

Exercises in Agda

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August 11, 2005

In the library `/usr/local/share/summerschool/agda/SummerSchool105` (on the Summer School's computers) you will find a small library with some standard files. It is good to look around in the different files.

1. In the file with `Exercises/Logic.agda`, there are many basic exercises on the Curry-Howard correspondence between propositions and sets and in the file `Exercises/ListProps.agda` there is some exercises on lists
2. Formulate and prove in type theory

$$((\forall x : A)(\exists y : B)R x y) \rightarrow (\exists f : A \rightarrow B)(\forall x : A)R x (f x)$$

This is a possible formulation of the axiom of choice. It was stressed by Bishop that this form of axiom of choice is constructively valid.

3. In the file `Nat.agda` you can find the definition of natural numbers, `Nat`. Define the equality, `eqNat`, on `Nat` by double recursion and show that this equality is substitutive (if `eqNat x y` and `P x.`, then `P y`)
4. Define the set of well-founded trees (well-orderings), `W`: given a family of sets `B(x)` over a set `A`, the set `W A B` has one constructor `sup`, where `sup x f : W A B` if `x : A` and `f : B(x) → W A B`. Write the corresponding elimination rule in Agda. Prove that

$$W A B \rightarrow \neg(\forall x : A.B x)$$

that is, the two propositions `W A B` and `Π A B` are incompatible. Explain intuitively why

$$W A B \rightarrow \exists x : A.\neg(B x)$$

should not be provable in type theory.

5. Define the type

$$F A n = \underbrace{A \rightarrow \dots \rightarrow A}_n \rightarrow A$$

i.e. a function `F` that takes a set `A`, a natural number, `n`, and returns a set.

Use this type to define a tautology function, i.e. a function that takes as arguments a number n , a boolean function with n arguments, and returns *True* if and only if this boolean function is a tautology.

6. Let A be a set with a well-founded relation, $<$, on it. Show that if $f : \text{Nat} \rightarrow A$ then $\neg((\forall n : \text{Nat}) f (n + 1) < f n)$, i.e. there is no infinite decreasing sequence. Show that if $<$ is decidable then we have $(\exists n : \text{Nat}) \neg(f (n + 1) < f n)$
7. Let A be a set with a well-founded relation, $<$, on it (see `WellOrder.agda`). Prove that if A is inhabited we have

$$((\forall x : A)(P x \vee (f x < x))) \rightarrow (\exists x : A)P x$$

Explain how this proof corresponds to a “for-loop” program.