Inconsistent Type Systems

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Introduction

- System F [Girard 1971]
- 'A theory of types' (Type:Type) [Martin-Löf 1971]
- Inconsistency of system U [Girard 1971]
  Inconsistency of Type:Types comes as a consequence
- Inconsistency of System U^- [Coquand 1991]
- Simplification of Girard's paradox (system U^-) [Hurkens 1995]
- Russell's paradox in systems U/U^- [Miquel 2000]

System $\lambda*$ (Type:Type) [Martin-Löf 71]

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Terms	M, N, T, U	$::=$	$x \mid \lambda x : T \,.\, M \mid M N \mid \mathsf{Type} \mid \Pi x : T \,.\, U$
Contexts	Γ, Δ	$::=$	$[] \mid \Gamma,\ x : T$

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- Logically inconsistent: closed term of type $\bot \equiv \Pi X : \text{Type}\,.\,X$
- Non (weakly) normalising, since:

Fact: Closed terms of type $\bot \equiv \Pi X : \text{Type}\,.\,X$ have no head normal form

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   - Typing belongs to the meta-language
     $\Rightarrow$ Precondition for an expression to be well-formed
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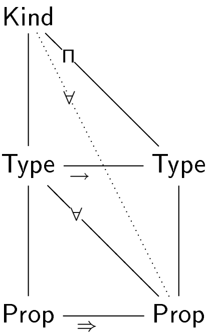
No cycle in the sorts (Prop : Type : Kind)...

... but two levels of impredicativity (Prop and Type)

Systems U and U^-



$U^- =$ copy of F glued on top of $F\omega$

$U =$ system $U^- +$ (Kind, Prop)-quantification

- Kind = sort for kinds
- Type = sort for constructors
- Prop = sort for proof-terms

Both Type and Prop are impredicative

Higher-level is isomorphic to F:
Type inference/checking is decidable

$$\begin{array}{rcl} \mathcal{S} & = & \{\text{Prop}, \text{Type}, \text{Kind}\} \\ \mathcal{A} & = & \{(\text{Prop} : \text{Type}), (\text{Type} : \text{Kind})\} \\ \mathcal{R} & = & \{\underbrace{(\text{Prop} : \text{Prop}), (\text{Type} : \text{Prop}), (\text{Type}, \text{Type}), (\text{Kind}, \text{Type})}_{\text{system } U^-}, \underbrace{(\text{Kind}, \text{Prop})}_{U \text{ only}}\} \end{array}$$

From system $F\omega$

$$\mathcal{S} = \text{Prop}, \quad \text{Type}$$
$$\mathcal{A} = \text{Prop} : \text{Type}$$
$$\mathcal{R} = (\text{Prop}, \text{Prop}), \quad (\text{Type}, \text{Prop}), \quad (\text{Type}, \text{Type})$$

Kinds	τ, σ	$::=$	Prop	
		$\mid$	$\tau \rightarrow \sigma$	(Type, Type)
Constructors	M, N	$::=$	ξ	
		$\mid$	$\lambda x : \tau . M \quad \mid \quad MN$	(Type, Type)
		$\mid$	$M \Rightarrow N$	(Prop, Prop)
		$\mid$	$\forall x : \tau . M$	(Type, Prop)
Proof-terms	t, u	$::=$	ξ	
		$\mid$	$\lambda \xi : M . t \quad \mid \quad tu$	(Prop, Prop)
		$\mid$	$\lambda x : \tau . t \quad \mid \quad tM$	(Type, Prop)

From system $F\omega$...

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Kinds	τ, σ	$::=$	$\text{Prop} \mid \alpha$	
		$\mid$	$\tau \to \sigma$	(Type, Type)
Constructors	M, N	$::=$	ξ	
		$\mid$	$\lambda x : \tau \,.\, M \mid MN$	(Type, Type)
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From system $F\omega$... to system U^-

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Kinds $\tau, \sigma ::= \text{Prop} \mid \alpha$

$\mid \tau \to \sigma$ (Type, Type)

$\mid \Pi\alpha : \text{Type} \,.\, \tau$ (Kind, Type)

Constructors $M, N ::= \xi$

$\mid \lambda x : \tau \,.\, M \mid MN$ (Type, Type)

$\mid \Lambda\alpha \,.\, M \mid M\tau$ (Kind, Type)

$\mid M \Rightarrow N$ (Prop, Prop)

$\mid \forall x : \tau \,.\, M$ (Type, Prop)

Proof-terms $t, u ::= \xi$

$\mid \lambda\xi : M \,.\, t \mid tu$ (Prop, Prop)

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& & | & \lambda x : \tau . M \quad | \quad MN & & (\text{Type}, \text{Type}) \\
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Examples

(Kind, Type)	$\Pi\alpha : \text{Type} \ldots$	Polymorphism in data types
(Type : Prop)	$\forall x : \tau \ldots$	Quantification over all objects (of a given type)
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Hurkens' paradox in system U^-

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For any kind τ : Type write: $\mathfrak{P}(\tau) \;:=\; \tau \to \mathrm{Prop}$

$\bot$	: Prop	:=	$\forall a : \mathrm{Prop} \,.\, a$
$\neg$	: Prop $\to$ Prop	:=	$\lambda a : \mathrm{Prop} \,.\, a \Rightarrow \bot$
$\mathbb{U}$	: Type	:=	$\Pi\alpha : \mathrm{Type} \,.\, \Big((\mathfrak{P}(\mathfrak{P}(\alpha)) \to \alpha) \to \mathfrak{P}(\mathfrak{P}(\alpha)) \Big)$
i	: $\mathfrak{P}(\mathfrak{P}(\mathbb{U})) \to \mathbb{U}$	:=	$\lambda q : \mathfrak{P}(\mathfrak{P}(\mathbb{U})) \,.\, \lambda\alpha : \mathrm{Type} \,.\, \lambda f : (\mathfrak{P}(\mathfrak{P}(\alpha)) \to \alpha) \,.$ $\lambda p : \mathfrak{P}(\alpha) \,.\; q \; (\lambda x : \mathbb{U} \,.\, p \; (f \; (x \; \alpha \; f)))$
j	: $\mathbb{U} \to \mathfrak{P}(\mathfrak{P}(\mathbb{U}))$	:=	$\lambda x : \mathbb{U} \,.\; x \; \mathbb{U} \; i$
Q	: $\mathfrak{P}(\mathfrak{P}(\mathbb{U}))$	:=	$\lambda p : \mathfrak{P}(\mathbb{U}) \,.\; \forall x : \mathbb{U} \,.\, (j \; x \; p \Rightarrow p \; x)$
C	: $\mathfrak{P}(\mathbb{U})$	:=	$\lambda y : \mathbb{U} \,.\; \neg\forall p : \mathfrak{P}(\mathbb{U}) \,.\, (j \; y \; p \Rightarrow p \; (i \; (j \; y)))$
B	: $\mathbb{U}$	:=	$i \; Q$
lem_1	: $Q \; C$	:=	$\lambda x : U \,.\, \lambda\xi^{jxC} \,.\, \lambda\zeta^{\forall p : \mathfrak{P}(\mathbb{U}) \,.\, (jxp \Rightarrow p(i(jx)))} \,.$ $\zeta \; C \; \xi \; (\lambda p : \mathfrak{P}(\mathbb{U}) \,.\, \zeta \; (\lambda y : \mathbb{U} \,.\, p \; (i \; (j \; y))))$
A	: Prop	:=	$\forall p : \mathfrak{P}(\mathbb{U}) \,.\; (Q \; p \Rightarrow p \; B)$
lem_2	: $\neg A$	:=	$\lambda\xi^{A} \,.\, \xi \; C \; \mathrm{lem}_1 \; (\lambda p : \mathfrak{P}(\mathbb{U}) \,.\, \xi \; (\lambda y : \mathbb{U} \,.\, p \; (i \; (j \; y))))$
lem_3	: A	:=	$\lambda p : \mathfrak{P}(\mathbb{U}) \,.\, \lambda\xi^{Qp} \,.\, \xi \; B \; (\lambda x : \mathbb{U} \,.\, \xi \; (i \; (j \; x)))$
paradox	: $\bot$	:=	$\mathrm{lem}_2 \; \mathrm{lem}_3$

Encoding sets as pointed graphs

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Pointed graph $=$ triple (X, A, a) where

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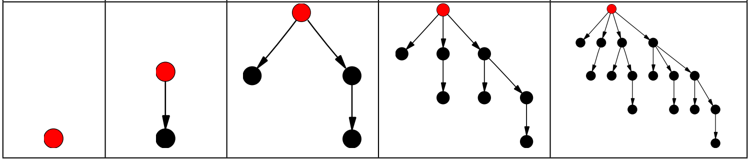
$0 = \varnothing$	$1 = \{0\}$	$2 = \{0; 1\}$	$3 = \{0; 1; 2\}$	$4 = \{0; 1; 2; 3\}$



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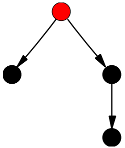

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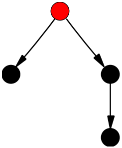

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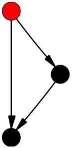

with sharing

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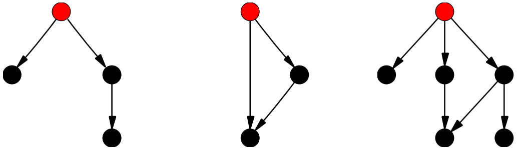
Example: the set $2 = \{\emptyset; \{\emptyset\}\}$



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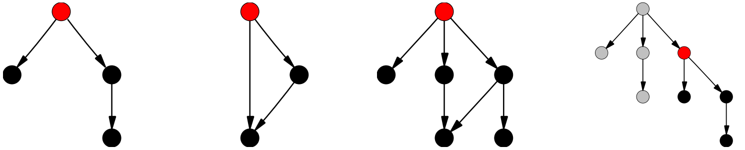

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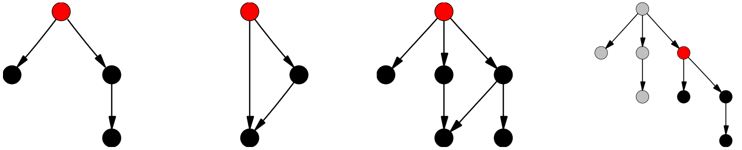

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+ Problems related to (possible) non well-foundedness

Extensional equality as bisimilarity

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$R : X \to Y \to \text{Prop}$ bisimulation between (X, A, a) and (Y, B, b) if:

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2. $\forall x : X \quad \forall y, y' : Y \quad \big(B(y', y) \wedge R(x, y) \;\Rightarrow\; \exists x' : X \; (R(x', y') \wedge A(x', x))\big)$
3. $R(a, b)$

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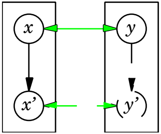
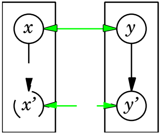


(2)



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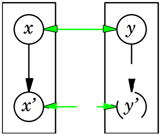
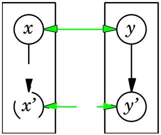
(1) 

(2) 

$(X, A, a) \approx (Y, B, b) \quad \equiv \quad \exists R : X \to Y \to \mathrm{Prop} \quad \text{bisimulation}$

Membership as shifted bisimilarity

$$(X, A, a) \in (Y, B, b) \quad \equiv \quad \exists b' : Y \ \ ((X, A, a) \approx (Y, B, b') \ \wedge \ B(b', b))$$

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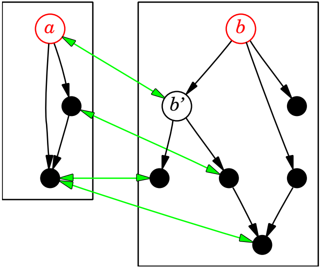

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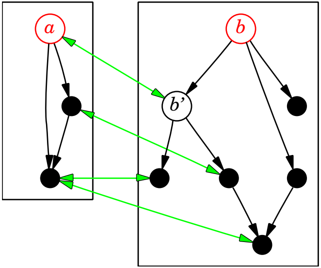

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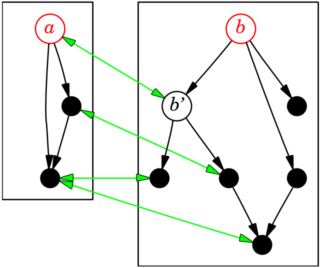

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Non well-founded sets

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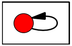 represents a set x such that $x = \{x\}$

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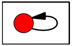
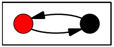
 represents a set x such that $x = \{x\}$

 represents a set y such that $y = \{z\}$ and $z = \{y\}$ for some z

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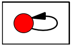
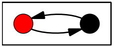
 represents a set x such that $x = \{x\}$

 represents a set y such that $y = \{z\}$ and $z = \{y\}$ for some z

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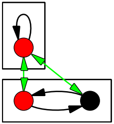

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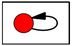
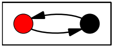
 represents a set x such that $x = \{x\}$

 represents a set y such that $y = \{z\}$ and $z = \{y\}$ for some z

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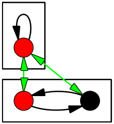

Sets as pointed graphs $+$ Equality as a bisimulation

$\Rightarrow$ Interprets the Anti-Foundation Axiom (*AFA*) [P. Aczel]

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$$\text{Let} \quad U \quad := \quad \Big(\Pi\, T : \text{Type}\,.\,(T \to T \to \text{Prop}) \to T \to \text{Prop}\Big) \to \text{Prop}$$

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- The map i is not surjective:

$$r : U \;=\; \lambda f \,.\, \bot \quad \text{is outside the codomain of } i$$

Translating equivalence and membership on U

$$u \approx v \quad := \quad \exists X, A, a \quad \exists Y, B, b \quad \big(u = i(X, A, a) \ \wedge \ v = i(Y, B, b) \ \wedge \ (X, A, a) \approx (Y, B, b)\big)$$

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- Exists some object $r : U$ such that $\neg\mathrm{set}(r)$

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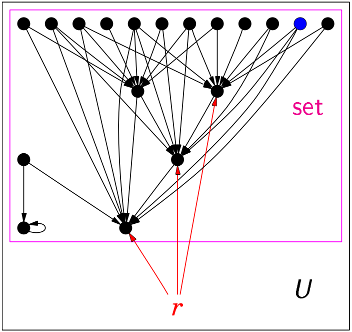

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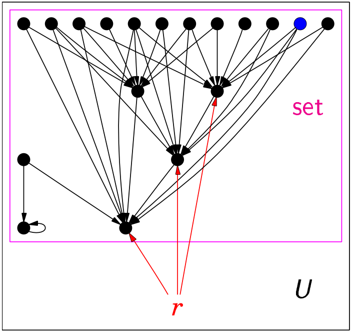

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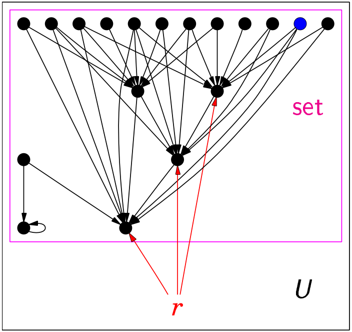

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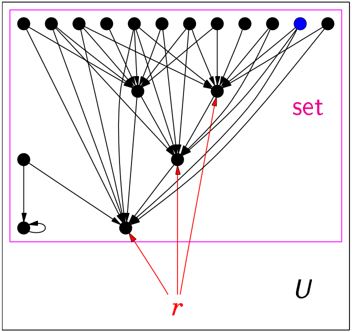

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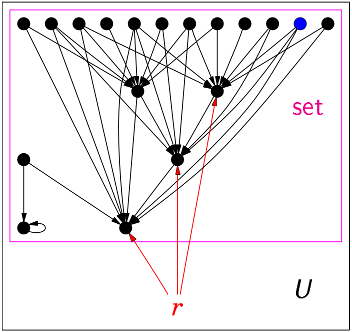

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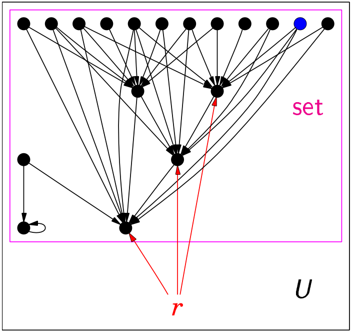

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Fact (Unbounded comprehension)

$\forall u : U \,.\, (u \in i(U, R_P, r) \Leftrightarrow P(u))$ (if P is extensional)

Cantor-Frege's set theory in systems U/U^{-}

Type U $+$ two relations $\approx$ and $\in$

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Remark: The formalization has been presented in system U

If we only consider pointed graphs based on $X = U$, we can drop the $(\text{Kind}, \text{Prop})$-quantification, thus restricting to system U^-

Why is system U^- inconsistent?

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$$
\begin{array}{llcll}
\textbf{Kinds} & \tau, \sigma & ::= & \mathsf{Prop} \quad | \quad \alpha & \\
 & & | & \tau \longrightarrow \sigma & (\mathsf{Type},\ \mathsf{Type}) \\
 & & | & \Pi\alpha : \mathsf{Type} \,.\, \tau & (\mathsf{Kind},\ \mathsf{Type}) \\
\textbf{Constructors} & M, N & ::= & \xi & \\
 & & | & \lambda x : \tau \,.\, M \quad | \quad MN & (\mathsf{Type},\ \mathsf{Type}) \\
 & & | & \Lambda\alpha \,.\, M \quad | \quad M\tau & (\mathsf{Kind},\ \mathsf{Type}) \\
 & & | & M \Rightarrow N & (\mathsf{Prop},\ \mathsf{Prop}) \\
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\textbf{Proof-terms} & t, u & ::= & \xi & \\
 & & | & \lambda x : M \,.\, t \quad | \quad tu & (\mathsf{Prop},\ \mathsf{Prop}) \\
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		$\mid$	$\Pi\alpha : \mathrm{Type} \,.\, \tau$	(Kind, Type)
Constructors	M, N	$::=$	ξ	
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		$\mid$	$\Lambda\alpha \,.\, M \mid M\tau$	(Kind, Type)
		$\mid$	$M \Rightarrow N$	(Prop, Prop)
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		$\mid$	$\tau \longrightarrow \sigma$	(Type, Type)
		$\mid$	$\Pi\alpha : \mathsf{Type} \,.\, \tau$	(Kind, Type)
Constructors	M, N	$::=$	ξ	
		$\mid$	$\lambda x : \tau \,.\, M \mid MN$	(Type, Type)
		$\mid$	$\Lambda\alpha \,.\, M \mid M\tau$	(Kind, Type)
		$\mid$	$M \Rightarrow N$	(Prop, Prop)
		$\mid$	$\forall x : \tau \,.\, M$	(Type, Prop)
Proof-terms	t, u	$::=$	ξ	
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$\Rightarrow$ (Kind, Type)-impredicativity is not parametric
i.e. cannot be reduced to an intersection