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Type Systems

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Lecture 3: Extensions of λHOL; the λ-cube; Pure Type Systems

(axiom) $\vdash \mathsf{Prop} : \mathsf{Type}$ $\qquad \vdash \mathsf{Type} : \mathsf{Type}'$

(var) $\dfrac{\Gamma \vdash A : s}{\Gamma, x{:}A \vdash x : A}$ (weak) $\dfrac{\Gamma \vdash A : s \quad \Gamma \vdash M : C}{\Gamma, x{:}A \vdash M : C}$

(Π) $\dfrac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2}$ if $(s_1, s_2) \in \{$ (Type, Type), (Prop, Prop), (Type, Prop) $\}$

(λ) $\dfrac{\Gamma, x{:}A \vdash M : B \quad \Gamma \vdash \Pi x{:}A.B : s}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B}$

(app) $\dfrac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B[N/x]}$

(conv) $\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma \vdash M : B}$ if $A =_\beta B$

λHOL contains $\lambda 2$ and $\lambda{\to}$.

(Π) $\dfrac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2}$ if $(s_1, s_2) \in \{$ (Type, Type), (Prop, Prop), (Type, Prop) $\}$

This rule allows to form

- $\to$-types on the Type-level (one copie of $\lambda{\to}$)
- $\to$-types on the Prop-level (second copie of $\lambda{\to}$)
- $\Pi\alpha$:Prop.$\alpha{\to}\alpha$: polymorphic types on the Prop-level (one copie of $\lambda 2$)

(Π) $\dfrac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2}$ if $(s_1, s_2) \in \{$ (Type, Type), (Prop, Prop), (Type, Prop) $\}$

Why not extend λHOL to include

- Higher order logic over polymorphic domains?
  like $\Pi A : \mathsf{Type}.A{\to}A$
- Quantification over all domains?
  like in $\Pi A : \mathsf{Type}.\Pi P{:}A{\to}\mathsf{Prop}.\Pi x{:}A.P\,x{\to}P\,x$

$$(\Pi)\ \frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2}\ \text{if } (s_1, s_2) \in \{\ (\text{Type}, \text{Type}),\ (\text{Prop}, \text{Prop}), (\text{Type}, \text{Prop})\ \}$$

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This can easily be done by allowing in the Π-rule

- $(s_1, s_2) \in \{\ (\text{Type}', \text{Type})\ \}$ to obtain higher order logic over polymorphic domains $\rightsquigarrow$ system λU^-
- $(s_1, s_2) \in \{\ (\text{Type}', \text{Prop})\ \}$ to allow quantification over all domains $\rightsquigarrow$ system λU

Problem:

- λU (λHOL + (Type',Type) and (Type',Prop)) is inconsistent (Girard)
- λU^- (λHOL + (Type',Type)) is inconsistent (Coquand, Hurkens)

NB λHOL + (Type',Prop) is consistent.

Problem:

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NB λHOL + (Type',Prop) is consistent.

Implications

- λU^- can't be used as a logic.
- In λU^-, there is a closed term M with $\vdash M : \bot$
- This M can not be in normal form (by some syntactic reasoning)
- So, λU^- is not SN

Type Checking in λU^- is still decidable:
All types (terms of type Prop, Type or Type$'$) are strongly normalizing

$$\begin{aligned}\text{Type}_\Gamma(MN) = \ & \text{if } \text{Type}_\Gamma(M) = C \text{ and } \text{Type}_\Gamma(N) = D \\ & \text{then} \quad \text{if } C \twoheadrightarrow_\beta \Pi x{:}A.B \text{ and } A =_\beta D \\ & \qquad \text{then } B[N/x] \text{ else 'false'} \\ & \text{else} \quad \text{'false'},\end{aligned}$$

In the type synthesis algorithm we only check equality of types

Variations on the rules of λHOL:

- There are many type system with (slightly) different rules
- Many (proofs of) properties are similar
- Plan: Study these type systems in one general framework:
  - The cube of typed λ-calculi (Barendregt)
  - Pure Type Systems (Terlouw, Berardi)

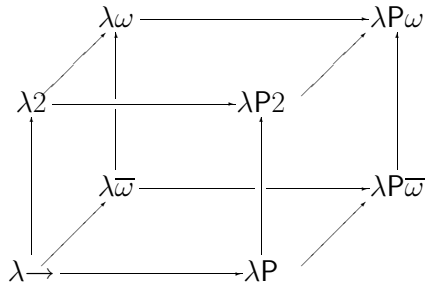
The cube of typed λ-calculi: (forget about Type$'$ for the moment)
Vary on all possible combinations for

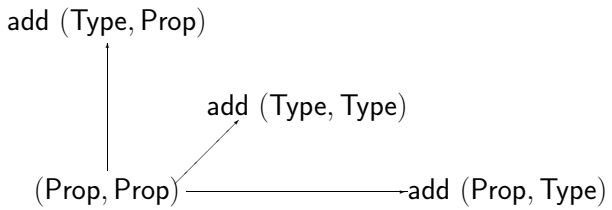
$\mathcal{R} \subseteq \{$ (Prop, Prop), (Type, Prop), (Type, Type), (Prop, Type) $\}$

in the Π-rule:

$$(\Pi)\ \frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2} \text{ if } (s_1, s_2) \in \mathcal{R}$$

We take (Prop, Prop) in every $\mathcal{R}$





$$(\Pi)\ \frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_2} \text{ if } (s_1, s_2) \in \mathcal{R}$$

System	$\mathcal{R}$			
$\lambda\!\!\rightarrow$	(Prop, Prop)			
$\lambda 2$ (system F)	(Prop, Prop)	(Type, Prop)		
λP (LF)	(Prop, Prop)		(Prop, Type)	
$\lambda\overline{\omega}$	(Prop, Prop)			(Type, Type)
λP2	(Prop, Prop)	(Type, Prop)	(Prop, Type)	
$\lambda\omega$ (system Fω)	(Prop, Prop)	(Type, Prop)		(Type, Type)
$\lambda P\overline{\omega}$	(Prop, Prop)		(Prop, Type)	(Type, Type)
λPω (CC)	(Prop, Prop)	(Type, Prop)	(Prop, Type)	(Type, Type)

$\lambda\!\!\rightarrow$ in this presentation is equivalent to $\lambda\!\!\rightarrow$ in the way we've presented before. Similarly for $\lambda 2$, λP, ...

This cube also gives a fine structure for the

Calculus of Constructions, CC (Coquand and Huet)

CC has:

- Polymorphic data types on the Prop-level,
  e.g. $\Pi\alpha{:}\mathsf{Prop}.\alpha{\to}(\alpha{\to}\alpha){\to}\alpha$.
- Predicate domains on the Type-level,
  e.g. $N{\to}N{\to}\mathsf{Prop}$
- Logic on the Prop-level,
  e.g. $\varphi \wedge \psi := \Pi\alpha{:}\mathsf{Prop}.(\varphi{\to}\psi{\to}\alpha){\to}\alpha$.
- Universal quantification (first and higher order),
  e.g. $\Pi P{:}N{\to}\mathsf{Prop}.\Pi x{:}N.Px{\to}Px$.

One can do higher order predicate logic in CC, in a slightly unusual way:

- 'propositions' and first order 'sets' are both of type Prop
- propositions and sets are completely mixed

Is it faithful to do higher order predicate logic in CC??

Answer: No!
There are non-provable formulas of $\mathbf{HOL}$ that become inhabited in CC

Consider extensionality of propostions:

$$\mathsf{EXT} := \forall\alpha,\beta{:}\mathsf{prop}.(\alpha \Leftrightarrow \beta) \Rightarrow (\alpha =_{\mathsf{prop}} \beta)$$

In CC, this becomes $\Pi\alpha,\beta{:}\mathsf{Prop}.(\alpha \leftrightarrow \beta) \to (\alpha =_{\mathsf{Prop}} \beta)$

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Suppose two base domains A and B and constants $a : A$, $b : B$.
In $\mathbf{HOL}$, the following formulas are consistent.

- $\varphi := \forall x{:}A.x = a$, $\psi := \forall x{:}B.\exists y{:}B.x \neq y$

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In **HOL**, the following formulas are consistent.

- $\varphi := \forall x{:}A.x = a$, $\psi := \forall x{:}B.\exists y{:}B.x \neq y$

But in CC, EXT also applies to the base sets A and B.
$A \leftrightarrow B$ (both are non-empty) so $A =_{\text{Prop}} B$
so property ψ (of B) also applies to A
so $\forall x{:}A.\exists y{:}A.x \neq y$
contradicting φ
So, in CC, φ and ψ are inconsistent

We have to be careful when doing higher order logic in CC.

Or: we may try to improve on this: taking the sets and the propositions apart:
System λPREDω:

- Sorts: Prop, Set, Type^p, Type^s
- Axioms for these sorts: Prop : Type^p, Set : Type^s
- Rules R:
  - (Prop, Prop): implication
  - (Set, Prop): first order quantification
  - (Type^p, Prop): higher order quantification
  - (Set, Set): function types
  - (Set, Type^p): predicate types
  - (Type^p, Type^p): higher order types

Pure Type Systems
Determined by a triple $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ with

- $\mathcal{S}$ the set of sorts
- $\mathcal{A}$ the set of axioms, $\mathcal{A} \subseteq \mathcal{S} \times \mathcal{S}$
- $\mathcal{R}$ the set of rules, $\mathcal{R} \subseteq \mathcal{S} \times \mathcal{S} \times \mathcal{S}$

If $s_2 = s_3$ in $(s_1, s_2, s_3) \in \mathcal{R}$, we write $(s_1, s_2) \in \mathcal{R}$.

pseudoterms:

$$\text{T} ::= \mathcal{S} \mid \text{Var} \mid (\Pi\text{Var:T.T}) \mid (\lambda\text{Var:T.T}) \mid \text{TT}.$$

(sort) $\vdash s_1 : s_2$ if $(s_1, s_2) \in \mathcal{A}$ (var) $\dfrac{\Gamma \vdash A : s}{\Gamma, x{:}A \vdash x : A}$ if $x \notin \Gamma$

(weak) $\dfrac{\Gamma \vdash A : s \quad \Gamma \vdash M : C}{\Gamma, x{:}A \vdash M : C}$ if $x \notin \Gamma$

(Π) $\dfrac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash \Pi x{:}A.B : s_3}$ if $(s_1, s_2, s_3) \in \mathcal{R}$

(λ) $\dfrac{\Gamma, x{:}A \vdash M : B \quad \Gamma \vdash \Pi x{:}A.B : s}{\Gamma \vdash \lambda x{:}A.M : \Pi x{:}A.B}$

(app) $\dfrac{\Gamma \vdash M : \Pi x{:}A.B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B[N/x]}$

(conv$_\beta$) $\dfrac{\Gamma \vdash M : A \quad \Gamma \vdash B : s}{\Gamma \vdash M : B}$ $A =_\beta B$

Examples of PTSs

λPREDω
$\mathcal{S}$ Set, Type^s, Prop, Type $\mathcal{A}$ Set : Type^s, Prop : Type $\mathcal{R}$ (Set, Set), (Set, Type), (Type, Type), (Prop, Prop), (Set, Prop), (Type, Prop)

CC
$\mathcal{S}$ Prop, Type $\mathcal{A}$ Prop : Type $\mathcal{R}$ (Prop, Prop), (Prop, Type), (Type, Prop), (Type, Type)

λHOL
$\mathcal{S}$ Prop, Type, Type$'$ $\mathcal{A}$ Prop : Type, Type : Type$'$ $\mathcal{R}$ (Prop, Prop), (Type, Type), (Type, Prop)

λU
$\mathcal{S}$ Prop, Type, Type$'$ $\mathcal{A}$ Prop : Type, Type : Type$'$ $\mathcal{R}$ (Prop, Prop), (Type, Type), (Type$'$, Type), (Type$'$, Prop), (Type, Prop)

$\lambda\star$
$\mathcal{S}$ $\star$ $\mathcal{A}$ $\star : \star$ $\mathcal{R}$ $(\star, \star)$

A PTS-morphism from $\lambda(\mathcal{S}, \mathcal{A}, \mathcal{R})$ to $\lambda(\mathcal{S}', \mathcal{A}', \mathcal{R}')$ is an $f : \mathcal{S} \to \mathcal{S}'$ that preserves the axioms and rules:

- if $(s_1, s_2) \in \mathcal{A}$ then $(f(s_1), f(s_2)) \in \mathcal{A}'$
- if $(s_1, s_2, s_3) \in \mathcal{R}$ then $(f(s_1), f(s_2), f(s_3)) \in \mathcal{R}'$

f extends to pseudoterms and contexts
Proposition:

If $\Gamma \vdash M : A$ then $f(\Gamma) \vdash f(M) : f(A)$

Examples:

- $f : \lambda(\mathcal{S}, \mathcal{A}, \mathcal{R}) \to \lambda\star$, $f(s) := \star$. ("initial" PTS)
- $g : \lambda$PRED$\omega \to$ CC, $g(\text{Prop}) = g(\text{Set}) := \text{Prop}$, $g(\text{Type}^p) = g(\text{Type}^s) := \text{Type}$.

Corollary: SN for CC $\Rightarrow$ SN for λPREDω

There are now two type systems for higher order predicate logic: λPREDω and λHOL.

λPREDω
$\mathcal{S}$ Set, Type^s, Prop, Type $\mathcal{A}$ Set : Type^s, Prop : Type $\mathcal{R}$ (Set, Set), (Set, Type), (Type, Type), (Prop, Prop), (Set, Prop), (Type, Prop)

λHOL
$\mathcal{S}$ Prop, Type, Type$'$ $\mathcal{A}$ Prop : Type, Type : Type$'$ $\mathcal{R}$ (Prop, Prop), (Type, Type), (Type, Prop)

They are equivalent:
The PTS-morphism $h : \lambda$PRED$\omega \to \lambda$HOL, given by

$$h(\text{Prop}) := \text{Prop} \quad h(\text{Set}) := \text{Type}$$
$$h(\text{Type}^p) := \text{Type} \quad h(\text{Type}^s) := \text{Type}'$$

constitutes an isomorphism between the derivable sequents.

CC^∞
$\mathcal{S}$ Prop, $\{\text{Type}_i\}_{i\in\mathbb{N}}$
$\mathcal{A}$ Prop : Type, $\text{Type}_i : \text{Type}_{i+1}$
$\mathcal{R}$ (Prop, Prop), (Prop, Type_i), (Type_i, Prop)
(Type_i, Type_j, $\text{Type}_{\max(i,j)}$)

NB: ($\text{Type}_{i+1}\text{Type}_i, \text{Type}_i$)) would be inconsistent.

CC^∞
$\mathcal{S}$ Prop, $\{\text{Type}_i\}_{i\in\mathbb{N}}$
$\mathcal{A}$ Prop : Type, $\text{Type}_i : \text{Type}_{i+1}$
$\mathcal{R}$ (Prop, Prop), (Prop, Type_i), (Type_i, Prop)
(Type_i, Type_j, $\text{Type}_{\max(i,j)}$)

NB: ($\text{Type}_{i+1}\text{Type}_i, \text{Type}_i$)) would be inconsistent.
The Extended Calculus of Constructions has in addition

- Cumulativity: $\text{Prop} \subseteq \text{Type}_0 \subseteq \text{Type}_1 \subseteq \ldots$
- Σ-types:

$$\frac{\Gamma \vdash A : \text{Prop} \quad \Gamma, x{:}A \vdash B : \text{Prop}}{\Gamma \vdash \Sigma x{:}A.B : \text{Prop}} \qquad \frac{\Gamma \vdash A : \text{Type}_i \quad \Gamma, x{:}A \vdash B : \text{Type}_j}{\Gamma \vdash \Sigma x{:}A.B : \text{Type}_{\max(i,j)}}$$

NB: We have $\Pi A{:}\text{Type}_i.\varphi$: Prop, but not $\Sigma A{:}\text{Type}_i.\varphi$: Prop
NB: Coq has in addition Set : Type and rules (Set, Set), (Type_i, Set), (Set, Prop) and inductive types.

What is the use of the abstract framework of PTSs?

- Present (the kernel of) systems in a uniform way
- Compare systems (e.g. λHOL, λPREDω, CC) within one framework
- Prove properties for many systems at once.

Properties of PTSs.

- Uniqueness of types
  If $\Gamma \vdash M : A$ and $\Gamma \vdash M : B$, then $A =_\beta B$.
  Holds if $\mathcal{A} \subseteq \mathcal{S} \times \mathcal{S}$ and $\mathcal{R} \subseteq (\mathcal{S} \times \mathcal{S}) \times \mathcal{S}$ are functions.
- Subject Reduction
  If $\Gamma \vdash M : A$ and $M \longrightarrow_\beta N$, then $\Gamma \vdash N : A$.
- Substitution property
  If $\Gamma, x : B, \Delta \vdash M : A$, $\Gamma \vdash P : B$, then $\Gamma, \Delta[P/x] \vdash M[P/x] : A[P/x]$.
- Thinning
  If $\Gamma \vdash M : A$ and $\Gamma \subseteq \Delta$, Δ well-formed, then $\Delta \vdash M : A$.
- Strengthening
  If $\Gamma, x : \tau, \Delta \vdash M : A$ and $x \notin \text{FV}(M, A, \Delta)$, then $\Gamma, \Delta \vdash M : A$.

Strong Normalization:

If $\Gamma \vdash M : A$, then all β-reductions from M terminate.

SN holds for some PTSs (all subsystems of CC,...), and for some not (λU^-, $\lambda\star$, ...).

SN for CC can be proved by a higher order extension of the saturated sets argument (for $\lambda 2$).