $G \leq^{\pi} G' \leftleftarrows \kappa$. (The reader is advised to play through the four cases for π in his mind.) Theoretically, the π-parameterized versions $\Delta \vdash F \leq^{\pi} F' \dots$ of higher-order subtyping could be defined from a non-parameterized version $\Delta \vdash F \leq F' \dots$, but to avoid the potential exponential blowup due to duplication of work in case of $\leq^{\circ}$, the π-parameterized versions are taken as primitive.

Exp	$\ni r, s, t$	$::= u \mid v \mid \lambda\vec{D}$	term
Intro	$\ni v$	$::= () \mid (t_1, t_2) \mid c\,t \mid {}^{G}t$	introduction
App	$\ni u$	$::= x \mid f \mid r\,e$	applicative
Fun	$\ni f, g$		function name
Elim	$\ni e$	$::= t \mid G \mid .d$	elimination
Pat	$\ni p$	$::= x \mid () \mid (p_1, p_2) \mid c\,p \mid {}^{Q}p$	pattern
TyPat	$\ni Q$	$::= X \mid \infty$	type pattern
Copat	$\ni q$	$::= p \mid Q \mid .d$	copattern
PatSp	$\ni \mathbf{q}$	$::= \vec{q}$	pattern spine
DCl	$\ni D$	$::= \{\mathbf{q} \to t\}$	def. clause
Def	$\ni \vec{D}$	$::= \{D_1; \dots; D_n\}$	def. clauses

Figure 4. Terms, (co)patterns, and clauses.

3.3 Terms and (co)patterns

Figure 4 presents the abstract syntax of $\mathsf{F}^{\mathsf{cop}}_{\omega}$ *terms* t, which are categorized into *introductions* v, *applicative terms* u, and *anonymous objects* $\lambda\vec{D}$. Introductions $()$, (t_1, t_2), $c\,t$ and ${}^{G}t$ construct tuples and inductive and existential types. Applicative terms x, f, and $r\,e$ are identifiers and generalized applications of a term r to an *elimination* e, which can be a term s for function elimination, a type G for instantiation of a polymorphic function, or a destructor $.d$ for projection from a coinductive type.

For each introduction form v we have the corresponding form of pattern p, and for each elimination form e there is a copattern q. Application copatterns are just patterns p to match the argument, type application copatterns Q are either type variables X or the special size pattern ∞, which matches anything, and projection copatterns are simply destructors d that match the same destructor in an elimination. A sequence of $\vec{q}$ of copatterns is called a pattern spine $\mathbf{q}$, in correspondence to an *elimination spine* $\vec{e}$.

Generalized lambda abstraction $\lambda\vec{D}$ introduces an object whose behavior is given by the clauses $\vec{D}$, each of which consists of a lhs, a (possibly empty) copattern sequence $\vec{q}$, and a rhs, a term t. Objects subsume both record and λ expressions of traditional functional languages. Here are a few simple examples:

$\lambda\{x \to t\}$	ordinary λ-abstraction $\lambda x t$
$\lambda\{X \to t\}$	type abstraction $\Lambda X t$
$\lambda\{(x, y) \to x\}$	first projection from pair
$\lambda\{{}^{X}x\,y \to y\,X\,x\}$	elimination of existential
$\lambda\{A\,x\,y\,.\mathsf{head}\;\infty \to x$	
$\;;A\,x\,y\,.\mathsf{tail}\;\;\infty \to y\}$	cons for $\mathsf{Stream}^{\infty} A$
$\lambda\{\cdot \to s;\ \cdot \to t\}$	non-deterministic choice $s \oplus t$

The meaning, given by the operational semantics, is that whenever $\lambda\vec{D}$ is applied to a sequence of eliminations $\vec{e}$ that match the copatterns $\vec{q}$ of a clause with rhs t under a substitution σ and a type substitution τ, then $(\lambda\vec{D})\,\vec{e}$ reduces to $t\sigma\tau$, the rhs instantiated by the substitutions computed from pattern matching. Using $\boxed{\vec{e} \,/\, \vec{q} \searrow \sigma; \tau}$ for *pattern matching*, the basic rule for *contraction* $\boxed{t \mapsto t'}$ becomes:

$$\frac{\vec{e} \,/\, \mathbf{q}_k \searrow \sigma; \tau}{\lambda\{\overrightarrow{\mathbf{q} \to t}\}\;\vec{e}\,\vec{e}\,' \mapsto t_k\sigma\tau\,\vec{e}\,'}$$

As usual, t is called a *redex* and t' its *reduct* if $t \mapsto t'$. We allow overlapping lhss, a spine $\vec{e}$ may match different pattern spines $\mathbf{q}$, resulting in different contractions of the same redex. Also, if no lhs in the clauses $\vec{D}$ matches $\vec{e}$, the expression $\lambda\vec{D}\;\vec{e}$ is *stuck*. While a coverage checker as described in previous work (Abel et al. 2013) could exclude overlapping and incomplete clauses in well-typed programs, we do not require coverage in this paper and confine ourselves to show *strong normalization*, i. e., the absence of infinite reduction sequences.

Not all stuck terms are pathological; since we are matching the whole pattern spine in one go, partially applied functions such as $\lambda\{xy \to t\}s$ are stuck, but can become unstuck if more arguments are supplied. The existence of partially applied functions will require careful treatment in the normalization proof, because non-contractibility of a non-introduction term is not preserved under application (as would be in the case of λ-calculus).

Decl	$\ni \delta$	$::= f : A = \vec{D}$	declaration
MDecl	$\ni {}'\delta$	$::= f : {}'A = \vec{D}$	declaration with measure
Block	$\ni \beta$	$::= \mathsf{mutual}_m\,\vec{{}'\delta}$	mutual block
Prg	$\ni P$	$::= \vec{\beta}; u$	program
Sig	$\ni \Sigma$	$::= \vec{\delta}$	signature

Figure 5. Declarations, blocks, and programs.

3.4 Declarations and programs

An $\mathsf{F}^{\mathsf{cop}}_{\omega}$ program consists of a sequence $\vec{\beta}$ of mutual blocks and an applicative term u, the *entry point* (this could be the name of the main function or a call to the main function with some initial arguments). Each *mutual block* $\mathsf{mutual}_m\vec{{}'\delta}$ is a sequence $\vec{{}'\delta}$ of mutually recursive declarations with a lexicographic termination measure of length m. Each *declaration* $f : {}'A = \vec{D}$ assigns to a function symbol f its measured type ${}'A$ and its clauses $\vec{D}$. Measures serve their purpose during checking of the mutual block and are discarded afterwards. Erasure of measure $(\!|{}'\delta|\!)$ yields a (unmeasured) declaration $f : A = \vec{D}$; after checking a mutual block and erasing the measures, the individual declarations of the block become part of the signature Σ which is used for type-checking and evaluation of the remainder of the program. An applied function $f\,\vec{e}$ reduces if one of its clauses does:

$$\frac{(\lambda\vec{D})\,\vec{e} \mapsto t}{f\,\vec{e} \mapsto t}\,(f : A = \vec{D}) \in \Sigma$$

The one-step reduction relation $\boxed{t \longrightarrow t'}$ is the compatible closure of the contraction relation $t \mapsto t'$, i. e., $t \longrightarrow t'$ if t' is the result of contracting exactly one redex in (an arbitrary subterm of) t. Strong normalization of reduction will be shown to hold for well-typed programs.

$\boxed{\Delta; \Gamma \vdash r \rightrightarrows C}$ Expression typing (inference mode). In: Δ, Γ, r with $\Delta \vdash \Gamma$. Out: C with $\Delta \vdash C$

$$\frac{}{\Delta; \Gamma \vdash f \rightrightarrows \Sigma(f)} \qquad \frac{(x{:}A) \in \Gamma}{\Delta; \Gamma \vdash x \rightrightarrows A} \qquad \frac{(x : \forall \Psi.\, \mathfrak{c} \Rightarrow A) \in \Gamma \qquad \Delta \vdash \vec{a} \leftleftarrows \Psi \qquad \tau = \vec{a}/\hat{\Psi} \qquad \Delta \vdash \mathfrak{c}\tau}{\Delta; \Gamma \vdash x\, \vec{a} \rightrightarrows A\tau}$$

$$\frac{\Delta; \Gamma \vdash r \rightrightarrows A \to B \qquad \Delta; \Gamma \vdash s \leftleftarrows A}{\Delta; \Gamma \vdash r\, s \rightrightarrows B} \qquad \frac{\Delta; \Gamma \vdash r \rightrightarrows \nu^a R}{\Delta; \Gamma \vdash r.d \rightrightarrows \forall j{<}a^{\uparrow}.\, R_d\, (\nu^j R)} \qquad \frac{\Delta; \Gamma \vdash r \rightrightarrows \forall_\kappa F \qquad \Delta \vdash G \leftleftarrows \kappa}{\Delta; \Gamma \vdash r\, G \rightrightarrows F \mathbin{@^\kappa} G}$$

Switching.

$$\frac{\Delta \vdash A \qquad \Delta; \Gamma \vdash t \leftleftarrows A}{\Delta; \Gamma \vdash (t : A) \rightrightarrows A} \qquad \frac{\Delta; \Gamma \vdash r \rightrightarrows A \qquad \Delta \vdash A \leq C}{\Delta; \Gamma \vdash r \leftleftarrows C}$$

$\boxed{\Delta; \Gamma \vdash t \leftleftarrows C}$ Expression typing (checking mode). In: $\Delta; \Gamma, t, C$ with $\Delta \vdash \Gamma$ and $\Delta \vdash C$. Out: success/failure.

$$\frac{}{\Delta; \Gamma \vdash () \leftleftarrows 1} \qquad \frac{\Delta; \Gamma \vdash t_1 \leftleftarrows A_1 \qquad \Delta; \Gamma \vdash t_2 \leftleftarrows A_2}{\Delta; \Gamma \vdash (t_1, t_2) \leftleftarrows A_1 \times A_2} \qquad \frac{\Delta; \Gamma \vdash t \leftleftarrows \exists j{<}a^{\uparrow}.\, S_c\, (\mu^j S)}{\Delta; \Gamma \vdash c\, t \leftleftarrows \mu^a S}$$

$$\frac{\Delta \vdash G \leftleftarrows \kappa \qquad \Delta; \Gamma \vdash t \leftleftarrows F \mathbin{@^\kappa} G}{\Delta; \Gamma \vdash {}^G t \leftleftarrows \exists_\kappa F} \qquad \frac{\Delta; \Gamma \vdash \vec{D} \leftleftarrows A}{\Delta; \Gamma \vdash \lambda \vec{D} \leftleftarrows A}$$

$\boxed{\Delta; \Gamma \vdash D \leftleftarrows A}$ and $\boxed{\Delta; \Gamma \vdash \vec{D} \leftleftarrows A}$ definition typing. In: Δ, Γ, A, D or $\vec{D}$ with $\Delta \vdash \Gamma$ and $\Gamma \vdash A$. Out: success/failure.

$$\frac{\Delta'; \Gamma' \mid A \vdash_\Delta \vec{q} \rightrightarrows C \qquad \Delta \vdash \exists \Delta' \qquad \Delta, \Delta'; \Gamma, \Gamma' \vdash t \leftleftarrows C}{\Delta; \Gamma \vdash \{\vec{q} \to t\} \leftleftarrows A} \qquad \frac{\Delta; \Gamma \vdash D_k \leftleftarrows A \text{ for all } k}{\Delta; \Gamma \vdash \vec{D} \leftleftarrows A}$$

Figure 6. Type checking rules.

$\boxed{\Delta; \Gamma \vdash_{\Delta_0} p \leftleftarrows A}$ Pattern typing (linear). In: Δ_0, p, A with $\Delta_0 \vdash A$. Out: Δ, Γ with $\Delta_0, \Delta; \Gamma \vdash p \leftleftarrows A$.

$$\frac{}{\cdot; x{:}A \vdash_{\Delta_0} x \leftleftarrows A} \qquad \frac{}{\cdot; \cdot \vdash_{\Delta_0} () \leftleftarrows 1} \qquad \frac{\Delta_1; \Gamma_1 \vdash_{\Delta_0} p_1 \leftleftarrows A_1 \qquad \Delta_2; \Gamma_2 \vdash_{\Delta_0} p_2 \leftleftarrows A_2}{\Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \vdash_{\Delta_0} (p_1, p_2) \leftleftarrows A_1 \times A_2}$$

$$\frac{\Delta; \Gamma \vdash_{\Delta_0} p \leftleftarrows \exists j{<}a^{\uparrow}.\, S_c\, (\mu^j S)}{\Delta; \Gamma \vdash_{\Delta_0} c\, p \leftleftarrows \mu^a S} \qquad \frac{\Delta; \Gamma \vdash_{\Delta_0, X:\kappa} p \leftleftarrows F \mathbin{@^\kappa} X}{X{:}\kappa, \Delta; \Gamma \vdash_{\Delta_0} {}^X p \leftleftarrows \exists_\kappa F}$$

$\boxed{\Delta; \Gamma \mid A \vdash_{\Delta_0} \vec{q} \rightrightarrows C}$ Pattern spine typing. In: $\Delta_0, A, \vec{q}$ with $\Delta_0 \vdash A$. Out: Δ, Γ, C with $\Delta_0, \Delta; \Gamma \vdash C$ and $\Delta_0, \Delta; \Gamma, z{:}A \vdash z\, \vec{q} \rightrightarrows C$.

$$\frac{}{\cdot; \cdot \mid A \vdash_{\Delta_0} \cdot \rightrightarrows A} \qquad \frac{\Delta_1; \Gamma_1 \vdash_{\Delta_0} p \leftleftarrows A \qquad \Delta_2; \Gamma_2 \mid B \vdash_{\Delta_0} \vec{q} \rightrightarrows C}{\Delta_1, \Delta_2; \Gamma_1, \Gamma_2 \mid A \to B \vdash_{\Delta_0} p\, \vec{q} \rightrightarrows C}$$

$$\frac{\Delta; \Gamma \mid \forall j{<}a^{\uparrow}.\, R_d\, (\nu^j R) \vdash_{\Delta_0} \vec{q} \rightrightarrows C}{\Delta; \Gamma \mid \nu^a R \vdash_{\Delta_0} .d\, \vec{q} \rightrightarrows C} \qquad \frac{\Delta; \Gamma \mid F \mathbin{@^\kappa} X \vdash_{\Delta_0, X:\kappa} \vec{q} \rightrightarrows C}{X{:}\kappa, \Delta; \Gamma \mid \forall_\kappa F \vdash_{\Delta_0} X\, \vec{q} \rightrightarrows C}$$

Figure 7. Pattern Typing.

$\Delta; \Gamma \vdash r \rightrightarrows C$	Infer type C for term r
$\Delta; \Gamma \vdash t \leftleftarrows C$	Term t checks against type C
$\Delta; \Gamma \vdash \{\mathbf{q} \to t\} \leftleftarrows A$	Clause $\{\mathbf{q} \to t\}$ checks against type A
$\Delta; \Gamma \vdash \vec{D} \leftleftarrows A$	Clauses D check against type A
$\Delta; \Gamma \vdash_{\Delta_0} p \leftleftarrows A$	Pattern p checks against type A
$\Delta; \Gamma \mid A \vdash_{\Delta_0} \mathbf{q} \rightrightarrows C$	Pattern spine $\mathbf{q}$ eliminates A into C

Table 4. Type checking.

3.5 Type checking

Table 4 lists the judgements involved in type checking $\mathsf{F}_\omega^{\mathsf{cop}}$ programs. Type-checking terms is bidirectional and a straightforward adaption of Abel et al. (2013) to polymorphism, bounded quantification, and constraints. The rules are given in figures 6 and 7, and we briefly explain them.

Inference $\boxed{\Delta; \Gamma \vdash r \rightrightarrows C}$. A function symbol f's type $\Sigma(f)$ is looked up in the signature, and a variable x's type $\Gamma(x)$ in the typing context. If $\Gamma(x)$ is a constrained type $\forall \Psi.\, \mathfrak{c} \Rightarrow A$, the variable x must be immediately applied to size arguments $\vec{a}$ satisfying both Ψ and the condition $\mathfrak{c}$; after all, a constrained type is, for consistency reasons, not a proper type for an expression. An application $r\, s$ of a function r of inferred type $A \to B$ has type B if the argument s checks against type A. Instantiation $r\, G$ of a polymorphic term r of inferred type $\forall_\kappa F$ has type $F \mathbin{@^\kappa} G$ if G has kind κ. In particular, r could be of type $\forall i{<}a.\, A$, then G must be a size expression $<\, a$ to succeed. If r is of coinductive type $\nu^a R$, then $r\, .d$ has type $\forall j{<}a^{\uparrow}.R_d\, (\nu^j R)$, see Section 2.3.

There are two rules to switch direction. Checking r against type C succeeds if r's type is inferred as A and A is a subtype of C. Also, we can add *type ascription* $(t : A)$ to the term language; then inference of $(t : A)$ succeeds and yields A if A is a well-formed type and t checks against A. While type ascription is needed to bidirectionally type check redexes or stuck terms, it is dispensable if one confines to checking normal terms (in the sense that no elimination is applied to a λ in the source program). We will consider type ascriptions be removed before execution of the program, so they do not pop up in the operational and denotational semantics.

Checking $\boxed{\Delta; \Gamma \vdash t \leftleftarrows C}$. Introductions and λs are checked against a given type. Checking a pair ${}^{G}t$ of a type expression G and a term t against an existential type $\exists_k F$ succeeds if G has kind κ and t is of the correct instance $F \mathbin{@^\kappa} G$. Checking a constructor term $c\,t$ against an inductive type $\mu^a S$ succeeds if t checks against $\exists j < a^{\uparrow}.\, S_c\,(\mu^j S)$. This means that t should be essentially a pair ${}^{b}t'$ of a size $b < a^{\uparrow}$ and t' be a correct argument to constructor c, i. e., having variant S_c instantiated to $\mu^j S$. If $a \geq \infty$, by bound normalization $b = \infty$ is a valid size index, which implies that in a value v in the fixpoint $\mu^\infty S$ all size witnesses can uniformly be ∞. To check $\lambda\vec{D}$ we check all clauses D_k.

Clause checking $\boxed{\Delta; \Gamma \vdash \{\mathbf{q} \to t\} \leftleftarrows A}$. We first check that pattern spine $\mathbf{q}$ eliminates indeed type A. As a result, we obtain a kinding context Δ' which binds the type variables X contained in $\mathbf{q}$ and a typing context Γ' which binds the pattern variables x contained in $\mathbf{q}$'s patterns, and a remaining type C of lhs and rhs. We now need to make sure that $\Delta \vdash \exists\Delta'$ such that any valuation of Δ can be extended to a valuation of Δ'. Complementing the original contexts $\Delta; \Gamma$ by the pattern contexts $\Delta'; \Gamma'$ we check the rhs t against C.

Pattern spine checking $\boxed{\Delta; \Gamma \mid A \vdash_{\Delta_0} \mathbf{q} \rightrightarrows C}$. We eliminate type A which is well-formed in Δ_0. If there are no copatterns in $\mathbf{q}$, thus, the clause has an empty lhs, we simply return A which must be the type of the rhs. If we encounter an application pattern p, the eliminated type must be a function type $A \to B$. We check p against A and obtain pattern contexts $\Delta_1; \Gamma_1$. We continue to check the remaining copatterns, obtaining more pattern contexts $\Delta_2; \Gamma_2$ and a result type C, which we return together with the concatenated pattern contexts. Concatenation, and thus, pattern spine checking fails if the contexts do not have disjoint domains. A common variable would mean a non-linear lhs, which we exclude.

If we encounter a projection pattern $.d$, the eliminated type must be a coinductive type $\nu^a R$. Taking projection $.d$ yields type $\forall j{<}a^{\uparrow}.\, R_d(\nu^j R)$, thus, we continue to eliminate this type. It could be eliminated by an ∞-pattern if a was $\geq \infty$, hence $a^{\uparrow} = \infty + 1$. In this case, we must additionally ensure that the coinductive type actually reached its fixed-point at ∞. This is the case if $R_d\,(\nu^j R)$ is antitone in j (we shall prove this in Section 4). In general, when eliminating $\forall_{\leq\infty} F$ by an ∞-pattern, we can continue with $F \mathbin{@} \infty$ if F is antitone, i. e., has kind $\mathsf{size} \overset{-}{\to} *$. The general form of eliminating a universal type $\forall_\kappa F$ is by a type variable pattern X; we record $X{:}\kappa$ in the type variable pattern context and continue eliminating $F \mathbin{@^\kappa} X$.

Pattern typing $\boxed{\Delta; \Gamma \vdash_{\Delta_0} p \leftleftarrows A}$. This judgement checks pattern p against type A which is valid in kinding context Δ_0, and returns pattern contexts $\Delta; \Gamma$. Pattern x succeeds against any type, returning singleton context $x{:}A$. The empty tuple $()$ succeeds against the unit type 1, binding no variables. The pair pattern (p_1, p_2) succeeds against the product type $A_1 \times A_2$ if each component p_i checks against its type A_i. The resulting pattern contexts are concatenated, checking for disjointness. A constructor pattern $c\,p$ checks against an inductive type $\mu^a S$ if p checks against $\exists j < a^{\uparrow}.\, S_c\,(\mu^j S)$. This can succeed if $p = {}^{\infty}p'$ and $S_c\,(\mu^j S)$ is monotone in j, meaning that $\mu^\infty S$ was indeed the fixed-point, and we continue checking p' against $S_c\,(\mu^\infty S)$. Or, $p = {}^{j}p'$, then we add size variable $j{<}a$ to the pattern context and continue checking p' against $S_c\,(\mu^j S)$. The last two cases were instances of checking against the general existential type $\exists_\kappa F$.

In the next section, we will validate all the typing rules by exhibiting a semantics of strongly normalizing terms based on Girard's reducibility candidates (Girard et al. 1989).

4. Semantics

In this section we show strong normalization of $\mathsf{F}^{\mathsf{cop}}_\omega$ by a term model. Types are interpreted as reducibility candidates à la Girard adapted to our needs. Our semantic constructions rely only on the terms and the operational semantics of $\mathsf{F}^{\mathsf{cop}}_\omega$, not to the types, kinds, or inference rules. Based on the operational semantics, semantic types and kinds are constructed that interpret the syntactic types, yet syntactic types are never used for semantic constructions.[5] We consider this conceptual hygiene important from a philosophic perspective: we use types just as a vehicle to assign properties to our programs; clearly, they have no run-time significance. While in the end we managed to keep syntactic types out of the semantic constructions, it was hard to get the semantic counterpart (Lemma 30) of pattern spine typing (Figure 7) right.

One clarification: Since $\mathsf{F}^{\mathsf{cop}}_\omega$ has Church-style polymorphism with explicit type abstraction and application, we can of course not talk about terms and operational semantics without mentioning syntactic types. However, we never refer to the structure of syntactic types, they remain abstract, and we could remove everything but type variables from our type language without altering the construction of semantic types and semantic typing "judgements". In particular, in the construction of the semantic universal type $\forall_{\mathcal{K}}\mathcal{F} = \{r \in \mathsf{SN} \mid r\,G \in \mathcal{F}(\mathcal{G})$ for all $G \in \mathsf{Type}, \mathcal{G} \in \mathcal{K}\}$ there is no connection between the syntactic type constructor G and the semantic type constructor $\mathcal{G}$ (of semantic kind $\mathcal{K}$). Type applications serve only to make type-checking decidable, they do not play any role in evaluation.

Preliminaries. We use partially applied relations to denote sets. For instance, we write $(t \longrightarrow \_)$ or simply $t\!\longrightarrow$ for the set $\{t' \mid t \longrightarrow t'\}$ of reducts of t. Similarly, ${<}\alpha = \{\beta \mid \beta < \alpha\}$. The identity substitution is denoted by σ_{id}.

Let $t \sqsubseteq t'$ be the compatible closure of $b \sqsubseteq \infty$.

Lemma 1 (Soundness and completeness of matching). *$s \;/\; p \searrow \tau; \sigma$ iff $s \sqsubseteq p\tau\sigma$.*

Strong normalization. Classically, a term t is strongly normalizing if it admits no infinite reduction sequences $t \longrightarrow t_1 \longrightarrow t_2$ starting with t. Inductively, we define $t \in \mathsf{SN}$ if all of its reducts are already in SN:

$$\frac{(t \longrightarrow \_) \subseteq \mathsf{SN}}{t \in \mathsf{SN}}$$

Naturally, if $t \in \mathsf{SN}$ then all its reducts and subterms are also strongly normalizing.

We extend the notion SN to other syntactic categories: An elimination e is strongly normalizing, $e \in \mathsf{SN}$, if it either is not a term (but a type G or a projection $.d$), or if it is a strongly normalizing term. A definition clause $D = \{\vec{q} \to t\}$ is strongly normalizing if $t \in \mathsf{SN}$.

Simulation. Our typing rules (see Figure 6) state that a definition $\lambda\vec{D} : A$ or $(f : A = \vec{D})$ is well-typed if each of the clauses

[5] Humbly following the masters (Vouillon and Melliès 2004).

D_k is of type A, individually. In the absence of a coverage check, there is no concept of "the clauses make sense *together*". We would like to see this independence of clauses reflected in our semantics. In particular, we would like to have *compositionality*, i. e., if each clause of a definition is semantically meaningful (in particular, does not lead to non-termination), then the clauses are meaningful together. For functions, our type-checker works exactly like that: each clause is checked individually, using the termination measure; an interaction between clauses need not be taken into account.[6]

One idea is to say that a defined function $f : A = \vec{D}$ reduces non-deterministically to one of its clauses D_k, however, this immediately destroys strong normalization, because D_k might mention f. We need to defer unfolding of f until the pattern of one of its clauses matches. Thus, instead we say that $f\,\vec{e}$ reduces if $(\lambda\vec{D})\vec{e}$ reduces; f is simulated by its clauses $\vec{D}$. In general, a term r is *simulated* by terms $\vec{r}$, written $\boxed{r \rhd \vec{r}}$, iff each of its contractions under some eliminations is accounted for by one of the terms $\vec{r}$, formally

$$\forall \vec{e}, t.\; r\,\vec{e} \mapsto t \implies \exists k.\, r_k\,\vec{e} \mapsto t.$$

Closing reducibility candidates by simulation is one of the new ideas of our proof.

Lemma 2 (Simulation)**.**

1. *$\lambda\{D_1; \ldots; D_n\} \rhd \lambda D_1, \ldots, \lambda D_n$.*
2. *If $(f : A = \vec{D}) \in \Sigma$ then $f \rhd \lambda\vec{D}$.*
3. *If $r \rhd r_1, \ldots, r_n$ then $r\,e \rhd r_1\,e, \ldots, r_n\,e$.*

Proof.

1. Assume $(\lambda\vec{D})\vec{e} \mapsto t$. By inversion, $(\lambda D_k)\vec{e} \mapsto t$ for some k.
2. Assume $f\,\vec{e} \mapsto t$. By inversion $(\lambda\vec{D})\vec{e} \mapsto t$.
3. We have to show $\forall \vec{e}, t.\; r\,e\,\vec{e} \mapsto t \implies \exists k.\; r_k\,e\,\vec{e} \mapsto t$. This holds directly by assumption $r \rhd \vec{r}$ with elimination vector $e, \vec{e}$. □

4.1 Semantic Types

In order to show strong normalization we model types as sets of strongly normalizing terms, more precisely, as reducibility candidates à la Girard. We choose reducibility candidates over Tait's saturated sets, since they allow us to show strong normalization in the absence of standardization and confluence. As a consequence, we can model definitions with incomplete and overlapping patterns.

A set of terms $\mathcal{A}$ is a *reducibility candidate* (Girard et al. 1989), written $\mathcal{A} \in \mathsf{CR}$, if the following conditions hold.

CR1 $\mathcal{A} \subseteq \mathsf{SN}$: "each term in $\mathcal{A}$ is strongly normalizing".

CR2 if $t \in \mathcal{A}$ then $(t \longrightarrow \_) \subseteq \mathcal{A}$: "$\mathcal{A}$ is closed under reduction".

CR3 if $t \in \mathsf{Ne}$ and $(t \longrightarrow \_) \subseteq \mathcal{A}$ then $t \in \mathcal{A}$: "$\mathcal{A}$ contains a neutral already if all its redexes are in $\mathcal{A}$".

CR4 if $t \notin \mathsf{Intro}$ and $(t \longrightarrow \_) \subseteq \mathcal{A}$ and $t \rhd \vec{t} \in \mathcal{A}$ then $t \in \mathcal{A}$: "$\mathcal{A}$ is closed under simulation".

Condition **CR4**, is new; it introduces multi-clause objects $\lambda\vec{D}$ and function symbols f into a semantic type (candidate).

Lemma 3 (Multi-clause objects)**.**

1. *If $\lambda D_1, \ldots, \lambda D_n \in \mathcal{A}$ then $\lambda\vec{D} \in \mathcal{A}$.*
2. *If $(f : A = \vec{D}) \in \Sigma$ and $\lambda\vec{D} \in \mathcal{A}$, then $f \in \mathcal{A}$.*

Proof.

1. We show $\lambda\vec{D} \in \mathcal{A}$ by induction on $\vec{D} \in \mathsf{SN}$. Since $\lambda\{\vec{D}\} \rhd \overrightarrow{\lambda D}$, we may use **CR4**. It remains to show that $\lambda\vec{D} \longrightarrow t$ implies $t \in \mathcal{A}$. If $\lambda\vec{D} \mapsto t$, then $\lambda D_k \mapsto t$ for some k, and since $\lambda D_k \in \mathcal{A}$ we infer $t \in \mathcal{A}$ by **CR2**. Otherwise the reduction takes place in some body and we have $t = \lambda\vec{D}'$ with $\vec{D} \longrightarrow \vec{D}'$. Since $\lambda\vec{D}' \rhd \overrightarrow{\lambda D'}$, we conclude by induction hypothesis.
2. Directly by **CR4**, since $f \rhd \lambda\vec{D}$ by Lemma 2 and all reducts of f are reducts of $\lambda\vec{D}$. □

In **CR3**, Ne is a suitable set of so-called *neutral* terms. These are "good", i. e., inhabit a candidate, as soon as all their redexes are good. For Girard's technique to work, neutral terms need to include redexes such as $(\lambda x.t)\,s$ and variables x, and need to be closed under application, i. e., r neutral implies $r\,s$ neutral. In case of pure lambda calculus, any term which is not a lambda-abstraction can be considered neutral.

In our setting of matching the whole pattern spine $\vec{q}$ against the eliminations $\vec{e}$, things are more subtle. For instance, the partial application $\lambda\{x\,y \to x\,x\}\,\delta$ with $\delta = \lambda\{x \to x\,x\}$ is stuck (even in normal form). However, it cannot be neutral and inhabit every candidate (following **CR3**), in particular semantic function types, since it reduces to the diverging term $\delta\,\delta$ if applied to any term t. Thus, we can only accept stuck terms as neutral which cannot become unstuck by extra eliminations. This leads to the following definition:

Definition 4 (Neutral term, terminally stuck)**.**

1. *A applicative term $u \in \mathsf{App}$ is* terminally stuck *if $u\,\vec{e}$ is not a redex for all eliminations $\vec{e}$.*
2. *A term r is* neutral*, written $r \in \mathsf{Ne}$, if it is a redex or terminally stuck.*

As Girard's, our refined notion of neutrality includes redexes, variables, and is closed under eliminations. Further, if $r \in \mathsf{Ne}$ then any reduction in $r\,e$ is either a reduction in r or in e. A reducibility candidate $\mathcal{A}$ is never empty since $\mathsf{Var} \subseteq \mathcal{A}$ by virtue of **CR3**.

Closure. For a set $\mathcal{A} \subseteq \mathsf{SN}$ which is closed under reduction let $\overline{\mathcal{A}}$ be the least reducibility candidate $\supseteq \mathcal{A}$. Inductively, $\overline{\mathcal{A}}$ is defined as the closure under neutrals and simulation:

$$\frac{t \in \mathcal{A}}{t \in \overline{\mathcal{A}}} \qquad \frac{t \in \mathsf{Ne} \qquad (t \longrightarrow \_) \subseteq \overline{\mathcal{A}}}{t \in \overline{\mathcal{A}}}$$

$$\frac{t \notin \mathsf{Intro} \qquad (t \longrightarrow \_) \subseteq \overline{\mathcal{A}} \qquad t \rhd \vec{t} \in \overline{\mathcal{A}}}{t \in \overline{\mathcal{A}}}$$

$\mathcal{A} \mapsto \overline{\mathcal{A}}$ is a *closure operation*, i. e., it is *monotone* ($\mathcal{A} \subseteq \mathcal{B}$ implies $\overline{\mathcal{A}} \subseteq \overline{\mathcal{B}}$), *extensive* ($\mathcal{A} \subseteq \overline{\mathcal{A}}$), and *idempotent* ($\overline{\overline{\mathcal{A}}} \subseteq \overline{\mathcal{A}}$). Note that the closure operator never adds introduction terms such as $()$, (t_1, t_2), $c\,t$, or ${}^{G}t$ to a term set $\mathcal{A}$. Thus, for introductions $v \in \overline{\mathcal{A}}$ we have $v \in \mathcal{A}$ already.

CR is closed under arbitrary intersections and forms, under the inclusion $\subseteq$ order, a complete lattice with greatest element SN and least element $\overline{\emptyset}$.

Semantic types. In the following, let $\mathcal{A}, \mathcal{B} \in \mathsf{CR}$ be candidates, P a proposition, $\mathcal{K}$ some index set and $\mathcal{F} \in \mathcal{K} \to \mathsf{CR}$ a family of reducibility candidates. The following operations, except the

[6] In general, normalization of rewriting is of course not compositional. E. g., the rule $\mathsf{f\,true} \longrightarrow \mathsf{f\,false}$ by itself terminates, but adding $\mathsf{f\,false} \longrightarrow \mathsf{f\,true}$ destroys normalization.

conditional $P \Rightarrow \mathcal{A}$, construct new candidates from existing ones.

$$\begin{array}{lcl}
\mathcal{A} \rightarrow \mathcal{B} & = & \{r \in \mathsf{SN} \mid \forall s \in \mathcal{A}.\, r\, s \in \mathcal{B}\} \\
\forall_{\mathcal{K}}\mathcal{F} & = & \{r \in \mathsf{SN} \mid \forall G \in \mathsf{Type}, \mathcal{G} \in \mathcal{K}.\, r\, G \in \mathcal{F}(\mathcal{G})\} \\
P \Rightarrow \mathcal{A} & = & \{r \in \mathsf{Exp} \mid r \in \mathcal{A} \text{ if } P\} \\
\mathbf{1} & = & \overline{\{()\}} \\
\mathcal{A}_1 \times \mathcal{A}_2 & = & \overline{\{(t_1, t_2) \mid t_1 \in \mathcal{A}_1 \text{ and } t_2 \in \mathcal{A}_2\}} \\
\exists_{\mathcal{K}}\mathcal{F} & = & \overline{\{{}^{G}t \mid \exists \mathcal{G} \in \mathcal{K}, t \in \mathcal{F}(\mathcal{G})\}}
\end{array}$$

Note that the condition $r \in \mathsf{SN}$ in the definition of $\mathcal{A} \rightarrow \mathcal{B}$ is redundant, since $x \in \mathcal{A}$ by **CR3** and $r\, x \in \mathsf{SN}$ implies $r \in \mathsf{SN}$. However, in the definition of $\forall_{\mathcal{K}}\mathcal{F}$ it is important since $\mathcal{K}$ could be empty, e. g., $\mathcal{K} = <0$. Conditional types are not first-class; $P \Rightarrow \mathcal{A}$ only forms a candidate if P is true, otherwise, it is just a set of expressions.

Lemma 5 (Function space candidate)**.** *If* $\mathsf{Var} \subseteq \mathcal{A} \subseteq \mathsf{SN}$ *and* $\mathcal{B} \in \mathsf{CR}$ *then* $\mathcal{A} \rightarrow \mathcal{B} \in \mathsf{CR}$.

Proof.

CR1 Strong normalization: Let $r \in \mathcal{A} \rightarrow \mathcal{B}$. Since $x \in \mathcal{A}$ we have $r\, x \in \mathcal{B} \subseteq \mathsf{SN}$, thus, $r \in \mathsf{SN}$.

CR2 Closure under reduction: Let $r \in \mathcal{A} \rightarrow \mathcal{B}$ and $r \longrightarrow r'$. Assume $s \in \mathcal{A}$ and show $r'\, s \in \mathcal{B}$, which we conclude by **CR2** on $r\, s \in \mathcal{B}$, since $r\, s \longrightarrow r'\, s$.

CR3 Closure under neutrals: Let $r \in \mathsf{Ne}$ and $(r \longrightarrow \_) \subseteq \mathcal{A} \rightarrow \mathcal{B}$. Since $\mathcal{A} \rightarrow \mathcal{B} \subseteq \mathsf{SN}$ we have $r \in \mathsf{SN}$. Assume $s \in \mathcal{A}$. We show $r\, s \in \mathcal{B}$ by **CR3**, exploiting $r\, s \in \mathsf{Ne}$. Consider $r\, s \longrightarrow t$; we show $t \in \mathcal{B}$ by induction on $r, s \in \mathsf{SN}$. Since $r \in \mathsf{Ne}$, either $t = r'\, s$ with $r \longrightarrow r'$ and we conclude by induction hypothesis on $r' \in \mathsf{SN}$, or $t = r\, s'$ with $s \longrightarrow s'$ and we conclude by induction hypothesis on $s' \in \mathsf{SN}$.

CR4 Closure under simulation: Let $r \notin \mathsf{Intro}$ and $(r \longrightarrow \_) \subseteq \mathcal{A} \rightarrow \mathcal{B}$ and $r \rhd \vec{r} \in \mathcal{A} \rightarrow \mathcal{B}$. Assume $s \in \mathcal{A}$ and show $r\, s \in \mathcal{B}$ by **CR4**, exploiting that $r\, s \notin \mathsf{Intro}$ and $r\, s \rhd r_1\, s, \ldots, r_n\, s \in \mathcal{B}$. Assume $r\, s \longrightarrow t$ and show $t \in \mathcal{B}$ by induction on $r, s \in \mathsf{SN}$. In cases $t = r'\, s$ or $t = r\, s'$ we conclude by induction hypothesis. In the remaining case $r\, s \mapsto t$ we have $r_k\, s \mapsto t$ for some $k \in 1..n$. Since $r_k\, s \in \mathcal{B}$ we conclude $t \in \mathcal{B}$ by **CR2**. □

Lemma 6 (Semantic typing rules)**.** *The following inferences are trivial consequences of the construction of semantic types:*

$$\frac{r \in \mathcal{A} \rightarrow \mathcal{B} \qquad s \in \mathcal{A}}{r\, s \in \mathcal{B}} \qquad \frac{r \in \forall_{\mathcal{K}}\mathcal{F} \qquad \mathcal{G} \in \mathcal{K}}{r\, G \in \mathcal{F}(G)}$$

$$\frac{}{() \in \mathbf{1}} \qquad \frac{t_1 \in \mathcal{A}_1 \qquad t_2 \in \mathcal{A}_2}{(t_1, t_2) \in \mathcal{A}_1 \times \mathcal{A}_2} \qquad \frac{\mathcal{G} \in \mathcal{K} \qquad t \in \mathcal{F}(\mathcal{G})}{{}^{G}t \in \exists_{\mathcal{K}}\mathcal{F}}$$

Besides definitions (which we will treat in Section 4.5), rules for constructors and destructors are missing. We will describe semantic (co)inductive types in the next section.

4.2 Ordinals and Fixed-Points

Previous approaches to type-based termination (Hughes et al. 1996; Amadio and Coupet-Grimal 1998; Barthe et al. 2004; Blanqui 2004; Barthe et al. 2008; Sacchini 2013) have defined approximants of least $\boldsymbol{\mu}^{\alpha}\mathcal{F}$ and greatest fixed-points $\boldsymbol{\nu}^{\alpha}\mathcal{F}$ of monotone type constructors $\mathcal{F} \in \mathsf{CR} \xrightarrow{+} \mathsf{CR}$ by conventional induction on ordinal α, distinguishing zero (0), successor $(\alpha + 1)$, and limit ordinals (λ).

$$\begin{array}{lcl@{\qquad}lcl}
\boldsymbol{\mu}^{0}\;\;\mathcal{F} & = & \overline{\emptyset} & \boldsymbol{\nu}^{0}\;\;\mathcal{F} & = & \mathsf{SN} \\
\boldsymbol{\mu}^{\alpha+1}\mathcal{F} & = & \mathcal{F}(\boldsymbol{\mu}^{\alpha}\mathcal{F}) & \boldsymbol{\nu}^{\alpha+1}\mathcal{F} & = & \mathcal{F}(\boldsymbol{\nu}^{\alpha}\mathcal{F}) \\
\boldsymbol{\mu}^{\lambda}\;\;\mathcal{F} & = & \overline{\bigcup_{\alpha<\lambda}\boldsymbol{\mu}^{\alpha}\mathcal{F}} & \boldsymbol{\nu}^{\lambda}\;\;\mathcal{F} & = & \bigcap_{\alpha<\lambda}\boldsymbol{\nu}^{\alpha}\mathcal{F}
\end{array}$$

In this work, we adopt the approach of Sprenger and Dam (2003) for approximations in μ-calculus and use *well-founded induction* instead, which amounts to construct $\boldsymbol{\mu}^{\alpha}\mathcal{F}$ by *inflationary iteration* and $\boldsymbol{\nu}^{\alpha}\mathcal{F}$ by *deflationary iteration*.

$$\boldsymbol{\mu}^{\alpha}\mathcal{F} = \overline{\bigcup_{\beta<\alpha}\mathcal{F}(\boldsymbol{\mu}^{\beta}\mathcal{F})} \qquad \boldsymbol{\nu}^{\alpha}\mathcal{F} = \bigcap_{\beta<\alpha}\mathcal{F}(\boldsymbol{\nu}^{\beta}\mathcal{F})$$

In this definition, $\mathcal{F}$ does not have to be monotone to obtain an ascending chain of approximants in case of $\boldsymbol{\mu}$ and a descending chain for $\boldsymbol{\nu}$. However, if $\mathcal{F}$ is monotone, one can derive above equations as special cases for α being zero, successor, or limit ordinal, *if such a distinction on ordinals exists*. Intuitionistically, this distinction is not valid (Taylor 1996); by building on well-founded induction, we remain within constructive foundations.

Let α, β, γ range over ordinals. We write $\forall_{\beta<\alpha}\mathcal{F}(\beta)$ for $\forall_{<\alpha}\mathcal{F}$ and analogously for $\exists$. Let $\mathcal{S} \in \mathsf{Cons} \rightharpoonup \mathsf{CR} \rightarrow \mathsf{CR}$ and $\mathcal{R} \in \mathsf{Proj} \rightharpoonup \mathsf{CR} \rightarrow \mathsf{CR}$ where we write the first argument, the constructor c, or the destructor d, resp., as index, thus, $\mathcal{S}_c$ and $\mathcal{R}_d$ resp. We define the αth approximants $\boldsymbol{\mu}^{\alpha}\mathcal{S}, \boldsymbol{\nu}^{\alpha}\mathcal{R} \in \mathsf{CR}$ of recursive variant and record type as follows.

$$\begin{array}{lcl}
\boldsymbol{\mu}^{\alpha}\mathcal{S} & = & \overline{\{c\, t \mid c \in \mathsf{dom}(\mathcal{S}) \text{ and } t \in \exists_{\beta<\alpha}\mathcal{S}_c(\boldsymbol{\mu}^{\beta}\mathcal{S})\}} \\
\boldsymbol{\nu}^{\alpha}\mathcal{R} & = & \{r \in \mathsf{SN} \mid \forall d \in \mathsf{dom}(\mathcal{R}).\, r.d \in \forall_{\beta<\alpha}\mathcal{R}_d(\boldsymbol{\nu}^{\beta}\mathcal{R})\}
\end{array}$$

Since $\exists_{<\alpha}\mathcal{F}$ is monotonic in α for any $\mathcal{F}$, so is $\boldsymbol{\mu}^{\alpha}\mathcal{S}$. Dually $\forall_{<\alpha}\mathcal{F}$ and $\boldsymbol{\nu}^{\alpha}\mathcal{R}$ are antitonic in α. We obtain chains:

$$\begin{array}{l}
\overline{\emptyset} \;\; = \boldsymbol{\mu}^{0}\mathcal{S} \subseteq \boldsymbol{\mu}^{1}\mathcal{S} \subseteq \ldots \subseteq \boldsymbol{\mu}^{\gamma}\mathcal{S} \subseteq \boldsymbol{\mu}^{\gamma+1}\mathcal{S} \subseteq \ldots \\
\mathsf{SN} = \boldsymbol{\nu}^{0}\mathcal{R} \supseteq \boldsymbol{\nu}^{1}\mathcal{R} \supseteq \ldots \supseteq \boldsymbol{\nu}^{\gamma}\mathcal{R} \supseteq \boldsymbol{\nu}^{\gamma+1}\mathcal{R} \supseteq \ldots
\end{array}$$

If $\boldsymbol{\mu}^{\alpha}\mathcal{S} = \boldsymbol{\mu}^{\gamma}\mathcal{S}$ for some $\alpha > \gamma$ then $\boldsymbol{\mu}^{\beta}\mathcal{S} = \boldsymbol{\mu}^{\gamma}\mathcal{S}$ for all $\beta > \gamma$ and we say that the chain has become *stationary* at γ. Since the set Exp of expressions is countable and all elements of these chains are subsets of Exp, the chains must become stationary latest at the first uncountable ordinal Ω. We call the ordinal at which all such chains of our language are stationary the *closure ordinal* and denote it by $\boldsymbol{\infty}$.

Since it does not make sense to inspect chains beyond the closure ordinal, we introduce *bound normalization*

$$\alpha^{\uparrow} = \begin{cases} \boldsymbol{\infty} + 1 & \text{if } \alpha \geq \boldsymbol{\infty}, \\ \alpha & \text{otherwise.} \end{cases}$$

Note that $\boldsymbol{\mu}^{\alpha}\mathcal{S} = \boldsymbol{\mu}^{\alpha^{\uparrow}}\mathcal{S}$ and $\boldsymbol{\nu}^{\alpha}\mathcal{R} = \boldsymbol{\nu}^{\alpha^{\uparrow}}\mathcal{R}$. In the following we will talk about ordinals that are as big as $\boldsymbol{\infty} + n$ for finite n, but not bigger ones, so all ordinals will be in $\mathsf{O} = \{\alpha \mid \alpha < \boldsymbol{\infty} + \omega\}$, a set closed under successor. As size index to a least or greatest fixed point, only the ordinals in $\mathsf{Size} = \{\alpha \mid \alpha \leq \boldsymbol{\infty}\}$ are interesting. Thus, if no bound for an ordinal β is given, we assume $\beta \in \mathsf{Size}$, for instance, we write $\exists_{\beta}\mathcal{F}(\beta)$ instead of $\exists_{\beta\in\mathsf{Size}}\mathcal{F}(\beta)$ or $\exists_{\mathsf{Size}}\mathcal{F}$.

The stationary point $\boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S}$ is a pre-fixed point in the sense that $t \in \mathcal{S}_c(\boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S})$ implies $c^{\,\infty}t \in \boldsymbol{\mu}^{\boldsymbol{\infty}+1}\mathcal{S} = \boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S}$. Dually, $\boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R}$ is a post-fixed point as $r \in \boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R} = \boldsymbol{\nu}^{\boldsymbol{\infty}+1}\mathcal{R}$ implies $r.d\,\infty \in \mathcal{R}_d(\boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R})$. Note that we do not require $\mathcal{R}$ or $\mathcal{S}$ to be monotone for these directions. Yet we do if we want $\boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S}$ and $\boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R}$ to be fixed-points.

Lemma 7 (Pre/post-fixed points)**.**

1. *If* $t \in \exists_{\beta\leq\boldsymbol{\infty}}\mathcal{S}_c(\boldsymbol{\mu}^{\beta}\mathcal{S})$ *then* $c\, t \in \boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S}$.
2. *If* $r \in \boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R}$ *then* $r.d \in \forall_{\beta\leq\boldsymbol{\infty}}\mathcal{R}_d(\boldsymbol{\nu}^{\beta}\mathcal{R})$.

Proof.

1. By definition $c\,t \in \boldsymbol{\mu}^{\boldsymbol{\infty}+1}\mathcal{S} = \boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S}$.
2. By definition, since $r \in \boldsymbol{\nu}^{\boldsymbol{\infty}+1}\mathcal{R}$.

Lemma 8 (Fixed-points)**.** *If $\mathcal{S}_c, \mathcal{R}_d$ be monotone for all $c \in \mathsf{dom}(\mathcal{S})$ and $d \in \mathsf{dom}(\mathcal{R})$, then*

1. $\boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S} = \overline{\{c\,{}^b t \mid c \in \mathsf{dom}(\mathcal{S}), b \in \mathsf{Type}, t \in \mathcal{S}_c(\boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S})\}}$*, and*
2. $\boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R} = \{r \mid \forall d \in \mathsf{dom}(\mathcal{R}), b \in \mathsf{Type}.\ r.d\ b \in \mathcal{R}_d(\boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R})\}$.

Proof. For 1, it is sufficient to show $\subseteq$, meaning that $\boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S}$ is a post-fixed point. Note that by definition

$$\boldsymbol{\mu}^{\boldsymbol{\infty}} = \overline{\bigcup_{\beta<\boldsymbol{\infty}} \{c\,{}^b t \mid c \in \mathsf{dom}(\mathcal{S}), b \in \mathsf{Type}, t \in \mathcal{S}_c(\boldsymbol{\mu}^{\beta}\mathcal{S})\}},$$

so we conclude by monotonicity of $\mathcal{S}_c$ and the closure operator, using $\boldsymbol{\mu}^{\beta}\mathcal{S} \subseteq \boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S}$.

For 2, it is sufficient to show that $\boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R}$ is a pre-fixed point. So, if $r.d\ b \in \mathcal{R}_d(\boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R})$ for all $d \in \mathsf{dom}(\mathcal{R})$ and $b \in \mathsf{Type}$, then $r \in \boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R}$. It is sufficient to show $r.d\ b \in \mathcal{R}_d(\boldsymbol{\nu}^{\beta}\mathcal{R})$ for all $\beta < \boldsymbol{\infty}$, and this follows from $\boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R} \subseteq \boldsymbol{\nu}^{\beta}\mathcal{R}$ by monotonicity of $\mathcal{R}_d$. □

Corollary 9.

1. *If $c\,{}^b t \in \boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S}$ and $\mathcal{S}_c$ is monotone, then $t \in \mathcal{S}_c(\boldsymbol{\mu}^{\boldsymbol{\infty}}\mathcal{S})$.*
2. *If $r.d\,b \in \mathcal{R}_d(\boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R})$ and $\mathcal{R}_d$ is monotone, then $r \in \boldsymbol{\nu}^{\boldsymbol{\infty}}\mathcal{R}$.*

4.3 Kinds

Simple kinds ι are interpreted as sets of semantic types, ordinals, or semantic type constructors.

$$\begin{array}{lcl} [\![*]\!] &=& \mathsf{CR} \\ [\![o]\!] &=& \mathsf{O} \\ [\![\iota \to \iota']\!] &=& [\![\iota]\!] \to [\![\iota']\!] \end{array}$$

A simple function kind $\iota \to \iota'$ is interpreted as the function space $[\![\iota]\!] \to [\![\iota']\!]$ of the meta-language (e.g. the set-theoretical function space).

With each simple kind ι we associate a set $\mathsf{KI}(\iota)$ of *semantic kinds* $\mathcal{K} \subseteq [\![\iota]\!]$. Semantic kinds $\mathcal{K}$ are pointed preorders. We write $\bot_{\mathcal{K}}$ for the least element of $\mathcal{K}$ and $\mathcal{F} \le \mathcal{F}' \in \mathcal{K}$ for the preorder relation, omitting "$\in \mathcal{K}$" when clear from the context of discourse. Also let

$$\begin{array}{lll} \mathcal{F} \le^{\circ} \mathcal{F}' & :\iff & \mathcal{F} \le \mathcal{F}' \text{ and } \mathcal{F}' \le \mathcal{F} \\ \mathcal{F} \le^{+} \mathcal{F}' & :\iff & \mathcal{F} \le \mathcal{F}' \\ \mathcal{F} \le^{-} \mathcal{F}' & :\iff & \mathcal{F}' \le \mathcal{F} \\ \mathcal{F} \le^{\top} \mathcal{F}' & :\iff & \text{true.} \end{array}$$

For the special case of posets, $\le^{\circ}$ coincides with equality, but we will later encounter preordered sets, where $\le^{\circ}$ is just an equivalence relation and not identity.

Lemma 10 (Soundness of variance ordering)**.** *If $\pi \le \pi'$ and $\mathcal{F} \le^{\pi} \mathcal{F}'$ then $\mathcal{F} \le^{\pi'} \mathcal{F}'$.*

If $\mathcal{K} \in \mathsf{KI}(\iota)$ and $\mathcal{K}' \in \mathsf{KI}(\iota')$ is a pointed preorder then the function space

$$\begin{array}{l} \mathcal{K} \to \mathcal{K}' \in \mathsf{KI}(\iota \to \iota') \\ \mathcal{K} \to \mathcal{K}' = \{\mathcal{F} \in [\![\iota]\!] \to [\![\iota']\!] \mid \forall \mathcal{G} \in \mathcal{K}.\ \mathcal{F}(\mathcal{G}) \in \mathcal{K}'\} \end{array}$$

is a pointed preorder with least element $\bot_{\mathcal{K}\to\mathcal{K}'}(\mathcal{G}) = \bot_{\mathcal{K}'}$, pointwise ordered by $\mathcal{F} \le \mathcal{F}' \in \mathcal{K} \to \mathcal{K}'$ iff $\mathcal{F}(\mathcal{G}) \le \mathcal{F}'(\mathcal{G}) \in \mathcal{K}'$ for all $\mathcal{G} \in \mathcal{K}$.

For posets $\mathcal{K}, \mathcal{K}'$ let $\mathcal{K} \xrightarrow{\circ} \mathcal{K}'$ be just $\mathcal{K} \to \mathcal{K}'$, the full function space, $\mathcal{K} \xrightarrow{+} \mathcal{K}'$ denote the subspace of monotone functions, $\mathcal{K} \xrightarrow{-} \mathcal{K}'$ the antitone ones and $\mathcal{K} \xrightarrow{\top} \mathcal{K}'$ the constant functions. Clearly, if $\mathcal{F} \in \mathcal{K} \xrightarrow{\pi} \mathcal{K}'$ and $\mathcal{G} \le^{\pi} \mathcal{G}' \in \mathcal{K}$, then $\mathcal{F}(\mathcal{G}) \le \mathcal{F}(\mathcal{G}') \in \mathcal{K}'$.

Let $(<\beta) = \{\alpha \mid \alpha < \beta\}$. We define the type of semantic kinds $\mathsf{KI}(\iota)$ associated to simple kind ι inductively by the rules

$$\frac{}{\mathsf{CR} \in \mathsf{KI}(*)} \qquad \frac{\beta \in \mathsf{O}}{(<\beta) \in \mathsf{KI}(o)} \qquad \frac{\mathcal{K} \in \mathsf{KI}(\iota) \qquad \mathcal{K}' \in \mathsf{KI}(\iota')}{\mathcal{K} \xrightarrow{\pi} \mathcal{K}' \in \mathsf{KI}(\iota \to \iota')}.$$

Note that $\bot_{\mathsf{CR}} = \overline{\emptyset}$ and $\bot_{\mathsf{O}} = 0$.

Type environments. We extend the kind erasure $|\kappa| = \iota$ to kinding contexts Δ in the obvious way: $|\cdot| = \cdot$ and $|\Delta, X{:}\pi\kappa| = |\Delta|, X{:}|\kappa|$. Erased kinding contexts are interpreted as sets of environments $\rho \in [\![|\Delta|]\!]$ inductively defined by

$$\frac{}{\cdot \in [\![\cdot]\!]} \qquad \frac{\rho \in [\![|\Delta|]\!] \qquad \mathcal{G} \in [\![\iota]\!]}{(\rho, \mathcal{G}/X) \in [\![|\Delta|, X{:}\iota]\!]}.$$

Environments ρ can be understood as finite maps from type constructor variables X to an appropriate semantic object $\mathcal{G} \in [\![|\Delta(X)|]\!]$ (an ordinal, a semantic type or a type operator). We will also use the notation ρ, ρ' for environment concatenation.

Semantic kinding contexts. In the following, we define semantic kinding contexts $\mathcal{D} \in \mathsf{KICXT}(|\Delta|)$ as counterparts of syntactic kinding contexts Δ. Each $\mathcal{D}$ induces a preordered subset $[\mathcal{D}] \subseteq [\![|\Delta|]\!]$ of (semantic) type environments $\rho \in [\mathcal{D}]$ (written just $\rho \in \mathcal{D}$). Analogously to syntactic contexts, semantic kinding contexts are finite maps from type constructor variables X to a pair of variance π and semantic kind $\mathcal{K}$ which may depend on "earlier variables" of mixed variance only. This dependency is expressed by $\mathcal{D}' \in \circ^{-1}\mathcal{D} \to \mathsf{KICXT}(|\Delta'|)$ in the rule for $\Sigma_{\mathcal{D}}\mathcal{D}' \in \mathsf{KICXT}(|\Delta, \Delta'|)$ below. It means that $\mathcal{D}'$ respects equivalence in $\mathcal{D}$ given by $\rho \le^{\circ} \rho' \in \mathcal{D}$, meaning that then $\mathcal{D}'(\rho) = \mathcal{D}'(\rho')$. Semantic kinding contexts $\boxed{\mathcal{D} \in \mathsf{KICXT}(|\Delta|)}$ are defined inductively by the rules

$$\frac{}{\cdot \in \mathsf{KICXT}(\cdot)} \qquad \frac{\mathcal{K} \in \mathsf{KI}(\iota)}{(X{:}\pi\mathcal{K}) \in \mathsf{KICXT}(X{:}\iota)}$$

$$\frac{\mathcal{D} \in \mathsf{KICXT}(|\Delta|) \qquad \mathcal{D}' \in \circ^{-1}\mathcal{D} \to \mathsf{KICXT}(|\Delta'|)}{\Sigma_{\mathcal{D}}\mathcal{D}' \in \mathsf{KICXT}(|\Delta, \Delta'|)}.$$

Simultaneously with $\mathcal{D} \in \mathsf{KICXT}(|\Delta|)$, we construct a preordered set of type environments; we define $\boxed{\rho \le \rho' \in \mathcal{D}}$ by recursion on $\mathcal{D} \in \mathsf{KICXT}(|\Delta|)$—an inductive-recursive definition (Dybjer 2000).

$$\begin{array}{ccccccl} \cdot & \le & \cdot & \in & \cdot & :\iff & \text{true} \\ (\mathcal{G}/X) & \le & (\mathcal{G}'/X) & \in & (X{:}\pi\mathcal{K}) & :\iff & \mathcal{G} \le^{\pi} \mathcal{G}' \in \mathcal{K} \\ (\rho_1, \rho_2) & \le & (\rho_1', \rho_2') & \in & \Sigma_{\mathcal{D}}\mathcal{D}' & :\iff & \rho_1 \le \rho_1' \in \mathcal{D} \\ & & & & & & \text{and } \rho_2 \le \rho_2' \in \mathcal{D}'(\rho_1) \end{array}$$

The last line shows that it is important that $\mathcal{D}'$ respects $\mathcal{D}$, because how would we otherwise know that $\mathcal{D}'(\rho_1) = \mathcal{D}'(\rho_1')$, and thus $\rho_2' \in \mathcal{D}'(\rho_1)$?

Lemma 11 (Well-definedness of partial order on type environments)**.** *If $\mathcal{D} \in \mathsf{KICXT}(|\Delta|)$ then $\rho \le \rho' \in \mathcal{D}$ is well-defined. Further, if $\rho \le \rho' \in \mathcal{D}$ then $\rho \le \rho' \in \circ^{-1}\mathcal{D}$ and even $\rho \le^{\circ} \rho' \in \circ^{-1}\mathcal{D}$.*

Proof. By induction on $\mathcal{D} \in \mathsf{KICXT}(|\Delta|)$. It is even true that $\rho \le \rho' \in \pi_1^{-1}\mathcal{D}$ implies $\rho \le \rho' \in \pi_2^{-1}\mathcal{D}$ for any $\pi_1 \ge \pi_2$. Instantiating this with $\pi_1 = +$ and $\pi_2 = \circ$ yields the first statement on orders; for the second we observe that $\rho \le^{\pi} \rho' \in \mathcal{D}$ iff $\rho \le \rho' \in \pi\mathcal{D}$ and that $\circ\circ^{-1}\mathcal{D} = \circ^{-1}\mathcal{D}$. Well-definedness follows in case $\Sigma_{\mathcal{D}}\mathcal{D}'$ since $\rho_1 \leq \rho_1' \in \mathcal{D}$ implies $\rho_1 \leq^\circ \rho_1' \in \circ^{-1}\mathcal{D}$ and thus $\mathcal{D}'(\rho_1) = \mathcal{D}'(\rho_1')$. ☐

$\boxed{\rho \in \mathcal{D}}$ is now simply defined as $\rho \leq \rho \in \mathcal{D}$. If in singleton contexts $(X{:}\pi\mathcal{K})$ we restrict $\mathcal{K}$ to be of the form $<\beta$, we obtain *semantic size contexts* $\boxed{\mathcal{D} \in \mathsf{SICXT}(|\Psi|)}$. Clearly, these are special semantic kinding contexts.

We shall omit $|\Delta|$ from $\mathsf{KICXT}(|\Delta|)$ when inessential or inferable. We drop singleton function domains, e.g., we identify $\cdot \to \mathsf{KICXT}$ with KICXT. Given a parametrized kinding context $\mathcal{D}_2 \in \mathcal{D} \to \mathsf{KICXT}$ we can weaken it to $\mathsf{W}_{\mathcal{D}_1}\mathcal{D}_2 \in (\mathcal{D}_1 \times \mathcal{D}) \to \mathsf{KICXT}$ where $(\mathsf{W}_{\mathcal{D}_1}\mathcal{D}_2)(\rho_1 \in \mathcal{D}_1, \rho) = \mathcal{D}_2(\rho)$. For non-dependent concatenation of semantic kinding contexts $\mathcal{D}_1$ and $\mathcal{D}_2$ we introduce the notation $(\mathcal{D}_1, \mathcal{D}_2)$ defined as $\Sigma_{\mathcal{D}_1}(\mathsf{W}_{\mathcal{D}_1}\mathcal{D}_2)$. As a derived rule we have:

$$\frac{\mathcal{D}_1 \in \mathsf{KICXT}|\Delta_1| \qquad \mathcal{D}_2 \in \mathsf{KICXT}|\Delta_2|}{(\mathcal{D}_1, \mathcal{D}_2) \in \mathsf{KICXT}|\Delta_1, \Delta_2|}$$

Lemma 12 (Preservation of context well-formedness)**.** *If* $\mathcal{D} \in \mathsf{KICXT}(|\Delta|)$ *then* $\pi^{-1}\mathcal{D} \in \mathsf{KICXT}(|\Delta|)$.

Proof. By induction on $\mathcal{D} \in \mathsf{KICXT}(|\Delta|)$. The interesting case is concatenation:

$$\frac{\mathcal{D} \in \mathsf{KICXT}(|\Delta|) \qquad \mathcal{D}' \in \circ^{-1}\mathcal{D} \to \mathsf{KICXT}(|\Delta'|)}{\Sigma_{\mathcal{D}}\mathcal{D}' \in \mathsf{KICXT}(|\Delta, \Delta'|)}$$

By induction hypothesis $\pi^{-1}\mathcal{D} \in \mathsf{KICXT}(|\Delta|)$ and $\pi^{-1}\mathcal{D}'(\rho) \in \mathsf{KICXT}(|\Delta'|)$ for all $\rho \in \circ^{-1}\mathcal{D}$. It remains to show that $\pi^{-1}\mathcal{D}'$ respects $\pi^{-1}\circ^{-1}\mathcal{D}$ which is equal to $(\pi\circ)^{-1}\mathcal{D}$. Assume $\rho \leq^\circ \rho' \in (\pi\circ)^{-1}\mathcal{D}$. Since $\rho \leq^\circ \rho' \in \circ^{-1}\mathcal{D}$ by antitonicity ($\pi\circ \geq \circ$), we have $\mathcal{D}'(\rho) = \mathcal{D}'(\rho')$ and, thus, $\pi^{-1}\mathcal{D}'(\rho) = \pi^{-1}\mathcal{D}'(\rho')$ as desired. ☐

Interpretation of sizes, measures, kinds, and kinding contexts. In the following let $\rho \in [\![|\Delta_0|]\!]$ for some erased kinding context $|\Delta_0|$. (Extended) sizes a^+ are interpreted as ordinals $[\![a^+]\!]_\rho \in \mathsf{O}$ and measures $\mathfrak{m}$ as ordinal tuples $[\![\mathfrak{m}]\!]_\rho \in \mathsf{O}^*$.

$$\begin{array}{rcl} [\![i+n]\!]_\rho &=& [\![i]\!]_\rho + n \\ [\![\infty+n]\!]_\rho &=& \boldsymbol{\infty} + n \\ [\![n]\!]_\rho &=& n \\ [\![a^+, \mathfrak{m}]\!]_\rho &=& ([\![a^+]\!]_\rho, [\![\mathfrak{m}]\!]_\rho) \end{array}$$

Kinds κ are interpreted as semantic kinds $[\![\kappa]\!]_\rho \in \mathsf{KI}(|\kappa|)$ and kinding contexts Δ as semantic kinding contexts $[\![\Delta]\!]_\rho \in \mathsf{KICXT}(|\Delta|)$.

$$\begin{array}{rcl} [\![*]\!]_\rho &=& \mathsf{CR} \\ [\![<b]\!]_\rho &=& <[\![b]\!]_\rho \\ [\![\pi\kappa \to \kappa']\!]_\rho &=& [\![\kappa]\!]_\rho \xrightarrow{\pi} [\![\kappa']\!]_\rho \\ [\![\cdot]\!]_\rho &=& \cdot \\ [\![\Delta, X{:}\pi\kappa]\!]_\rho &=& \Sigma_{\mathcal{D}}(X{:}\pi\mathcal{K}) \\ \text{where } \mathcal{D} && = [\![\Delta]\!]_\rho \\ \text{and } \quad \mathcal{K}(\rho' \in \mathcal{D}) = [\![\kappa]\!]_{(\rho,\rho')} && \end{array}$$

The structurally recursive interpretation $[\![O]\!]_\rho$ for a kind-level object $O ::= a^+ \mid \mathfrak{m} \mid \kappa$ as given above is well-defined if $\rho(i) \in \mathsf{O}$ for all $i \in \mathsf{FV}(O)$. In the following, we show that the interpretations fit into the appropriate semantic concepts.

Lemma 13 (Soundness of size (context) formation)**.** *Let* $\vdash \Psi$.

1. *Then* $[\![\Psi]\!] \in \mathsf{SICXT}(|\Psi|)$.
2. *If* $\Psi \vdash a$ *then* $[\![a]\!] \in [\![\Psi]\!] \xrightarrow{+} \mathsf{O}$.
3. *If* $\Psi \vdash i < a$ *and* $\rho \leq \rho' \in [\![\Psi]\!]$ *then* $[\![a]\!]_\rho \leq [\![a]\!]_{\rho'} \in \mathsf{O}$ *and* $\rho(i) \leq \rho'(i) < [\![a]\!]_\rho$.

Proof. By induction on the length of context Ψ. We demonstrate the case for context extension.

$$\frac{\vdash \Psi \qquad \circ^{-1}\Psi \vdash a}{\vdash \Psi, i{:}\pi(<a)}$$

By induction hypothesis 1, $\mathcal{D} := [\![\Psi]\!] \in \mathsf{SICXT}(|\Psi|)$. By induction hypothesis 2, $[\![a]\!] \in [\![\circ^{-1}\Psi]\!] \to \mathsf{O}$, thus $[\![a]\!] \in \circ^{-1}\mathcal{D} \to \mathsf{O}$, entailing respect, and $\Sigma_{\mathcal{D}}(i{:}\pi(<[\![a]\!])) \in \mathsf{SICXT}(|\Psi|, i{:}o)$. ☐

Theorem 14 (Soundness of kind-level judgements)**.** *Let* $\vdash \Psi$ *and let* $\rho \leq \rho' \in \mathcal{D} := [\![\Psi]\!]$.

1. *If* $\Psi \vdash a^+$ *then* $[\![a^+]\!]_\rho \leq [\![a^+]\!]_{\rho'} \in \mathsf{O}$.
2. *If* $\Psi \vdash a^+ \leq b^+$ *then* $[\![a^+]\!]_\rho \leq [\![b^+]\!]_{\rho'} \in \mathsf{O}$.
3. *If* $\Psi \vdash a^+ < b^+$ *then* $[\![a^+]\!]_\rho < [\![b^+]\!]_{\rho'} \in \mathsf{O}$.
4. *If* $\Psi \vdash_m \mathfrak{m}$ *then* $[\![\mathfrak{m}]\!]_\rho \leq [\![\mathfrak{m}]\!]_{\rho'} \in \mathsf{O}^m$.
5. *If* $\Psi \vdash \mathfrak{m} \leq \mathfrak{m}'$ *then* $[\![\mathfrak{m}]\!]_\rho \leq [\![\mathfrak{m}]\!]_{\rho'} \in \mathsf{O}^*$.
6. *If* $\Psi \vdash \mathfrak{m} < \mathfrak{m}'$ *then* $[\![\mathfrak{m}]\!]_\rho < [\![\mathfrak{m}]\!]_{\rho'} \in \mathsf{O}^*$.
7. *If* $\Psi \vdash \kappa$ *then* $[\![\kappa]\!]_\rho \leq [\![\kappa]\!]_{\rho'} \in \mathsf{KI}(|\kappa|)$.
8. *If* $\Psi \vdash \kappa \leq \kappa'$ *then* $|\kappa| = |\kappa'|$ *and* $[\![\kappa]\!]_\rho \leq [\![\kappa]\!]_{\rho'} \in \mathsf{KI}(|\kappa|)$.

Proof. Each by induction on the derivation. ☐

The following theorem is the reason that we do not allow finitely bounded size variables $i < n$ in kinding contexts.

Theorem 15 (Context satisfaction)**.** *If* $\vdash \Delta$ *then there is some* $\rho_0 \in [\![\Delta]\!]$.

Proof. We prove the following stronger theorem by induction on Δ: For each $\alpha < \boldsymbol{\infty}$ there is some $\rho \in [\![\Delta]\!]$ such that $\rho(i) \geq \alpha$ for each size variable i declared in Δ.

Case

$$\frac{\vdash \Delta \qquad \circ^{-1}\Delta \vdash <(\infty+n)}{\vdash \Delta,\ i : \pi(<(\infty+n))}$$

By induction hypothesis there is some $\rho \in [\![\Delta]\!]$, thus, $\rho[i \mapsto \alpha]$ is the desired environment.

Case

$$\frac{\vdash \Delta \qquad \circ^{-1}\Delta \vdash <(j+n)}{\vdash \Delta,\ i : \pi(<(j+n))}$$

By induction hypothesis there is some $\rho \in [\![\Delta]\!]$ with $\rho(j) \geq \alpha + 1$, thus, $\alpha < \rho(j) + n$ and $\rho[i \mapsto \alpha]$ is the desired environment.

Case

$$\frac{\vdash \Delta \qquad \circ^{-1}\Delta \vdash \kappa}{\vdash \Delta, X{:}\pi\kappa}\ \kappa \neq (<a)$$

Return $\rho[X \mapsto \bot_{[\![\kappa]\!]\rho}]$ where ρ is obtained by induction hypothesis. ☐

Theorem 16 (Conditional context satisfaction)**.**

1. *If* $\Psi \vdash \exists\Psi'$ *and* $\rho \in [\![\Psi]\!]$ *then there is some* $\rho' \in [\![\Psi']\!]_\rho$.
2. *If* $\Delta \vdash \exists\Delta'$ *and* $\rho \in [\![\Delta]\!]$ *then there is some* $\rho' \in [\![\Delta']\!]_\rho$.

4.4 Type Constructors

In order to interpret type constructors semantically, we need to restrict to well-kinded ones. However, we do not wish to define the semantics of a type constructor by recursion on its kinding derivation. After all, since we have subkinding, the kinding derivation is not unique. This dilemma can be solved by interpreting all type constructors which have a *simple kind*. Using simple kind annotations in type function $\lambda X{:}\iota.\,F$, we obtain a deterministic simple kinding judgement $\boxed{|\Delta| \vdash F : \iota}$. By induction on this judgement, whose derivation is in one-to-one correspondence with F, we can then define type (constructor) interpretation $[\![F]\!]_\rho \in [\![\iota]\!]$ for $\rho \in [\![|\Delta|]\!]$.

Simple kinding is standard, we only present some of the rules to convey the idea. Here, $|\Delta|$ shall denote a simple kinding context.

$$\frac{}{|\Delta| \vdash X : |\Delta|(X)} \qquad \frac{|\Delta|, X{:}\iota \vdash F : \iota'}{|\Delta| \vdash \lambda X{:}\iota.\, F : \iota \to \iota'}$$

$$\frac{|\Delta| \vdash F : \iota \to \iota' \qquad |\Delta| \vdash G : \iota}{|\Delta| \vdash F\,G : \iota'}$$

$$\frac{}{|\Delta| \vdash \forall_\kappa : (|\kappa| \to *) \to *}$$

Simple kinding is unique, so we have a partial computable function taking a simple kinding context $|\Delta|$ and a type constructor F and computing its simple kind ι, if it exists.

Now given a derivation $J :: |\Delta| \vdash F : \iota$ and an environment $\rho \in [\![|\Delta|]\!]$ we define the type interpretation $[\![J]\!]_\rho \in [\![|\iota|]\!]$ by recursion on J. Since J is completely determined by F and $|\Delta|$, we simply write $[\![F]\!]_\rho$, hiding $|\Delta|$ as it is implicit in the typing of ρ.

$$\begin{array}{lcl}
[\![X]\!]_\rho & = & \rho(X) \\
[\![\lambda X{:}\iota.\, F]\!]_\rho(\mathcal{G} \in [\![\iota]\!]) & = & [\![F]\!]_{\rho[X\mapsto\mathcal{G}]} \\
[\![F\,G]\!]_\rho & = & [\![F]\!]_\rho([\![G]\!]_\rho) \\
[\![1]\!]_\rho & = & \mathbf{1} \\
[\![\times]\!]_\rho(\mathcal{A},\mathcal{B}) & = & \mathcal{A} \boldsymbol{\times} \mathcal{B} \\
[\![\to]\!]_\rho(\mathcal{A},\mathcal{B}) & = & \mathcal{A} \boldsymbol{\to} \mathcal{B} \\
[\![\forall_\kappa]\!]_\rho(\mathcal{F} \in [\![|\kappa|]\!] \to \mathsf{CR}) & = & \boldsymbol{\forall}_{[\![\kappa]\!]_\rho} \mathcal{F} \\
[\![\exists_\kappa]\!]_\rho(\mathcal{F} \in [\![\kappa|]\!] \to \mathsf{CR}) & = & \boldsymbol{\exists}_{[\![\kappa]\!]_\rho} \mathcal{F} \\
[\![\mu^a S]\!]_\rho & = & \boldsymbol{\mu}^{[\![a]\!]_\rho} [\![S]\!]_\rho \\
[\![\nu^a R]\!]_\rho & = & \boldsymbol{\nu}^{[\![a]\!]_\rho} [\![R]\!]_\rho \\
 & & \\
([\![S]\!]_\rho)_c & = & [\![S_c]\!]_\rho \\
([\![R]\!]_\rho)_d & = & [\![R_d]\!]_\rho
\end{array}$$

The interpretation of F depends only on the value of ρ for the free variables of F:

Lemma 17 (Well-definedness)**.** *If $|\Delta| \vdash F : \iota$ then $(|\Delta| \upharpoonright \mathsf{FV}(F)) \vdash F : \iota$. If $\rho \in [\![|\Delta|]\!]$ and $\rho' = (\rho \upharpoonright \mathsf{FV}(F))$ then $[\![F]\!]_\rho = [\![F]\!]_{\rho'}$.*

Theorem 18 (Soundness of type-level judgements)**.** *Let $\vdash \Delta$ and $\mathcal{D} := [\![\Delta]\!]$ and $\rho \le \rho' \in \mathcal{D}$.*

1. *If $\Delta \vdash F \rightrightarrows \kappa$ or $\Delta \vdash F \leftleftarrows \kappa$ then $|\Delta| \vdash F : |\kappa|$ and $[\![F]\!]_\rho \le [\![F]\!]_{\rho'} \in [\![\kappa]\!]_\rho$.*
2. *If $\Delta \vdash F \le^\pi F' \rightrightarrows \kappa$ or $\Delta \vdash F \le^\pi F' \leftleftarrows \kappa$ then $|\Delta| \vdash F, F' : |\kappa|$ and $[\![F]\!]_\rho \le^\pi [\![F']\!]_{\rho'} \in [\![\kappa]\!]_\rho$.*

Proof. By induction on the derivation. □

Lemma 19 (Soundness of normalizing substitution and application)**.** *Let $|\Delta| \vdash G : \iota_1$.*

1. *If $|\Delta|, X{:}\iota_1 \vdash F : \iota_2$ then $[\![[G/X]^{\iota_1} F]\!]_\rho = [\![F]\!]_{\rho[X\mapsto[\![G]\!]_\rho]}$.*
2. *If $|\Delta| \vdash F : \iota_1 \to \iota_2$ then $[\![F \mathbin{@}^{\iota_1} G]\!]_\rho = [\![F]\!]_\rho([\![G]\!]_\rho)$.*

Lemma 20 (Soundness of substitution)**.** *If $|\Delta'| \vdash F : \iota$ and $|\Delta| \vdash \tau : |\Delta'|$ then $[\![F\tau]\!]_\rho = [\![F]\!]_{[\![\tau]\!]_\rho}$.*

The interpretation can be extended to constraint types ${}^?\!A$ by adding the case:

$$[\![\mathfrak{m}{<}\mathfrak{m}' \Rightarrow A]\!]_\rho \quad = \quad [\![\mathfrak{m}]\!]_\rho {<} [\![\mathfrak{m}']\!]_\rho \Rightarrow [\![A]\!]_\rho$$

4.5 Patterns, copatterns, λ-abstractions

In this section, we explain patterns and copatterns by developing semantic notions of pattern and pattern spine typing. These provide us with semantic conditions when a definition $\lambda\vec{D}$ inhabits a semantic type $\mathcal{A}$. As a consequence, we can prove soundness of syntactic pattern, pattern spine, and expression typing.

Semantic typing. We want to isolate conditions under which objects $\lambda\vec{D}$ are member or a semantic type $\mathcal{A} \in \mathsf{CR}$. Let us recapitulate the proof for lambda calculus:

Lemma 21 (Lambda abstraction)**.** *The following implication, written as a rule, holds for $\mathcal{A},\mathcal{B} \in \mathsf{CR}$.*

$$\frac{\forall s \in \mathcal{A}.\, t[s/x] \in \mathcal{B}}{\lambda x.t \in \mathcal{A} \boldsymbol{\to} \mathcal{B}}$$

Proof. First note that $t \in \mathcal{B}$ because $x \in \mathcal{A}$, so $t \in \mathsf{SN}$. By definition of $\mathcal{A} \boldsymbol{\to} \mathcal{B}$, it is sufficient to show $(\lambda x.t)\, s$ for arbitrary $s \in \mathcal{A}$. Since $(\lambda x.t)\, s$ is neutral, by **CR3** we only need to show that each of its reducts r is in $\mathcal{B}$. We do this by induction on $t \in \mathsf{SN}$ and $s \in \mathsf{SN}$.

Case $r = (\lambda x.t')\, s$ where $t \longrightarrow t'$: By induction hypothesis on $t' \in \mathsf{SN}$.

Case $r = (\lambda x.t)\, s'$ where $s \longrightarrow s'$: By i.h. on $s' \in \mathsf{SN}$.

Case $r = t[s/x]$: By assumption. □

Next, we turn to the slightly more general case $\lambda\{p \to t\}$.

Semantic typing contexts and semantic pattern typing. A semantic typing context $\mathcal{E} \in \mathsf{CXT}(\cdot)$ ($\mathcal{E}$ for typing *environment*) is a finite map from term variables to semantic types, so $\mathcal{E} \in \mathsf{Var} \rightharpoonup \mathsf{CR}$. We write $\cdot$ for the empty semantic typing context, $x{:}\mathcal{A}$ for the singleton and $\mathcal{E},\mathcal{E}'$ for the disjoint union. *Semantic substitution typing* $\sigma \in \mathcal{E}$ is defined as $\sigma(x) \in \mathcal{E}(x)$ for all $x \in \mathsf{dom}(\mathcal{E})$.

A parameterized semantic typing context $\mathcal{E} \in \mathsf{CXT}(\mathcal{D})$ is a family $\mathcal{E}(\rho)$ of semantic typing contexts indexed by semantic type substitutions ρ that belong to a semantic kinding context $\mathcal{D}$. Each instance $\mathcal{E}(\rho)$ is a partial function from variables to semantic types. We overload the notation for non-parameterized semantic typing contexts by setting $\cdot(\rho) = \cdot$ and $(x{:}\mathcal{A})(\rho) = x{:}\mathcal{A}(\rho)$ and $(\mathcal{E},\mathcal{E}')(\rho) = \mathcal{E}(\rho),\mathcal{E}'(\rho)$ with $\mathsf{dom}(\mathcal{E}(\rho)) \cap \mathsf{dom}(\mathcal{E}'(\rho)) = \emptyset$.

For two differently parameterized semantic typing contexts $\mathcal{E}_1 \in \mathsf{CXT}(\mathcal{D}_1)$ and $\mathcal{E}_2 \in \mathsf{CXT}(\mathcal{D}_2)$ we let their disjoint union $\mathcal{E}_1 * \mathcal{E}_2 \in \mathsf{CXT}(\mathcal{D}_1,\mathcal{D}_2)$ be defined by $(\mathcal{E}_1 * \mathcal{E}_2)(\rho_1 \in \mathcal{D}_1, \rho_2 \in \mathcal{D}_2) = (\mathcal{E}_1(\rho_1), \mathcal{E}_2(\rho_2))$. Further, if $\mathcal{E} \in \mathsf{CXT}(\Sigma_{\mathcal{D}}\mathcal{D}')$ and $\rho \in \mathcal{D}$ we let the partial application $\mathcal{E}(\rho,\_) \in \mathsf{CXT}(\mathcal{D}'(\rho))$ be defined by $\mathcal{E}(\rho,\_)(\rho') = \mathcal{E}(\rho,\rho')$.

If $\mathcal{C}(\mathcal{G})(\rho)$ is a type parameterized by another type $\mathcal{G}$ and a type substitution ρ, we let $\mathcal{C}X$ be defined by $(\mathcal{C}X)(\rho) = \mathcal{C}(\rho(X))(\rho \setminus X)$. In particular, $(\mathcal{C}X)(\mathcal{G}/X,\rho) = \mathcal{C}(\mathcal{G})(\rho)$. The notations $\mathcal{D}X$ and $\mathcal{E}X$ are defined analogously.

A pattern p is semantically of type $\mathcal{A}$ in context $\mathcal{E}$ if it acts as a bidirectional (invertible) map from $\mathcal{E}$ to $\mathcal{A}$, i. e., $p\sigma \in \mathcal{A}$ for all $\sigma \in \mathcal{E}$, and, for any substitution σ with $p\sigma \in \mathcal{A}$ we have $\sigma \in \mathcal{E}$. Extending this to type substitutions we define *semantic pattern typing* by

$$\boxed{\mathcal{A} \,/\, p \searrow \mathcal{D};\mathcal{E}} \;:\Longleftrightarrow\; \forall \tau,\sigma.\, (\exists \rho \in \mathcal{D}.\, \sigma \in \mathcal{E}(\rho)) \iff p\tau\sigma \in \mathcal{A}.$$

Here, and in the following, τ denotes a syntactic type substitution. Note that it is unconstrained, it needs not bear a relationship with the semantic type substitution ρ.

One could have expected that semantic pattern typing implies that p matches any introduction term $v \in \mathcal{A}$. But since we are not interested in pattern coverage, but merely strong normalization, we do not require this strong guarantee.[7]

Lemma 22 (Semantic pattern typing). *The following implications, written as rules, hold.*

$$\frac{}{\mathcal{A} \,/\, x \searrow \cdot; (x{:}\mathcal{A})} \qquad \frac{}{\mathbf{1} \,/\, () \searrow \cdot; \cdot}$$

$$\frac{\mathcal{A}_1 \,/\, p_1 \searrow \mathcal{D}_1; \mathcal{E}_1 \qquad \mathcal{A}_2 \,/\, p_2 \searrow \mathcal{D}_2; \mathcal{E}_2}{\mathcal{A}_1 \times \mathcal{A}_2 \,/\, (p_1, p_2) \searrow \mathcal{D}_1, \mathcal{D}_2; \mathcal{E}_1 * \mathcal{E}_2}$$

$$\frac{\exists_{\beta<\alpha^\uparrow} \mathcal{S}_c(\boldsymbol{\mu}^\beta \mathcal{S}) \,/\, p \searrow \mathcal{D}; \mathcal{E}}{\boldsymbol{\mu}^\alpha \mathcal{S} \,/\, c\,p \searrow \mathcal{D}; \mathcal{E}}$$

$$\frac{\mathcal{F}(\mathcal{G}) \,/\, p \searrow \mathcal{D}(\mathcal{G}); \mathcal{E}(\mathcal{G}) \text{ for all } \mathcal{G} \in \mathcal{K}}{\exists_{\mathcal{K}} \mathcal{F} \,/\, {}^X p \searrow \Sigma_{X:\mathcal{K}} \mathcal{D}; \mathcal{E} X}$$

Proof. We consider a few rules.

$$\frac{\exists_{\beta<\alpha^\uparrow} \mathcal{S}_c(\boldsymbol{\mu}^\beta \mathcal{S}) \,/\, p \searrow \mathcal{D}; \mathcal{E}}{\boldsymbol{\mu}^\alpha \mathcal{S} \,/\, c\,p \searrow \mathcal{D}; \mathcal{E}}$$

For $\boldsymbol{\mu}^\alpha \mathcal{S} \,/\, c\,p \searrow \mathcal{D}; \mathcal{E}$, first assume $(c\,p)\tau\sigma \in \boldsymbol{\mu}^\alpha \mathcal{S}$ and derive $\sigma \in \mathcal{E}(\rho)$ for some $\rho \in \mathcal{D}$. Note that $\boldsymbol{\mu}^\alpha \mathcal{S} = \boldsymbol{\mu}^{\alpha^\uparrow} S$, thus, by definition, $p\tau\sigma \in \exists_{\beta<\alpha^\uparrow} \mathcal{S}_c(\boldsymbol{\mu}^\beta \mathcal{S})$. Using the assumption $\exists_{\beta<\alpha^\uparrow} \mathcal{S}_c(\boldsymbol{\mu}^\beta \mathcal{S}) \,/\, p \searrow \mathcal{D}; \mathcal{E}$, we conclude $\sigma \in \mathcal{E}(\rho)$ for some $\rho \in \mathcal{D}$. For the opposite direction, assume $\rho \in \mathcal{D}$ and $\sigma \in \mathcal{E}(\rho)$. By the hypothesis, $p\tau\sigma \in \exists_{\beta<\alpha^\uparrow} \mathcal{S}_c(\boldsymbol{\mu}^\beta \mathcal{S})$, hence $(c\,p)\tau\sigma \in \boldsymbol{\mu}^\alpha \mathcal{S}$.

$$\frac{\mathcal{F}(\mathcal{G}) \,/\, p \searrow \mathcal{D}(\mathcal{G}); \mathcal{E}(\mathcal{G}) \text{ for all } \mathcal{G} \in \mathcal{K}}{\exists_{\mathcal{K}} \mathcal{F} \,/\, {}^X p \searrow \Sigma_{X:\mathcal{K}} \mathcal{D}; \mathcal{E} X}$$

To prove $\exists_{\mathcal{K}} \mathcal{F} \,/\, {}^X p \searrow \Sigma_{X:\mathcal{K}} \mathcal{D}; \mathcal{E} X$, first assume $({}^X p)\tau\sigma \in \exists_{\mathcal{K}} \mathcal{F}$ and show $\sigma \in (\mathcal{E} X)(\mathcal{G}/X, \rho) = \mathcal{E}(\mathcal{G})(\rho)$ for some $\mathcal{G} \in \mathcal{K}$ and $\rho \in \mathcal{D}(\mathcal{G})$. Since $p\tau\sigma \in \mathcal{F}(\mathcal{G})$ for some $\mathcal{G} \in \mathcal{K}$, we can apply the hypothesis to obtain $\sigma \in \mathcal{E}(\mathcal{G})(\rho)$ for some $\rho \in \mathcal{D}(\mathcal{G})$. For the other direction, assume $\mathcal{G} \in \mathcal{K}$ and $\rho \in \mathcal{D}(\mathcal{G})$ and $\sigma \in (\mathcal{E} X)(\mathcal{G}/X, \rho)$ and show $({}^X p)\tau\sigma \in \exists_{\mathcal{K}} \mathcal{F}$. Since $\sigma \in \mathcal{E}(\mathcal{G})(\rho)$, by hypothesis, $p\tau\sigma \in \mathcal{F}(\mathcal{G})$, yielding $({}^X p)\tau\sigma \in \exists_{\mathcal{K}} \mathcal{F}$.

$$\frac{\mathcal{F}(\infty) \,/\, p \searrow \mathcal{D}; \mathcal{E}}{\exists_\beta \mathcal{F}(\beta) \,/\, {}^\infty p \searrow \mathcal{D}; \mathcal{E}}\ \mathcal{F}\text{ mon.}$$

First, assume $({}^\infty p)\tau\sigma \in \exists_\beta \mathcal{F}(\beta)$ which entails $p\tau\sigma \in \mathcal{F}(\beta)$ for some $\beta \le \infty$. Since $\mathcal{F}$ is monotone, thus, $p\tau\sigma \in \mathcal{F}(\infty)$, we can apply the hypothesis to infer $\sigma \in \mathcal{E}(\rho)$ for some $\rho \in \mathcal{D}$.

For the other direction, assume $\rho \in \mathcal{D}$ and $\sigma \in \mathcal{E}(\rho)$. By the hypothesis, $p\tau\sigma \in \mathcal{F}(\infty)$, thus, $({}^\infty p)\tau\sigma \in \exists_\beta \mathcal{F}(\beta)$. □

Theorem 23 (Soundness of pattern typing). *Let* $\vdash \Delta_0, \Delta$ *and* $\Delta_0, \Delta \vdash \Gamma$. *If* $\Delta; \Gamma \vdash_{\Delta_0} p \Leftarrow A$ *and* $\rho_0 \in [\![\Delta_0]\!]$ *then* $[\![A]\!]_{\rho_0} \,/\, p \searrow [\![\Delta]\!]_{\rho_0}; [\![\Gamma]\!]_{(\rho_0, \_)}$.

Proof. By induction on $\Delta; \Gamma \vdash_{\Delta_0} p \Leftarrow A$ using the inferences of Lemma 22. □

Lemma 24 (Single pattern abstraction). *Let* $\mathcal{E} \in \mathsf{Var} \to \mathsf{CR}$ *and* $\mathcal{B} \in \mathsf{CR}$.

$$\frac{\mathcal{A} \,/\, p \searrow \mathcal{D}; \mathcal{E} \qquad \forall \rho \in \mathcal{D}, \sigma \in \mathcal{E}(\rho).\, t\tau\sigma \in \mathcal{B}}{\lambda\{p \to t\} \in \mathcal{A} \to \mathcal{B}}$$

Proof. Assume $s \in \mathcal{A}$ and show $\lambda\{p \to t\}\, s \longrightarrow r$ implies $r \in \mathcal{B}$. The interesting case is $s \,/\, p \searrow \tau; \sigma$ and $r = t\tau\sigma$. Since $s = p\tau\sigma \in \mathcal{A}$, we have $\sigma \in \mathcal{E}(\rho)$ for some $\rho \in \mathcal{D}$, hence $r \in \mathcal{B}$ by assumption. □ Example: If $\mathcal{C} \in [\![*]\!]$ and $k \in \forall_{\mathcal{X} \in [\![*]\!]} (\mathcal{X} \to \mathcal{C})$ then $\lambda\{{}^X x \mapsto k\, X\, x\} \in (\exists_{\mathcal{X} \in [\![*]\!]} \mathcal{X}) \to \mathcal{C}$.

Lemma 25 (Case). *Let* $p_{1..n}$ *be patterns (not necessarily disjoint),* $\mathcal{E}_k \in \mathsf{Var} \to \mathsf{CR}$ *for* $k = 1..n$ *and* $\mathcal{B} \in \mathsf{CR}$.

$$\frac{\forall k:\ \mathcal{A} \,/\, p_k \searrow \mathcal{D}_k; \mathcal{E}_k \text{ and } \forall \rho \in \mathcal{D}_k, \sigma \in \mathcal{E}_k.\, t_k\sigma \in \mathcal{B}}{\lambda\{p_1 \to t_1, \dots, p_n \to t_n\} \in \mathcal{A} \to \mathcal{B}}$$

Proof. Assume $s \in \mathcal{A}$ and $r := \lambda\{p_1 \to t_1, \dots, p_n \to t_n\}\, s$. Since r is neutral it is sufficient to show $r \longrightarrow r'$ implies $r' \in \mathcal{B}$. We proceed by induction on $\vec{t}, s \in \mathsf{SN}$. If s matches none of $\vec{p}$, the only redexes are in $\vec{t}, s$. The interesting case is $s \,/\, p_k \searrow \tau; \sigma$ and $r' = t_k\tau\sigma$ for some (not necessarily unique) k. Since $s = p_k\tau\sigma \in \mathcal{A}$, we have $\sigma \in \mathcal{E}_k(\rho)$ for some $\rho \in \mathcal{D}_k$, hence $r' \in \mathcal{B}$ by assumption. □

Lemma 26 (Single destructor pattern).

$$\frac{t \in \forall_{\beta<\alpha^\uparrow} \mathcal{R}_d(\boldsymbol{\nu}^\beta \mathcal{R})}{\lambda\{.d \to t\} \in \boldsymbol{\nu}^\alpha \mathcal{R}}$$

Proof. It is sufficient to show $\lambda\{.d \to t\}.d' \in \forall_{\beta<\alpha^\uparrow} \mathcal{R}_{d'}(\boldsymbol{\nu}^\beta \mathcal{R})$ for all $d' \in \mathsf{dom}(\mathcal{R})$, by analyzing the reducts of this neutral term. If $d' \neq d$ the redex is stuck, only reductions in t are possible which are covered by $t \in \mathsf{SN}$. Otherwise, $\lambda\{.d \to t\}.d \longrightarrow t$ and we conclude by the assumption. □

Lemma 27 (Records). *Let* $d_{1..n}$ *be projections (not necessarily distinct ones).*

$$\frac{\text{for all } k = 1..n:\ \ t_k \in \forall_{\beta<\alpha^\uparrow} \mathcal{R}_{d_k}(\boldsymbol{\nu}^\beta \mathcal{R})}{\lambda\{.d_1 \to t_1, \dots, .d_n \to t_n\} \in \boldsymbol{\nu}^\alpha \mathcal{R}}$$

Proof. Assume an arbitrary $d \in \mathsf{dom}(\mathcal{R})$ and let $r := \lambda\{.d_1 \to t_1, \dots, .d_n \to t_n\}.d$. We show $r \in \forall_{\beta<\alpha^\uparrow} \mathcal{R}_d(\boldsymbol{\nu}^\beta \mathcal{R})$ by analyzing the reducts of this neutral term. If $d \notin \vec{d}$ the redex is stuck, only reductions in $\vec{t}$ are possible which are covered by $\vec{t} \in \mathsf{SN}$. Otherwise, some t_k is a possible reduct of r and we conclude by the hypothesis for t_k. □

In the following, we work our way up to the general case of multiple clauses with multiple patterns per clause.

Semantic typing in context. Given a parameterized semantic type $\mathcal{C} \in \mathcal{D}' \to \mathsf{CR}$ we define weakening $\mathsf{W}_{\mathcal{D}}\mathcal{C} \in (\mathcal{D}, \mathcal{D}') \to \mathsf{CR}$ of $\mathcal{C}$ by semantic kinding context $\mathcal{D}$ as $(\mathsf{W}_{\mathcal{D}}\mathcal{C})(\rho \in \mathcal{D}, \rho') = C(\rho')$. Given a semantic type family $\mathcal{C} \in (\mathcal{D}, \mathcal{D}') \to \mathsf{CR}$ and a semantic type substitution $\rho \in \mathcal{D}$, we let the partial application $\mathcal{C}(\rho, \_) \in \mathcal{D}' \to \mathsf{CR}$ be defined by $\mathcal{C}(\rho, \_)(\rho') = \mathcal{C}(\rho, \rho')$. Semantic typing under a context is defined by

$$\boxed{\mathcal{D}; \mathcal{E} \vdash t \in \mathcal{C}} \;:\Longleftrightarrow\; \forall \rho \in \mathcal{D}, \sigma \in \mathcal{E}(\rho), \tau.\ t\tau\sigma \in \mathcal{C}(\rho)$$

Lemma 28 (Partial instantiation). *The following implications hold:*

$$\frac{\mathcal{D}, \mathcal{D}'; \mathcal{E} * \mathcal{E}' \vdash t \in \mathcal{C}}{\mathcal{D}'; \mathcal{E}' \vdash t\tau\sigma \in \mathcal{C}(\rho, \_)}\ \rho \in \mathcal{D},\ \sigma \in \mathcal{E}(\rho)$$

$$\frac{\Sigma_{X:\mathcal{K}} \mathcal{D}; \mathcal{E} X \vdash t \in \mathcal{C} X}{\mathcal{D}(\mathcal{G}); \mathcal{E}(\mathcal{G}) \vdash t[G/X] \in \mathcal{C}(\mathcal{G})}\ \mathcal{G} \in \mathcal{K}$$

Let P be a proposition depending on the pattern variables and pattern type variables of a copattern spine $\vec{q}$. We define the following shorthand for the replacement of the pattern variables by expressions obtained from matching $\vec{q}$ against an elimination list $\vec{e}$:

$$\boxed{P[\vec{e}/\vec{q}]} \;:\Longleftrightarrow\; \exists \tau, \sigma.\ \vec{e} \,/\, \vec{q} \searrow \tau; \sigma \ \wedge\ P\tau\sigma$$

[7] On the contrary, we can live with junk introductions in our semantic types. For instance, it would not endanger normalization to throw the empty tuple into each semantic type.

Semantic pattern spines. [8] A pattern spine $\vec{q}$ has to be understood by its purpose, to serve as the lhs of a definition. Semantically, q eliminates type $\mathcal{A}$ into $\mathcal{C}$ at contexts $\mathcal{D};\mathcal{E}$ if any definition $\lambda\{\vec{q} \to t\}$ that can be formed with $\vec{q}$ is in $\mathcal{A}$ as long as the rhs t is in $\mathcal{C}$ under contexts $\mathcal{D};\mathcal{E}$. We further generalize this to partially applied definitions $\lambda\{\vec{q}'\,\vec{q} \to t\}\vec{e}$ where $\vec{e}$ matches $\vec{q}'$. We let

$$\boxed{\mathcal{A} \mid \vec{q} \searrow \mathcal{D};\mathcal{E};\mathcal{C}} \;:\Longleftrightarrow\; \forall t, \vec{e} \in \mathsf{SN}. \forall \vec{q}'.$$
$$\mathcal{D};\mathcal{E} \vdash t[\vec{e}/\vec{q}'] \in \mathcal{C} \implies \lambda\{\vec{q}'\vec{q} \to t\}\vec{e} \in \mathcal{A}.$$

For reasoning about semantic pattern spines we will expand the definition of pattern substitution so the implication becomes

$$\forall \tau, \sigma.\ \vec{e} \,/\, \vec{q}' \searrow \tau;\sigma \,\wedge\, \mathcal{D};\mathcal{E} \vdash t\tau\sigma \in \mathcal{C}$$
$$\implies \lambda\{\vec{q}'\vec{q} \to t\}\vec{e} \in \mathcal{A}.$$

Lemma 29 (Semantic clause typing)**.** *The following implication holds:*

$$\frac{\mathcal{A} \mid \vec{q} \searrow \mathcal{D};\mathcal{E};\mathcal{C} \qquad \mathcal{D};\mathcal{E} \vdash t \in \mathcal{C} \qquad \rho \in \mathcal{D}}{\lambda\{\vec{q} \to t\} \in \mathcal{A}}$$

Proof. With $\sigma_{\mathsf{id}} \in \mathcal{E}(\rho)$ we have $t = t\sigma_{\mathsf{id}} \in \mathcal{C}(\rho) \subseteq \mathsf{SN}$. The rest follows by definition of semantic pattern spine typing with empty $\vec{e}$ and empty $\vec{q}'$. Note that we cannot proceed if $\mathcal{D}$ is inconsistent. $\square$

Lemma 30 (Semantic pattern spine typing)**.** *The following implications hold.*

$$\frac{}{\mathcal{A} \mid \cdot \searrow \cdot;\cdot;\mathcal{A}}$$

$$\frac{\mathcal{A}_1 \,/\, p \searrow \mathcal{D}_1;\mathcal{E}_1 \qquad \mathcal{A}_2 \mid \vec{q} \searrow \mathcal{D}_2;\mathcal{E}_2;\mathcal{C}}{\mathcal{A}_1 \twoheadrightarrow \mathcal{A}_2 \mid p\,\vec{q} \searrow \mathcal{D}_1,\mathcal{D}_2;\mathcal{E}_1 * \mathcal{E}_2;\mathsf{W}_{\mathcal{D}_1}\mathcal{C}}$$

$$\frac{\forall_{\beta<\alpha^{\uparrow}} \mathcal{R}_d\,(\boldsymbol{\nu}^{\beta}\mathcal{R}) \mid \vec{q} \searrow \mathcal{D};\mathcal{E};\mathcal{C}}{\boldsymbol{\nu}^{\alpha}\mathcal{R} \mid .d\,\vec{q} \searrow \mathcal{D};\mathcal{E};\mathcal{C}}$$

$$\frac{\forall \mathcal{G} \in \mathcal{K}.\ \mathcal{F}(\mathcal{G}) \mid \vec{q} \searrow \mathcal{D}(\mathcal{G});\mathcal{E}(\mathcal{G});\mathcal{C}(\mathcal{G})}{\forall_{\mathcal{K}}\mathcal{F} \mid X\,\vec{q} \searrow \Sigma_{X:\mathcal{K}}\mathcal{D};\mathcal{E}X;\mathcal{C}X}$$

$$\frac{\mathcal{F}(\infty) \mid \vec{q} \searrow \mathcal{D};\mathcal{E};\mathcal{C}}{\forall_{\beta}\mathcal{F}(\beta) \mid \infty\,\vec{q} \searrow \mathcal{D};\mathcal{E};\mathcal{C}}\ \mathcal{F}\ \textit{antitone}$$

Proof. Let us consider a few of these statements:

$$\frac{\mathcal{A}_1 \,/\, p \searrow \mathcal{D}_1;\mathcal{E}_1 \qquad \mathcal{A}_2 \mid \vec{q} \searrow \mathcal{D}_2;\mathcal{E}_2;\mathcal{C}}{\mathcal{A}_1 \twoheadrightarrow \mathcal{A}_2 \mid p\,\vec{q} \searrow \mathcal{D}_1,\mathcal{D}_2;\mathcal{E}_1 * \mathcal{E}_2;\mathsf{W}_{\mathcal{D}_1}\mathcal{C}}$$

Assume $\vec{e} \in \mathsf{SN}$ with $\vec{e} \,/\, \vec{q}' \searrow \tau;\sigma$ and $\mathcal{D}_1,\mathcal{D}_2;\mathcal{E}_1,\mathcal{E}_2 \vdash t\tau\sigma \in \mathsf{W}\mathcal{C}$ and show $\lambda\{\vec{q}'p\vec{q} \to t\}\vec{e} \in \mathcal{A}_1 \twoheadrightarrow \mathcal{A}_2$. Assume $s \in \mathcal{A}_1$.

Case $s \,/\, p \searrow \tau_1;\sigma_1$. Then $\sigma_1 \in \mathcal{E}_1(\rho_1)$ for some $\rho_1 \in \mathcal{D}_1$ by the first premise of the "rule". Since $\vec{e}s \,/\, \vec{q}'p \searrow \tau,\tau_1;\sigma,\sigma_1$ and $(\mathsf{W}\mathcal{C})(\rho_1) = \mathcal{C}$, we have $\mathcal{D}_2;\mathcal{E}_2 \vdash t(\tau,\tau_1)(\sigma,\sigma_1) \in \mathcal{C}$. Thus, by the second premise, $\lambda\{\vec{q}'p\vec{q} \to t\}\vec{e}s \in \mathcal{A}_2$.

Case s does not match p. Then $\lambda\{\vec{q}'p\vec{q} \to t\}\vec{e}s \in \overline{\emptyset} \subseteq \mathcal{A}_2$ because it is terminally stuck.

$$\frac{\forall \mathcal{G} \in \mathcal{K}.\ \mathcal{F}(\mathcal{G}) \mid \vec{q} \searrow \mathcal{D}(\mathcal{G});\mathcal{E}(\mathcal{G});\mathcal{C}(\mathcal{G})}{\forall_{\mathcal{K}}\mathcal{F} \mid X\,\vec{q} \searrow \Sigma_{X:\mathcal{K}}\mathcal{D};\mathcal{E}X;\mathcal{C}X}$$

Assume $\vec{e} \,/\, \vec{q}' \searrow \tau;\sigma$ and $\Sigma_{X:\mathcal{K}}\mathcal{D};\mathcal{E}X \vdash t\tau\sigma \in \mathcal{C}X$ and show $r := \lambda\{\vec{q}'X\vec{q} \to t\}\vec{e} \in \forall_{\mathcal{K}}\mathcal{F}$. First $r \in \mathsf{SN}$ since $t, \vec{e} \in \mathsf{SN}$ and r is not a redex. Now assume G and $\mathcal{G} \in \mathcal{K}$. Since $\vec{e}G \,/\, \vec{q}'\,X \searrow \tau, G/X;\sigma$ and $\mathcal{D}(\mathcal{G});\mathcal{E}(\mathcal{G}) \vdash t(\tau, G/X)\sigma \in \mathcal{C}(\mathcal{G})$, we can conclude $\lambda\{\vec{q}'X\vec{q} \to t\}\vec{e}G \in \mathcal{F}(\mathcal{G})$ by the premise.

$$\frac{\mathcal{F}(\infty) \mid \vec{q} \searrow \mathcal{D};\mathcal{E};\mathcal{C}}{\forall_{\beta}\mathcal{F}(\beta) \mid \infty\,\vec{q} \searrow \mathcal{D};\mathcal{E};\mathcal{C}}\ \mathcal{F}\ \text{antitone}$$

Assume $\vec{e} \,/\, \vec{q}' \searrow \tau;\sigma$ and $\mathcal{D};\mathcal{E} \vdash t\tau\sigma \in \mathcal{C}$ and show $\lambda\{\vec{q}'\infty\vec{q} \to t\}\vec{e} \in \forall_{\beta}\mathcal{F}(\beta)$. Assume b and $\beta \leq \infty$. Since $\vec{e}b \,/\, \vec{q}'\infty \searrow \tau;\sigma$ we can conclude $\lambda\{\vec{q}'\infty\vec{q} \to t\}\vec{e}b \in \mathcal{F}(\infty) \subseteq \mathcal{F}(\beta)$ by the premise and antitonicity of $\mathcal{F}$. $\square$

Theorem 31 (Soundness of pattern spine typing)**.** *Let* $\vdash \Delta_0, \Delta$ *and* $\Delta_0, \Delta \vdash \Gamma$. *If* $\Delta;\Gamma \mid A \vdash_{\Delta_0} \vec{q} \rightrightarrows C$ *and* $\rho_0 \in [\![\Delta_0]\!]$ *then* $[\![A]\!]_{\rho_0} \mid \vec{q} \searrow [\![\Delta]\!]_{\rho_0};[\![\Gamma]\!]_{(\rho_0,\text{-})};[\![C]\!]_{(\rho_0,\text{-})}$.

Proof. By induction on $\Delta;\Gamma \mid A \vdash_{\Delta_0} \vec{q} \rightrightarrows C$ using Lem. 30. $\square$

Semantic declaration and signature well-formedness. Having understood definitons by clauses $\lambda\vec{D}$ we can now show that any well-typed term inhabits its corresponding semantic type. For function symbols f, we simply assume it, by postulating a sematically well-formed signature Σ. We define $\boxed{\models \delta}$ and $\boxed{\models \Sigma}$ by

$$\begin{array}{lll} \models (f : A = \vec{D}) & :\Longleftrightarrow & f \in [\![A]\!] \\ \models \Sigma & :\Longleftrightarrow & \forall \delta \in \Sigma.\ \models \delta. \end{array}$$

Theorem 32 (Soundness of expression typing)**.** *Assume* $\models \Sigma$. *Let* $\vdash \Delta$ *and* $\Delta \vdash \Gamma$ *and* $\Delta \vdash C$ *and* $\mathcal{D} = [\![\Delta]\!]$ *and* $\mathcal{E}(\rho) = [\![\Gamma]\!]_{\rho}$ *and* $\mathcal{C}(\rho) = [\![C]\!]_{\rho}$.

1. *If* $\Delta;\Gamma \vdash r \rightrightarrows C$ *in* Σ *then* $\mathcal{D};\mathcal{E} \vdash r \in \mathcal{C}$.
2. *If* $\Delta;\Gamma \vdash t \leftleftarrows C$ *in* Σ *then* $\mathcal{D};\mathcal{E} \vdash t \in \mathcal{C}$.
3. *If* $\Delta;\Gamma \vdash \vec{D} \leftleftarrows C$ *in* Σ *then* $\mathcal{D};\mathcal{E} \vdash \lambda\vec{D} \in \mathcal{C}$.

Proof. Simultaneously by induction on the typing derivation.

Case Function symbol.

$$\frac{}{\Delta;\Gamma \vdash f \rightrightarrows \Sigma(f)}$$

Follows directly by well-formedness of the signature.

Case Subsumption.

$$\frac{\Delta;\Gamma \vdash r \rightrightarrows A \qquad \Delta \vdash A \leq C}{\Delta;\Gamma \vdash r \leftleftarrows C}$$

Follows by soundness of subtyping.

Case Definition clause.

$$\frac{\begin{array}{c}\Delta';\Gamma' \mid A \vdash_{\Delta} \vec{q} \rightrightarrows C \qquad \Delta \vdash \exists\Delta' \\ \Delta,\Delta';\Gamma,\Gamma' \vdash t \leftleftarrows C\end{array}}{\Delta;\Gamma \vdash \{\vec{q} \to t\} \leftleftarrows A}$$

Let $\mathcal{A}(\rho) = [\![A]\!]_{\rho}$ and $\rho \in \mathcal{D}$ and $\sigma \in \mathcal{E}(\rho)$ and τ arbitrary and show $\lambda\{\vec{q} \to t\tau\sigma\} \in \mathcal{A}(\rho)$. We set $\mathcal{D}' = [\![\Delta']\!]_{\rho}$ and $\mathcal{E}'(\rho') = [\![\Gamma']\!]_{(\rho,\rho')}$ and $\mathcal{C}'(\rho') = [\![C]\!]_{(\rho,\rho')}$. By induction hypothesis $\mathcal{D}';\mathcal{E}' \vdash t\tau\sigma \in \mathcal{C}'$, and by Theorem 31 $\mathcal{A}(\rho) \mid \vec{q} \searrow \mathcal{D}';\mathcal{E}';\mathcal{C}'$ entailing $\lambda\{\vec{q} \to t\tau\sigma\} \in \mathcal{A}(\rho)$ by Lemma 29. The lemma can be applied since $\Delta \vdash \exists\Delta'$ guarantees that for each $\rho \in \mathcal{D}$ there is some $\rho' \in \mathcal{D}'(\rho)$.

What remains to be proven is that well-typed programs yield, after measure erasure, semantically well-formed signatures. This is shown mutual block by mutual block using a lexicographic induction on ordinals as given by the termination measure assigned to each block. A formal description of program typing and its soundness proof is given in the next section.

[8] This definition is tricky to get right in a term-centered semantics. It might be easier in a semantics based on orthogonality where eliminations are first-class citizens (Pitts 2000).

$\boxed{'A^{<\mathfrak{m}} = A}$ Measure replacement and $\boxed{(\!|'A|\!) = A}$ $\boxed{(\!|'\delta|\!) = \delta}$ $\boxed{(\!|'\vec{\delta}|\!) = \Sigma}$ $\boxed{(\!|\beta|\!) = \Sigma}$ deletion.

$$(\forall\Psi.\,\mathfrak{m}' \Rightarrow A)^{<\mathfrak{m}} = \forall\Psi.\,\mathfrak{m}'{<}\mathfrak{m} \Rightarrow A \qquad (\!|\forall\Psi.\,\mathfrak{m}' \Rightarrow A|\!) = \forall\Psi.\,A \qquad (\!|'\vec{\delta}|\!) = \overrightarrow{(\!|'\delta|\!)}$$
$$(\!|f : {'A} = \vec{D}|\!) = f : (\!|'A|\!) = \vec{D} \qquad (\!|\mathsf{mutual}_m\,'\vec{\delta}|\!) = \overrightarrow{(\!|'\delta|\!)}$$

$\boxed{\vdash_m {'A}}$ Measured-type well-formedness and $\boxed{'\vec{\delta} \vdash {'\delta}}$ declaration typing.

$$\frac{\vdash \Psi \quad \Psi \vdash A \quad \Psi \vdash_m \mathfrak{m}}{\vdash_m \forall\Psi.\,\mathfrak{m} \Rightarrow A} \qquad \frac{\vdash\ \hat{\Psi}\vec{\mathsf{x}_f}\vec{D} \ \Leftarrow\ \forall\Psi.\,\vec{'A}^{<\mathfrak{m}} \to A}{\overrightarrow{f : {'A} = \mathbf{D}} \vdash f : (\forall\Psi.\,\mathfrak{m} \Rightarrow A) = \hat{\Psi}\vec{D}[\vec{f}/\vec{\mathsf{x}_f}]}$$

$\boxed{\vdash \beta}$ Block, $\boxed{\vdash \vec{\beta} \text{ in } \Sigma}$ blocks, and $\boxed{\vdash P}$ program typing.

$$\frac{\vdash_m {'A} \text{ for all } (f{:}'A = \vec{D}) \in {'\vec{\delta}} \qquad '\vec{\delta} \vdash {'\delta_k} \text{ for all } k}{\vdash \mathsf{mutual}_m\,'\vec{\delta}} \qquad \frac{}{\vdash \cdot \text{ in } \Sigma} \qquad \frac{\vdash \beta \text{ in } \Sigma \qquad \vdash \vec{\beta} \text{ in } \Sigma, (\!|\beta|\!)}{\vdash \beta, \vec{\beta} \text{ in } \Sigma} \qquad \frac{\vdash \vec{\beta} \text{ in } \cdot \qquad \vdash u \rightrightarrows A \text{ in } (\!|\vec{\beta}|\!)}{\vdash \vec{\beta}; u}$$

Figure 8. Program and signature typing.

5. Program typing and soundness

5.1 Program typing

Figure 8 presents the operations and judgements needed to type-check programs. The rules describe a type-checking process that is at the core of MiniAgda (Abel 2010).

The interesting rule is how to type-check a mutual block $'\vec{\delta}$ with measure annotations in the function types. First, we check well-formedness of the measured function types $'A$ and ensure that all measures have the same length m. Then we check each individual declaration $'\delta$ in the mutual block. The form of such a declaration is

$$f : (\forall\Psi.\,\mathfrak{m} \Rightarrow A) = \hat{\Psi}\vec{D}[\vec{f}/\vec{\mathsf{x}_f}].$$

This means that f should consist of a list of clauses $\hat{\Psi}\vec{D}$ that all start by abstracting over the size variables $\hat{\Psi}$ declared in size context Ψ. These are the size variables that can be used in measure $\mathfrak{m}$. Further, before type-checking is completed, the recursive occurrences of the mutually defined functions $\vec{f}$ are represented as special variables $\vec{\mathsf{x}_f}$ in the clauses $\vec{D}$; after type-checking, they get substituted by the actual function symbols. This trick allows us to type-check the clauses where we give constrained types $\mathsf{x}_{f'} : \forall\Psi'.\,\mathfrak{m}'{<}\mathfrak{m} \Rightarrow A'$ to the mutually defined functions $f' : \forall\Psi'.\,\mathfrak{m}' \Rightarrow A'$. Thus, we ensure that recursive-call sequences are well-founded w. r. t. the termination measure.

After a mutual block $'\vec{\delta}$ has been checked, its measure-erased declarations $|'\vec{\delta}|$ are added to the signature Σ. The entry point u of program $\vec{\beta}; u$ is finally checked in the signature $|\vec{\beta}|$ created from all mutual blocks $\vec{\beta}$.

5.2 Soundness of program typing

In the following we prove program typing correct by giving a meaning to measured types and declarations. The correctness of mutually recursive definitions will follow from a lexicographic induction on ordinals.

A measured type $'A$ is not a proper type, it does not have a meaning by itself. *Bounded type interpretation* $\boxed{[\![{'A}]\!]^{<\vec{\alpha}}}$ assigns it a meaning relative to a tuple of ordinals which has the same length as the measure $\mathfrak{m}$ in $'A$.

$$[\![\forall\Psi.\,\mathfrak{m} \Rightarrow A]\!]^{<\vec{\alpha}} = \forall_{\rho\in[\![\Psi]\!]}\,([\![\mathfrak{m}]\!]_\rho < \vec{\alpha}) \Rightarrow [\![A]\!]_\rho$$

$[\![{'A}]\!]^{<\vec{\alpha}}$ denotes a constrained type. It is the semantic counterpart of $'A^{<\mathfrak{m}}$, as the following lemma proves:

Lemma 33 (Soundness of measure replacement)**.** *Let* $'A = \forall\Psi.\,\mathfrak{m} \Rightarrow A$*. If* $\vdash_n {'A}$ *and* $\rho \in [\![\Psi]\!]$ *then* $[\![{'A}^{<\mathfrak{m}}]\!]_\rho = [\![{'A}]\!]^{<[\![\mathfrak{m}]\!]_\rho}$*.*

Proof. Let $\vec{\alpha} = [\![\mathfrak{m}]\!]_\rho$. Recall that $'A^{<\mathfrak{m}} = \forall\Psi'.\,\mathfrak{m}' < \mathfrak{m} \Rightarrow A'$ where Ψ' is a renaming of Ψ and $\mathfrak{m}'$, A' are the corresponding renamings of of $\mathfrak{m}$, A. We thus have $[\![{'A}^{<\mathfrak{m}}]\!]_\rho = (\forall_{\rho'\in[\![\Psi']\!]}([\![\mathfrak{m}']\!]_{\rho'} < \vec{\alpha}) \Rightarrow [\![A']\!]_{\rho'}) = [\![{'A}]\!]^{<\vec{\alpha}}$. □

Erasure of the measure in $'A$ turns a bounded quantification into an unbounded one:

Lemma 34 (Soundness of measure erasure)**.** *Let m be the length of the measure in measure-decorated type $'A$. Then* $[\![|'A|]\!] = \bigcap_{\vec{\alpha}\in\mathsf{O}^m} [\![{'A}]\!]^{<\vec{\alpha}}$*.*

Proof. For "$\subseteq$", assume $r \in [\![|'A|]\!] = [\![\forall\Psi.\,A]\!]$ and $\vec{\alpha} \in \mathsf{O}^m$ and $\rho \in [\![\Psi]\!]$ and $\vec{b} : \Psi$ and show $r\,\vec{b} \in ([\![\mathfrak{m}]\!]_\rho < \vec{\alpha}) \Rightarrow [\![A]\!]_\rho$. This follows from $r\,\vec{b} \in [\![A]\!]_\rho$, since by definition $\mathcal{A} \subseteq (P \Rightarrow \mathcal{A})$ for all $P, \mathcal{A}$.

For "$\supseteq$", assume $r \in \bigcap_{\vec{\alpha}} \forall_{\rho\in[\![\Psi]\!]}\,([\![\mathfrak{m}]\!]_\rho < \vec{\alpha}) \Rightarrow [\![A]\!]_\rho$ and $\rho \in [\![\Psi]\!]$ and $\vec{b} : \Psi$ and show $r\,\vec{b} \in [\![A]\!]_\rho$. Choosing some $\vec{\alpha} > [\![\mathfrak{m}]\!]_\rho$ (this is always possible due to the open nature of O), we conclude by instantiation of the first assumption. □

In order to justify a block of mutually recursive functions, we perform an lexicographic induction over over a tuple $\vec{\alpha}$ of ordinals. This requires us to interpret the declarations of the mutual block relative to the upper bound $\vec{\alpha}$ on the measure of the recursive calls. *Bounded semantic declaration typing* $\boxed{'\vec{\delta} \models^{\vec{\alpha}} {'\delta}}$ is defined by

$$\boxed{f_1{:}'A_1 = \vec{D}_1, \ldots, f_n{:}'A_n = \vec{D}_n \models^{\vec{\alpha}} f : (\forall\Psi.\,\mathfrak{m} \Rightarrow A) = \vec{D}}$$
$$\begin{aligned} :\Longleftrightarrow\ & \text{if } f_i \in [\![{'A_i}]\!]^{<\vec{\alpha}} \text{ for } i = 1..n \\ & \text{and } \rho \in [\![\Psi]\!] \text{ with } [\![\mathfrak{m}]\!]_\rho \leq \alpha \\ & \text{and } \vec{b} : \Psi \\ & \text{then } f\,\vec{b} \in [\![A]\!]_\rho \end{aligned}$$

In this definition $\vec{b} : \Psi$ shall mean that $\vec{b}$ is a list of size expressions that has the same length as size context Ψ.

Corollary 35 (Soundness of measure erasure in declarations)**.** $\models |'\delta|$ *iff* $\models^{\vec{\alpha}} {'\delta}$ *for all* $\alpha \in \mathsf{O}^m$*.*

Lemma 36 (Soundness of declaration typing)**.** *Let m be the length of the measure in block ${}'\vec{\delta}$ and declaration ${}'\delta$. If ${}'\vec{\delta} \vdash {}'\delta$ then ${}'\vec{\delta} \models^{\vec{\alpha}} {}'\delta$ for all $\vec{\alpha} \in \mathsf{O}^m$.*

Proof. Declaration typing ${}'\vec{\delta} \vdash {}'\delta$ is derived by rule:

$$\frac{\vdash\ \hat{\Psi}\overrightarrow{\mathsf{x}_f}\vec{D} \ \Leftarrow\ \forall\Psi.\, {}'\vec{A}^{<\mathfrak{m}} \to A}{\overrightarrow{f : {}'A = \mathbf{D}} \vdash f : (\forall\Psi.\, \mathfrak{m} \Rightarrow A) = \hat{\Psi}\vec{D}[\vec{f}/\overrightarrow{x_f}]}$$

We show $\overrightarrow{f : {}'A = \mathbf{D}} \models^{\vec{\alpha}} f : (\forall\Psi.\, \mathfrak{m} \Rightarrow A) = \hat{\Psi}\vec{D}[\vec{f}/\overrightarrow{x_f}]$. By assumption $f_i \in [\![{}'A_i]\!]^{<\vec{\alpha}}$ for all i. Assume $\rho \in [\![\Psi]\!]$ with $[\![\mathfrak{m}]\!]_\rho \leq \vec{\alpha}$ and $\vec{b} : \Psi$, and show $f\,\vec{b} \in [\![A]\!]_\rho$. By soundness of declaration typing, $\lambda\hat{\Psi}\overrightarrow{x_f}\vec{D} \in \forall_{\rho\in[\![\Psi]\!]}([\![{}'\vec{A}]\!]^{<[\![\mathfrak{m}]\!]_\rho} \to [\![A]\!]_\rho)$. Since $f_i \in [\![{}'A_i]\!]^{<\vec{\alpha}} \subseteq [\![{}'A_i]\!]^{<[\![\mathfrak{m}]\!]_\rho}$ for all i by contravariance, this implies that $(\lambda\hat{\Psi}\overrightarrow{x_f}\vec{D})\,\vec{b}\,\vec{f} \in [\![A]\!]_\rho$. By the simulation $f\,\vec{b} \rhd (\lambda\hat{\Psi}\overrightarrow{x_f}\vec{D})\,\vec{b}\,\vec{f}$ we have $f\,\vec{b} \in [\![A]\!]_\rho$. □

Theorem 37 (Soundness of block typing)**.** *Let $\models \Sigma$. If $\vdash \beta$ in Σ then $\models \Sigma, |\beta|$.*

Proof. Let n the number of mutual declarations and ${}'\delta_k = (f_k : {}'A_k = \vec{D}_k)$ and ${}'A_k = \forall\Psi_k.\, \mathfrak{m}_k \Rightarrow A_k$ for $k = 1..n$. Note that $|{}'\vec{\delta}| = (f_k : \forall\Psi_k.\, A_k = \vec{D}_k)_{k=1..n}$ in this case.

$$\frac{\vdash_m {}'A_k \text{ for } k = 1..n \qquad {}'\vec{\delta} \vdash {}'\delta_k \text{ in } \Sigma \text{ for } k = 1..n}{\vdash \mathsf{mutual}_m\, {}'\vec{\delta} \text{ in } \Sigma}$$

By soundness of declaration typing (Lemma 36) we have ${}'\vec{\delta} \models^{\vec{\alpha}} {}'\delta_k$ for $\vec{\alpha} \in \mathsf{O}^m$ and $k = 1..n$. By lexicographic induction on $\vec{\alpha} \in \mathsf{O}^m$ this entails $\models^{\vec{\alpha}} {}'\delta_k$ for $k = 1..n$, using the reduction rules for $f_{1..n}$ in the extended signature $\Sigma, |{}'\vec{\delta}|$. This entails $\models |{}'\delta_k|$ by Corollary 35.

We spell out the induction in more detail. Assume $\vec{\alpha} \in \mathsf{O}^m$ and $k \in \{1..n\}$ and show $\models^{\vec{\alpha}} {}'\delta_k$. By induction hypothesis $\models^{\vec{\beta}} {}'\vec{\delta}$ for all $\vec{\beta} < \vec{\alpha}$ (lexicographic comparison). Using ${}'\vec{\delta} \models^{\vec{\alpha}} {}'\delta_k$ solves the goal, but to apply it we have to show $f_k \in [\![{}'A_k]\!]^{<\vec{\alpha}}$ for all k. Assume $\rho \in [\![\Psi_k]\!]$ with $[\![\mathfrak{m}_k]\!]_\rho < \vec{\alpha}$ and $\vec{b} : \Psi$ and show $f_k\,\vec{b} \in [\![A_k]\!]_\rho$. We conclude by induction hypothesis for $\vec{\beta} = [\![\mathfrak{m}_k]\!]_\rho$. □

Corollary 38 (Soundness of program typing)**.**

1. *If $\models \Sigma$ and $\vdash \vec{\beta}$ in Σ then $\models \Sigma, |\vec{\beta}|$.*
2. *If $\vdash \vec{\beta}; t$ then $t \in \mathsf{SN}$ in signature $|\vec{\beta}|$.*

6. Further Examples

6.1 On the context extension check

Here is an example what can go wrong when we omit the check $\Delta \vdash \exists\Delta'$ from definition typing.

$$\begin{array}{lcl} \mathsf{kUnit} & : & \forall A.\, A \to 1 \\ \mathsf{kUnit}\ A\ a & = & () \\[1ex] \mathsf{badLam} & : & \forall i.\, |i| \Rightarrow 1 \\ \mathsf{badLam}\ i & = & \mathsf{kUnit}\ (\forall j{<}i.\, 1)\ \lambda\{j \to \mathsf{badLam}\ j\} \end{array}$$

Without a separate context check, badLam type-checks since $j < i$, thus the recursive call $\mathsf{badLam}\,j$ is valid. But surely, $\mathsf{badLam}\,i$ is the start of an infinite reduction sequence, leading to an infinite descending chain of sizes $i > j > j_1 > j_2 > \dots$. The context check $i \leq \infty \vdash \exists j{<}i$ however fails, since for $i = 0$ there is no instance for j. Thus badLam is rejected, rightfully so.

6.2 On first-class constrained types

Treating conditional types $\mathfrak{c} \Rightarrow A$ as first-class would jeopardize strong normalization, as the following example shows:

$$\begin{array}{lcl} \mathsf{badCond} & : & \forall i.\, |i| \Rightarrow 1 \\ \mathsf{badCond}\ i & = & \mathsf{kUnit}\ (|i| < |i| \to 1)\ (\mathsf{badCond}\ i) \end{array}$$

The recursive call $\mathsf{badCond}\,i$ makes the promise $i < i$ which can never be fulfilled. Thus $\mathsf{badCond}\,i$ should not appear on the rhs. However, types that combine a quantifier with a constraint should be fine, e. g., $\forall j.\, |j|{<}|i| \to 1$, which is equivalent to $\forall j{<}i.\, 1$, but also constraints that can never be fulfilled are fine under a quantifier, e. g., $\forall j.\, |0| < |0| \to 1$. Constraints need to be checked immediately after the quantifier has been eliminated (Blanqui and Riba 2006).

7. Conclusion

Our work provides a uniform type-based approach to proving termination of (co)inductive definitions. It is centered around patterns and copatterns which allow us to reason about both finite and infinite data by well-founded induction. Proving strong normalization for this language is a significant step towards understanding well-founded corecursion in terms of the depth of observation we can safely make.

As a next step, we plan to extend our work to full dependently typed systems to allow coinductive definitions to be defined and reasoned with by observations. This will put coinduction in these systems on a robust foundation. We have already implemented size-based type checking for patterns and copatterns in MiniAgda (Abel 2012) which gives us confidence in the approach.

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A. Appendix

The appendix contains a recapitulation of syntax and notational definitions of $\mathsf{F}_\omega^{\mathsf{cop}}$ and the detailed rules for size, kind, and type well-formedness, and size comparison, subkinding and subtyping. We also provide pattern matching and reduction rules in detail.

Language grammar.

	SizeVar	$\ni i, j$		size variable
	SizeExp	$\ni a, b$	$::= i + n \mid \infty + n$	size expression $(n \geq 0)$
SizeExp $\subseteq$	SizeExp$^+$	$\ni a^+, b^+$	$::= a \mid n$	extended size expression $(n \geq 0)$
SizeExp$^+$ $\subseteq$	Measure	$\ni \mathfrak{m}$	$::= a^+ \mid a^+, \mathfrak{m}$	measure expression
	Cond	$\ni \mathfrak{c}$	$::= \mathfrak{m} < \mathfrak{m}'$	condition
	Pol	$\ni \pi$	$::= \circ \mid + \mid - \mid \top$	variance
	SKind	$\ni \iota$	$::= * \mid o \mid \iota \to \iota'$	simple kinds
	Kind	$\ni \kappa$	$::= * \mid {<}a \mid \pi\kappa \to \kappa'$	kinds with variance information
SizeVar $\subseteq$	TyVar	$\ni X, Y, Z, i, j$		type and size variables
TyVar $\cup$ SizeExp $\subseteq$	TyAtom	$\ni K$	$::= X \mid a \mid 1 \mid \times \mid \to$	type atoms
TyAtom $\subseteq$	Type	$\ni F, G, A, B, C$	$::= K \mid \lambda X{:}\iota.\, F \mid F\, G$	type-level lambda-calculus
			$\mid \forall_\kappa \mid \exists_\kappa$	quantifiers
			$\mid \mu^a S \mid \nu^a R$	variant and record types
	MType	$\ni {}'\!A, {}'\!B$	$::= \forall\Psi.\, \mathfrak{m} \Rightarrow C$	type with measure
	CType	$\ni {}^?\!A, {}^?\!B$	$::= \forall\Psi.\, \mathfrak{c} \Rightarrow C$	constrained type
	Variant	$\ni S$	$::= \langle c_1{:}F_1; \dots; c_n{:}F_n \rangle$	variant row $(n \geq 0)$
	Record	$\ni R$	$::= \{d_1{:}F_1; \dots; d_n{:}F_n\}$	record row $(n \geq 0)$
	Cons	$\ni c$		constructor (variant label)
	Proj	$\ni d$		destructor (record label)
	Var	$\ni x, y, z$		term variable
TyVar $\subseteq$	TyPat	$\ni Q$	$::= X \mid \infty$	type pattern
Var $\subseteq$	Pat	$\ni p$	$::= x \mid () \mid (p_1, p_2) \mid c\, p \mid {}^Q p$	pattern
Pat $\cup$ TyPat $\subseteq$	Copat	$\ni q$	$::= p \mid Q \mid .d$	copattern
	PatSp	$\ni \mathbf{q}$	$::= \vec{q}$	pattern spine
	Fun	$\ni f, g, h$		defined function symbol
	Elim	$\ni e$	$::= t \mid G \mid .d$	eliminations
Var $\subseteq$	App	$\ni u$	$::= x \mid f \mid r\, e$	applicative expressions
	Intro	$\ni v$	$::= () \mid (t_1, t_2) \mid c\, t \mid {}^G t$	introductions (checkable)
App $\cup$ Intro $\subseteq$	Exp	$\ni r, s, t$	$::= u \mid (t : A)$	inferable expressions
			$\mid v \mid \lambda\vec{D}$	intros, anonymous object (checkable)
	DCl	$\ni D$	$::= \{\vec{q} \to t\}$	definition clause
	Def	$\ni \vec{D}, \mathbf{D}$	$::= \{D_1; \dots; D_n\}$	definition
	Decl	$\ni \delta$	$::= f : A = \vec{D}$	declaration
	MDecl	$\ni {}'\delta$	$::= f : {}'\!A = \vec{D}$	declaration with measure
	Block	$\ni \beta$	$::= \mathsf{mutual}_m\ {}'\vec{\delta}$	mutual block $(m \geq 1)$
	Prg	$\ni P$	$::= \vec{\beta}; t$	program
	Sig	$\ni \Sigma$	$::= \vec{\delta}$	signature
	SizeCxt	$\ni \Psi$	$::= \cdot \mid \Psi, i{:}\pi({<}a)$	size variable context
SizeCxt $\subseteq$	TyCxt	$\ni \Delta$	$::= \cdot \mid \Delta, X{:}\pi\kappa$	type/size variable context
	Cxt	$\ni \Gamma$	$::= \cdot \mid \Gamma, x{:}A \mid \Gamma, x{:}{}^?\!A$	term variable context

Figure 9. Syntax.

Notation.

$\kappa \xrightarrow{\pi} \kappa'$	for	$\pi\kappa \to \kappa'$	function kind
$\kappa \to \kappa'$	for	$\kappa \xrightarrow{\circ} \kappa'$	default variance
$\leq a$	for	$<(a+1)$	weak bound
size	for	$\leq\infty$	
$\lambda X F$	for	$\lambda X{:}\iota.\, F$	if ι inferable
$A \times B$	for	$(\times)\, A\, B$	product type
$A \to B$	for	$(\to)\, A\, B$	function type
$\forall X{:}\kappa.\, A$	for	$\forall_\kappa(\lambda X{:}\lvert\kappa\rvert.\, A)$	universal type
$\exists X{:}\kappa.\, A$	for	$\exists_\kappa(\lambda X{:}\lvert\kappa\rvert.\, A)$	existential type
$\forall j{<}a.\, A$	for	$\forall_{<a}\,(\lambda j{:}o.\, A)$	bounded universal
$\exists j{<}a.\, A$	for	$\exists_{<a}\,(\lambda j{:}o.\, A)$	bounded existential
S_c	for	F where $(c{:}F) \in S$	type of constructor
R_d	for	F where $(d{:}F) \in R$	type of destructor

$\Delta, X{:}\kappa$	for	$\Delta, X{:}\circ\kappa$	default variance
$\Delta, i{<}a$	for	$\Delta, i{:}\circ({<}a)$	default variance
$\cdot \to A$	for	A	context abstraction
$\forall\Delta, X{:}\kappa.\, A$	for	$\forall\Delta.\, \forall X{:}\kappa.\, A$	$\forall\Delta.\, A$
$\widehat{\cdot}$	for	$\cdot$	context domain
$\widehat{\Delta, X{:}\pi\kappa}$	for	$\hat{\Delta}, X$	$\hat{\Delta}$ (variable list)
$(t_1, t_2, \dots, t_n)$	for	$(t_1, (t_2, \dots, t_n))$	n-ary tuples
$\lambda x.\, t$	for	$\lambda\{x \to t\}$	lambda abstraction
$\lambda\vec{q}.\, t$	for	$\lambda\{\vec{q} \to t\}$	single-clause object
$\vec{q}\{\vec{q}' \to t\}$	for	$\{\vec{q}\vec{q}' \to t\}$	copattern prefix for clause
$\vec{q}\vec{D}$	for	$\{\vec{q}D_1; \dots; \vec{q}D_n\}$	copattern prefix for clauses

Figure 10. Notational definitions.

$\boxed{\pi \leq \pi'}$ Lattice of variances and $\boxed{\pi\pi'}$ variance composition (commutative).

$$\frac{}{\pi \leq \pi} \qquad \frac{}{\circ \leq \pi} \qquad \frac{}{\pi \leq \top} \qquad\qquad \top\pi = \top \qquad \circ\pi = \circ \text{ (if } \pi \neq \top) \qquad +\pi = \pi \qquad -- = +$$

$\boxed{\pi^{-1}\pi'}$ Variance inverse composition.

$$\top^{-1}\pi = \circ \qquad \circ^{-1}\circ = \circ \qquad \circ^{-1}\pi = \top \text{ (if } \pi \neq \top) \qquad +^{-1}\pi = \pi \qquad -^{-1}\pi = -\pi$$

$\boxed{\pi^{-1}\Delta}$ Variance inverse composition with kinding context.

$$\pi^{-1}\cdot = \cdot \qquad \pi^{-1}(\Delta, X{:}\pi'\kappa) = (\pi^{-1}\Delta), X{:}(\pi^{-1}\pi')\kappa \qquad -\Delta = -^{-1}\Delta$$

Figure 11. Variances (polarities).

$\boxed{\Psi \vdash a}$ Well-formed sizes, $\boxed{\vdash \Psi}$ well-formed size contexts, and $\boxed{\Psi \vdash i < a}$ size bound lookup.

$$\frac{}{\Psi \vdash \infty + n} \qquad \frac{\Psi \vdash i < a}{\Psi \vdash i + n} \qquad\qquad \frac{}{\vdash \cdot} \qquad \frac{\vdash \Psi \qquad \circ^{-1}\Psi \vdash a}{\vdash \Psi, i{:}\pi({<}a)} \qquad\qquad \frac{(i{:}\pi({<}a)) \in \Psi}{\Psi \vdash i < a}\ \pi \leq +$$

$\boxed{\Delta \vdash \vec{a} \leftleftarrows \Psi}$ Well-formed size substitution

$$\frac{}{\Psi \vdash \cdot \leftleftarrows \cdot} \qquad \frac{\Psi' \vdash \vec{a} \leftleftarrows \Psi \qquad \tau = \vec{a}/\hat{\Psi} \qquad \Psi' \vdash a < b\tau}{\Psi' \vdash \vec{a}\ a \leftleftarrows \Psi, i : \pi({<}b)}$$

$\boxed{\Psi \vdash a < b}$ Strict and $\boxed{\Psi \vdash a \leq b}$ weak size comparison.

$$\frac{n < m}{\Psi \vdash \infty + n < \infty + m} \qquad \frac{n < m \qquad \Psi \vdash i < a}{\Psi \vdash i + n < i + m} \qquad \frac{\Psi \vdash i < \infty}{\Psi \vdash i + n < \infty + m} \qquad \frac{\Psi \vdash i < \infty + m}{\Psi \vdash i + n < \infty + (m + n)}$$

$$\frac{\Psi \vdash a + n \leq b}{\Psi, i{:}\pi({<}a), \Psi' \vdash i + n < b}\ \pi \leq + \qquad \frac{\Psi \vdash a < b + 1}{\Psi \vdash a \leq b}$$

$\boxed{\Psi \vdash_1 a^+}$ Extended size and $\boxed{\Psi \vdash_k \mathfrak{m}}$ $\boxed{\Psi \vdash \mathfrak{m}}$ measure well-formedness.

$$\frac{}{\Psi \vdash_1 n} \qquad \frac{\Psi \vdash a}{\Psi \vdash_1 a} \qquad \frac{\Psi \vdash_1 a^+ \qquad \Psi \vdash_k \mathfrak{m}}{\Psi \vdash_{k+1} a^+, \mathfrak{m}} \qquad \frac{\Psi \vdash_k \mathfrak{m}}{\Psi \vdash \mathfrak{m}}\ k \text{ is length of } \mathfrak{m}$$

$\boxed{\Psi \vdash a^+ < b^+}$ Extending strict and $\boxed{\Psi \vdash a^+ \leq b^+}$ weak size comparison.

$$\frac{n_1 < n_2}{\Psi \vdash n_1 < n_2} \qquad \frac{n_1 < n_2}{\Psi \vdash n_1 < i + n_2} \qquad \frac{}{\Psi \vdash n_1 < \infty + n_2} \qquad \frac{\Psi \vdash a^+ < b^+ + 1}{\Psi \vdash a^+ \leq b^+}$$

$\boxed{\Psi \vdash \mathfrak{c}}$ $\boxed{\Psi \vdash \mathfrak{m} < \mathfrak{m}'}$ Strict and $\boxed{\Psi \vdash \mathfrak{m} \leq \mathfrak{m}'}$ weak measure comparison.

$$\frac{\Psi \vdash a_1^+ < a_2^+}{\Psi \vdash a_1^+, \mathfrak{m}_1 < a_2^+, \mathfrak{m}_2} \qquad \frac{\Psi \vdash a_1^+ \leq a_2^+ \qquad \Psi \vdash \mathfrak{m}_1 < \mathfrak{m}_2}{\Psi \vdash a_1^+, \mathfrak{m}_1 < a_2^+, \mathfrak{m}_2} \qquad \frac{\Psi \vdash a_1^+ \leq a_2^+ \qquad \Psi \vdash \mathfrak{m}_1 \leq \mathfrak{m}_2}{\Psi \vdash a_1^+, \mathfrak{m}_1 \leq a_2^+, \mathfrak{m}_2}$$

$\boxed{|\kappa| = \iota}$ Kind erasure defined by $|{*}| = *$ and $|{<}b| = o$ and $|\pi\kappa \to \kappa'| = |\kappa| \to |\kappa'|$.

$\boxed{\Psi \vdash \kappa}$ Wellformed kinds.

$$\frac{}{\Psi \vdash *} \qquad \frac{\Psi \vdash a}{\Psi \vdash {<}a} \qquad \frac{-\Psi \vdash \kappa \qquad \Psi \vdash \kappa'}{\Psi \vdash \pi\kappa \to \kappa'}$$

$\boxed{\Psi \vdash \kappa \leq \kappa'}$ Subkinding.

$$\frac{}{\Psi \vdash * \leq *} \qquad \frac{\Psi \vdash a \leq b}{\Psi \vdash ({<}a) \leq ({<}b)} \qquad \frac{\pi' \leq \pi \qquad -\Psi \vdash \kappa_1' \leq \kappa_1 \qquad \Psi \vdash \kappa_2 \leq \kappa_2'}{\Psi \vdash \pi\kappa_1 \to \kappa_2 \leq \pi'\kappa_1' \to \kappa_2'}$$

$\boxed{\Psi \vdash O \leq^\pi O' \text{ for } O ::= a \mid \mathfrak{m} \mid \kappa}$ Parametrized size, measure, and kind comparison.

$$\frac{\Psi \vdash O \leq O' \qquad \Psi \vdash O' \leq O}{\Psi \vdash O \leq^\circ O'} \qquad \frac{\Psi \vdash O \leq O'}{\Psi \vdash O \leq^+ O'} \qquad \frac{\Psi \vdash O' \leq O}{\Psi \vdash O \leq^- O'} \qquad \frac{}{\Psi \vdash O \leq^\top O'}$$

Figure 12. Sizes, measures, and kinds.

$\boxed{\Delta \vdash A}$ Well-formed types (entry point for kinding) and $\boxed{\Delta \vdash F \rightrightarrows \kappa}$ kinding (inference mode).

$$\frac{\Delta \vdash A \rightrightarrows *}{\Delta \vdash A} \qquad \frac{}{\Delta \vdash 1 \rightrightarrows *} \qquad \frac{}{\Delta \vdash \times \rightrightarrows * \xrightarrow{+} * \xrightarrow{+} *} \qquad \frac{}{\Delta \vdash \to \rightrightarrows * \xrightarrow{-} * \xrightarrow{+} *} \qquad \frac{\Delta \vdash a}{\Delta \vdash a \rightrightarrows \le a}$$

$$\frac{(X{:}\pi\kappa) \in \Delta}{\Delta \vdash X \rightrightarrows \kappa}\ \pi \le + \qquad \frac{\Delta \vdash F \rightrightarrows \kappa \xrightarrow{\pi} \kappa' \qquad \pi^{-1}\Delta \vdash G \leftleftarrows \kappa}{\Delta \vdash F\, G \rightrightarrows \kappa'}$$

$$\frac{-\Delta \vdash \kappa}{\Delta \vdash \forall_\kappa \rightrightarrows (\kappa \xrightarrow{\circ} *) \xrightarrow{+} *} \qquad \frac{\Delta \vdash \kappa}{\Delta \vdash \exists_\kappa \rightrightarrows (\kappa \xrightarrow{\circ} *) \xrightarrow{+} *} \qquad \frac{\Delta \vdash a \qquad \Delta \vdash S \leftleftarrows * \xrightarrow{\circ} *}{\Delta \vdash \mu^a S \rightrightarrows *} \qquad \frac{-\Delta \vdash a \qquad \Delta \vdash R \leftleftarrows * \xrightarrow{\circ} *}{\Delta \vdash \nu^a R \rightrightarrows *}$$

$\boxed{\Delta \vdash F \leftleftarrows \kappa}$ Kinding (checking mode).

$$\frac{\Delta \vdash F \rightrightarrows \kappa \qquad \Delta \vdash \kappa \le \kappa'}{\Delta \vdash F \leftleftarrows \kappa'} \qquad \frac{|\kappa| = \iota \qquad \Delta, X{:}\pi\kappa \vdash F \leftleftarrows \kappa'}{\Delta \vdash \lambda X{:}\iota.\, F \leftleftarrows \pi\kappa \to \kappa'}$$

$$\frac{\Delta \vdash S_c \leftleftarrows \kappa \text{ for all } c \in S}{\Delta \vdash S \leftleftarrows \kappa} \qquad \frac{\Delta \vdash R_d \leftleftarrows \kappa \text{ for all } d \in R}{\Delta \vdash R \leftleftarrows \kappa}$$

$\boxed{\Delta \vdash {}^{?}A}$ Well-formed constrained types.

$$\frac{\Delta \vdash \mathfrak{m} \qquad \Delta \vdash \Psi \qquad \Delta, \Psi \vdash \mathfrak{m}' \qquad \Delta, \Psi \vdash A}{\Delta \vdash \forall \Psi.\, \mathfrak{m}' {<} \mathfrak{m} \Rightarrow A}$$

$\boxed{\Delta \vdash \Delta'}$ Well-formed kinding and $\boxed{\Delta \vdash \Gamma}$ typing contexts.

$$\frac{}{\Delta \vdash \cdot} \qquad \frac{\circ^{-1}\Delta \vdash \kappa \qquad \Delta, X{:}\pi\kappa \vdash \Delta'}{\Delta \vdash X{:}\pi\kappa, \Delta'} \qquad \frac{}{\Delta \vdash \cdot} \qquad \frac{\Delta \vdash \Gamma \qquad \Delta \vdash A}{\Delta \vdash \Gamma, x{:}A} \qquad \frac{\Delta \vdash \Gamma \qquad \Delta \vdash {}^{?}A}{\Delta \vdash \Gamma, x{:}{}^{?}A}$$

Figure 13. Kinding

$\boxed{a^\uparrow}$ Bound normalization defined by $(\infty + n)^\uparrow = \infty + 1$ for $n \ge 0$ and $a^\uparrow = a$ for $a ::= i + n$.

$\boxed{\Delta \vdash F \le^\pi F' \rightrightarrows \kappa \text{ for } \pi \ne \top}$ Subtyping and type equality (inference mode).

$$\frac{\Delta \vdash K \rightrightarrows \kappa}{\Delta \vdash K \le^\pi K \rightrightarrows \kappa} \qquad \frac{\Delta \vdash F \le^\pi F' \rightrightarrows \pi_1\kappa_1 \to \kappa_2 \qquad \pi_1^{-1}\Delta \vdash G \le^{\pi_1\pi} G' \leftleftarrows \kappa_1}{\Delta \vdash F\, G \le^\pi F'\, G' \rightrightarrows \kappa_2}$$

$$\frac{-\Delta \vdash \kappa \le^{-\pi} \kappa' \qquad \kappa'' = \max^{-\pi}(\kappa, \kappa')}{\Delta \vdash \forall_\kappa \le^\pi \forall_{\kappa'} \rightrightarrows (\kappa'' \xrightarrow{\circ} *) \xrightarrow{-} *} \qquad \frac{\Delta \vdash \kappa \le^\pi \kappa' \qquad \kappa'' = \max^\pi(\kappa, \kappa')}{\Delta \vdash \exists_\kappa \le^\pi \exists_{\kappa'} \rightrightarrows (\kappa'' \xrightarrow{\circ} *) \xrightarrow{+} *} \qquad \begin{array}{rcl} \max^+ &=& \max^\circ \;=\; \max \\ \max^- &=& \min \end{array}$$

$$\frac{\Delta \vdash a^\uparrow \le^\pi a'^\uparrow \qquad \Delta \vdash S \le^\pi S' \leftleftarrows * \xrightarrow{\circ} *}{\Delta \vdash \mu^a S \le^\pi \mu^{a'} S' \rightrightarrows *} \qquad \frac{-\Delta \vdash a^\uparrow \le^{-\pi} a'^\uparrow \qquad \Delta \vdash R \le^\pi R' \leftleftarrows * \xrightarrow{\circ} *}{\Delta \vdash \nu^a R \le^\pi \nu^{a'} R' \rightrightarrows *}$$

$\boxed{\Delta \vdash F \le^\pi F' \leftleftarrows \kappa}$ Subtyping and type equality (checking mode) and $\boxed{\Delta \vdash A \le A'}$ entry point for subtyping.

$$\frac{}{\Delta \vdash F \le^\top F' \leftleftarrows \kappa} \qquad \frac{\Delta \vdash A \le^\pi A' \rightrightarrows *}{\Delta \vdash A \le^\pi A' \leftleftarrows *} \qquad \frac{\Delta, X{:}\pi_1\kappa_1 \vdash (F \mathbin{@} X) \le^\pi (F' \mathbin{@} X) \leftleftarrows \kappa_2}{\Delta \vdash F \le^\pi F' \leftleftarrows \pi_1\kappa_1 \to \kappa_2}$$

$$\frac{\Delta \vdash S_c \le^\pi S'_c \leftleftarrows \kappa \text{ for all } c \in S}{\Delta \vdash S \le^\pi S' \leftleftarrows \kappa} \qquad \frac{\Delta \vdash R_d \le^\pi R'_d \leftleftarrows \kappa \text{ for all } d \in R'}{\Delta \vdash R \le^\pi R' \leftleftarrows \kappa} \qquad \frac{\Delta \vdash A \le^+ A' \rightrightarrows *}{\Delta \vdash A \le A'}$$

Figure 14. Subtyping.

$\boxed{t \,/\, p \searrow \tau;\sigma}$ Pattern, $\boxed{e \,/\, q \searrow \tau;\sigma}$ destructor pattern, and $\boxed{\vec{e} \,/\, \vec{q} \searrow \tau;\sigma}$ pattern spine matching.

$$\frac{}{t \,/\, x \searrow \cdot; t/x} \qquad \frac{}{() \,/\, () \searrow \cdot;\cdot} \qquad \frac{t_1 \,/\, p_1 \searrow \tau_1;\sigma_1 \qquad t_2 \,/\, p_2 \searrow \tau_2;\sigma_2}{(t_1,t_2) \,/\, (p_1,p_2) \searrow \tau_1,\tau_2;\sigma_1,\sigma_2} \qquad \frac{t \,/\, p \searrow \tau;\sigma}{c\,t \,/\, c\,p \searrow \tau;\sigma}$$

$$\frac{t \,/\, p \searrow \tau;\sigma}{{}^{G}t \,/\, {}^{X}p \searrow G/X,\tau;\sigma} \qquad \frac{t \,/\, p \searrow \tau;\sigma}{{}^{G}t \,/\, {}^{\infty}p \searrow \tau;\sigma}$$

$$\frac{}{G \,/\, X \searrow G/X;\cdot} \qquad \frac{}{G \,/\, \infty \searrow \cdot;\cdot} \qquad \frac{}{.d \,/\, .d \searrow \cdot;\cdot} \qquad \frac{}{\cdot \,/\, \cdot \searrow \cdot;\cdot} \qquad \frac{e \,/\, q \searrow \tau;\sigma \qquad \vec{e} \,/\, \vec{q} \searrow \tau';\sigma'}{e\,\vec{e} \,/\, q\,\vec{q} \searrow \tau,\tau';\sigma,\sigma'}$$

$\boxed{t \mapsto t'}$ Weak head reduction.

$$\frac{\vec{e} \,/\, \vec{q} \searrow \tau;\sigma}{\lambda\{\vec{q} \to t\}\,\vec{e}\,\vec{e}\,' \mapsto t\tau\sigma\,\vec{e}\,'} \qquad \frac{\lambda D_k\,\vec{e} \mapsto t' \text{ for some } k}{\lambda\vec{D}\,\vec{e} \mapsto t'} \qquad \frac{\lambda\vec{D}\,\vec{e} \mapsto t'}{f\,\vec{e} \mapsto t'}\ (f{:}A = \vec{D}) \in \Sigma$$

$\boxed{t \longrightarrow t'}$ Reduction of terms, $\boxed{D \longrightarrow D'}$ clauses, and $\boxed{\vec{D} \longrightarrow \vec{D}'}$ definitions.

$$\frac{t \mapsto t'}{t \longrightarrow t'} \qquad \frac{t_1 \longrightarrow t_1'}{(t_1,t_2) \longrightarrow (t_1',t_2)} \qquad \frac{t_2 \longrightarrow t_2'}{(t_1,t_2) \longrightarrow (t_1,t_2')} \qquad \frac{t \longrightarrow t'}{c\,t \longrightarrow c\,t'} \qquad \frac{t \longrightarrow t'}{{}^{G}t \longrightarrow {}^{G}t'} \qquad \frac{r \longrightarrow r'}{r\,e \longrightarrow r'\,e} \qquad \frac{s \longrightarrow s'}{r\,s \longrightarrow r\,s'}$$

$$\frac{\vec{D} \longrightarrow \vec{D}'}{\lambda\vec{D} \longrightarrow \lambda\vec{D}'} \qquad \frac{t \longrightarrow t'}{\{\vec{q} \to t\} \longrightarrow \{\vec{q} \to t'\}} \qquad \frac{D \longrightarrow D'}{\vec{D}^1, D, \vec{D}^2 \longrightarrow \vec{D}^1, D', \vec{D}^2}$$

$\boxed{r \rhd \vec{r}}$ Term r is simulated by terms $\vec{r}$.

$$\frac{\forall \vec{e}, t.\ r\,\vec{e} \mapsto t \implies \exists k.\ r_k\,\vec{e} \mapsto t}{r \rhd \vec{r}}$$

Figure 15. Operational Semantics.