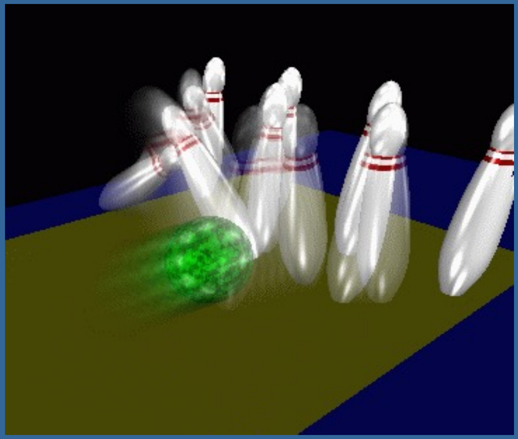
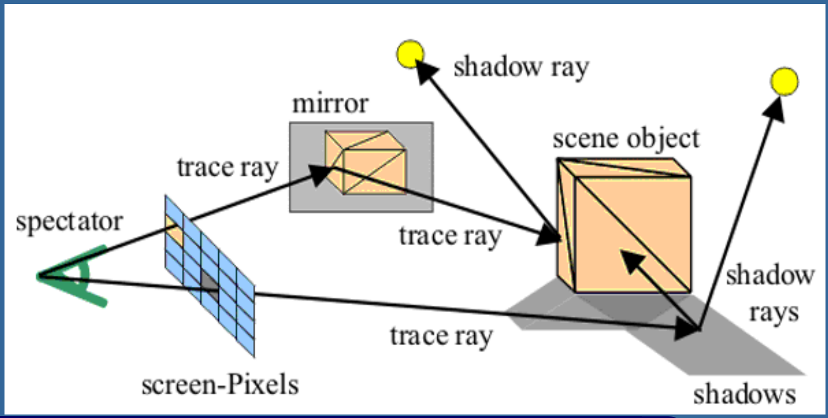
Intersection Testing
Chapter 16





Department of Computer Engineering

Chalmers University of Technology

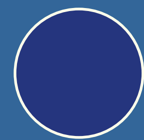
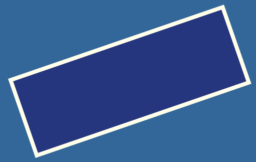
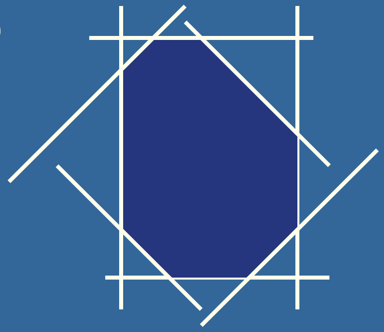
What for?

- A tool needed for the graphics people all the time…
- Very important components:
  - Need to make them fast!
- Finding if (and where) a ray hits an object
  - Picking
  - Ray tracing and global illumination
- For speed-up techniques
- Collision detection (treated in a later lecture)

Example

Midtown Madness 3, DICE

Some basic geometrical primitives

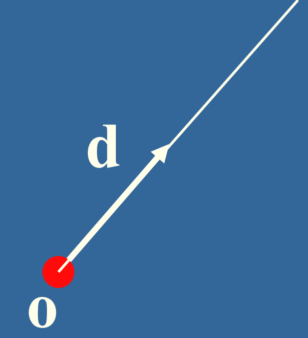
- Ray: 
- Sphere: 
- Box
  - Axis-aligned (AABB) 
  - Oriented (OBB) 
- *k*-DOP 

Four different techniques

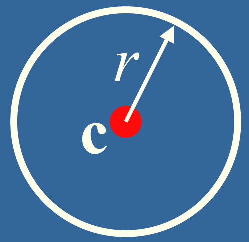
- Analytical
- Geometrical
- Separating axis theorem (SAT)
- Dynamic tests

- Given these, one can derive many tests quite easily
  - However, often tricks are needed to make them fast

Analytical: Ray/sphere test



- Sphere center: $\mathbf{c}$, and radius r
- Ray: $\mathbf{r}(t)=\mathbf{o}+t\mathbf{d}$
- Sphere formula: $\|\mathbf{p}-\mathbf{c}\|=r$
- Replace $\mathbf{p}$ by $\mathbf{r}(t)$, and square it:



$$(\mathbf{r}(t)-\mathbf{c})\cdot(\mathbf{r}(t)-\mathbf{c})-r^2=0$$

$$(\mathbf{o}+t\mathbf{d}-\mathbf{c})\cdot(\mathbf{o}+t\mathbf{d}-\mathbf{c})-r^2=0$$

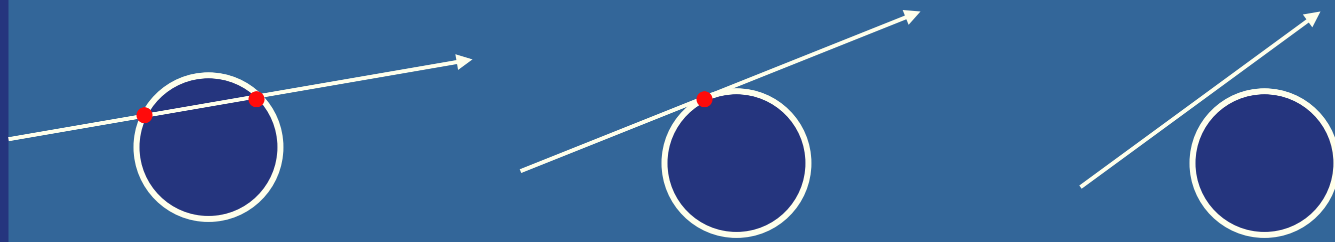
$$(t\mathbf{d}+(\mathbf{o}-\mathbf{c}))\cdot(t\mathbf{d}+(\mathbf{o}-\mathbf{c}))-r^2=0$$

$$(\mathbf{d}\cdot\mathbf{d})t^2+2((\mathbf{o}-\mathbf{c})\cdot\mathbf{d})t+(\mathbf{o}-\mathbf{c})\cdot(\mathbf{o}-\mathbf{c})-r^2=0$$

$$t^2+2((\mathbf{o}-\mathbf{c})\cdot\mathbf{d})t+(\mathbf{o}-\mathbf{c})\cdot(\mathbf{o}-\mathbf{c})-r^2=0 \qquad \|\mathbf{d}\|=1$$

Analytical, continued

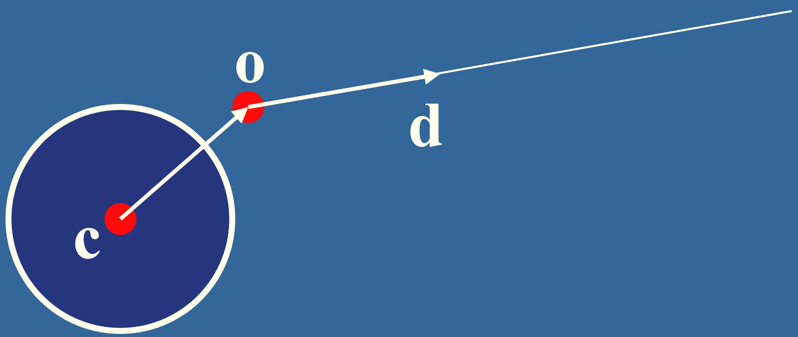
$$t^2 + 2((\mathbf{o}-\mathbf{c})\cdot\mathbf{d})t + (\mathbf{o}-\mathbf{c})\cdot(\mathbf{o}-\mathbf{c}) - r^2 = 0$$



- Be a little smart…

$$(\mathbf{o}-\mathbf{c})\cdot\mathbf{d} > 0 \ \ ?$$

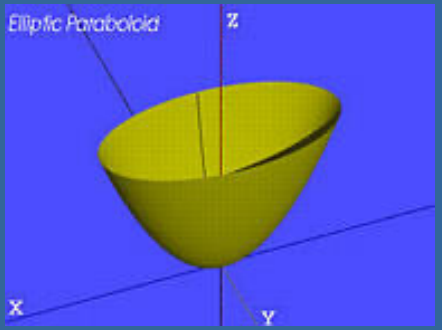
$$(\mathbf{o}-\mathbf{c})\cdot(\mathbf{o}-\mathbf{c}) - r^2 < 0 \ \ ?$$



- Such tests are called ”rejection tests”
- Other shapes: $p_x^2 + p_y^2 = r^2$

$$(p_x/a)^2 + (p_y/b)^2 + (p_z/c)^2 = 1$$

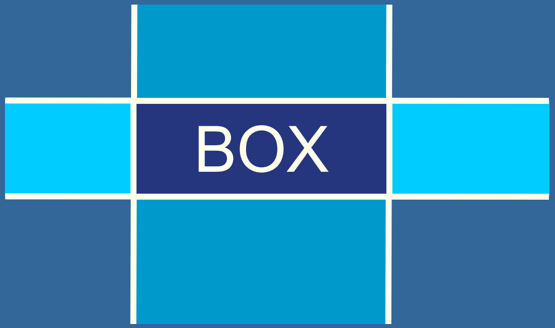
$$(p_x/a)^2 + (p_y/b)^2 - p_z = 0$$



Geometrical: Ray/Box Intersection

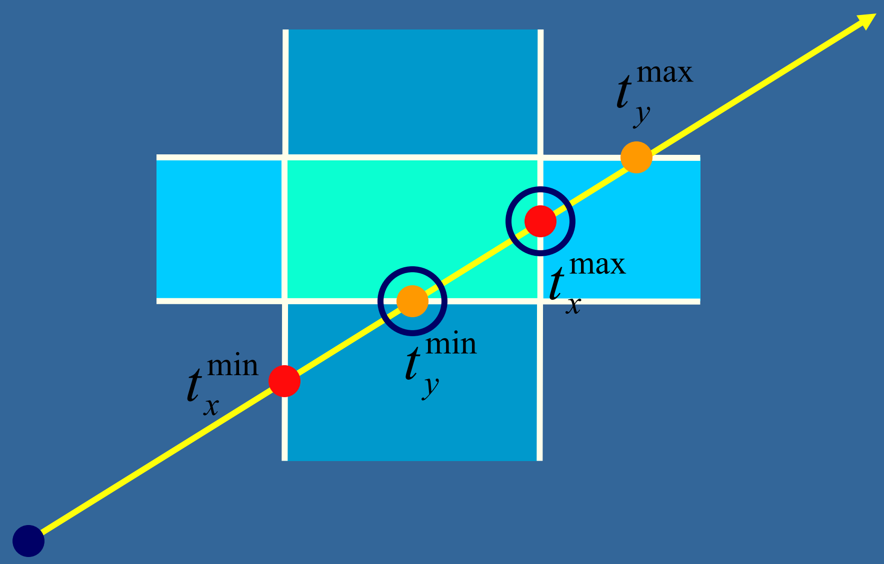
- Boxes and spheres often used as bounding volumes
- A slab is the volume between two parallell planes:

- A box is the logical intersection of three slabs (2 in 2D):



Geometrical: Ray/Box Intersection (2)

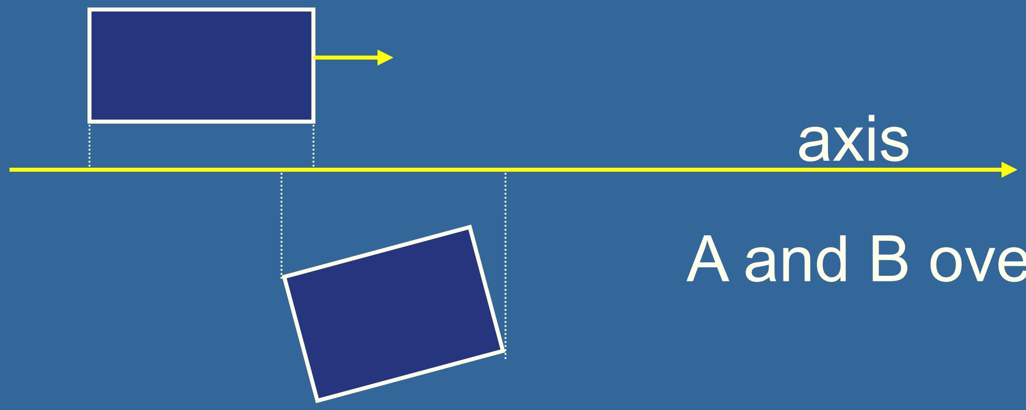
- Intersect the 2 planes of each slab with the ray



- Keep max of t^{min} and min of t^{max}
- If $t^{min} < t^{max}$ then we got an intersection
- Special case when ray parallell to slab

Separating Axis Theorem (SAT) Page 563 in book

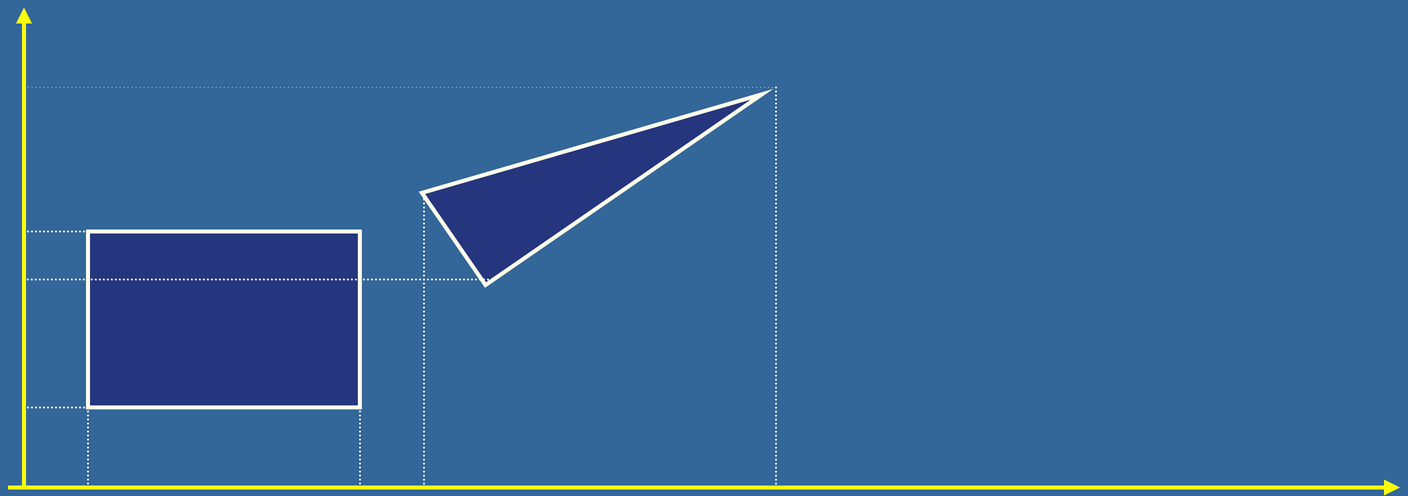
- Two convex polyhedrons, A and B, are disjoint if any of the following axes separate the objects' projections:
  - A face normal of A
  - A face normal of B
  - Any $edge_A$ cross $edge_B$



A and B overlaps on this axis

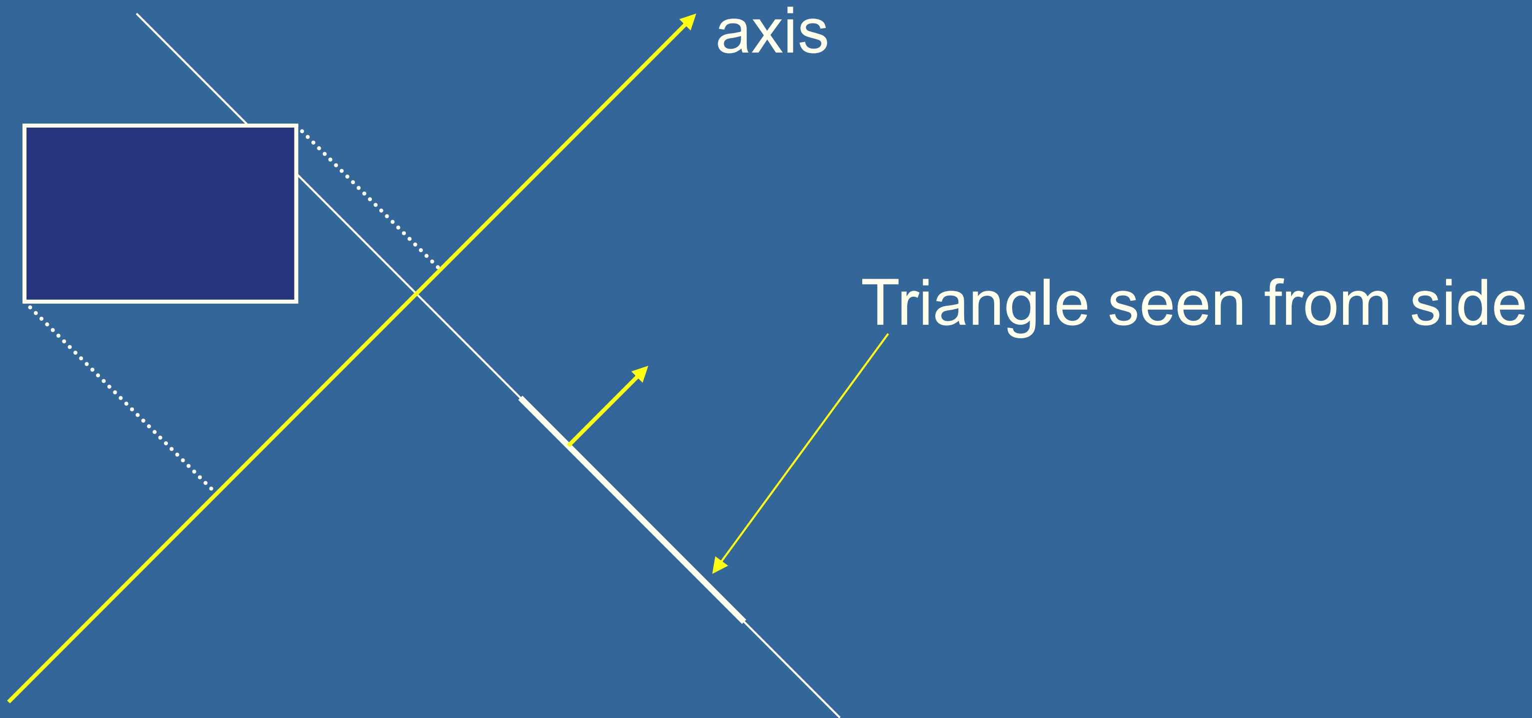
SAT example: Triangle/Box

- E.g an axis-aligned box and a triangle
- **1)** test the axes that are orthogonal to the faces of the box
- That is, x,y, and z



Triangle/Box with SAT (2)

- Assume that they overlapped on x,y,z
- Must continue testing
- **2)** Axis orthogonal to face of triangle



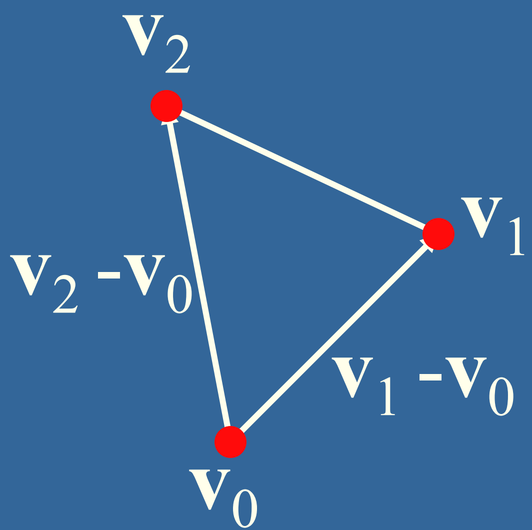
Triangle/Box with SAT (3)

- If still no separating axis has been found…
- **3)** Test axis: $\mathbf{t}=\mathbf{e}_{box} \times \mathbf{e}_{triangle}$
- Example:
  - x-axis from box: $\mathbf{e}_{box}=(1,0,0)$
  - $\mathbf{e}_{triangle}=\mathbf{v}_1-\mathbf{v}_0$
- Test all such combinations
- If there is at least one separating axis, then the objects do not collide
- Else they do overlap

Rules of Thumb for Intersection Testing

- Acceptance and rejection test
  - Try them early on to make a fast exit
- Postpone expensive calculations if possible
- Use dimension reduction
  - E.g. 3 one-dimensional tests instead of one complex 3D test, or 2D instead of 3D
- Share computations between objects if possible
- Timing!

Another analytical example: Ray/Triangle in detail



- Ray: $\mathbf{r}(t)=\mathbf{o}+t\mathbf{d}$
- Triangle vertices: $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$
- A point in the triangle:
- $\mathbf{t}(u,v)=\mathbf{v}_0+u(\mathbf{v}_1 - \mathbf{v}_0) + v(\mathbf{v}_2 - \mathbf{v}_0)=$
  $=(1-u-v)\mathbf{v}_0+u\mathbf{v}_1+v\mathbf{v}_2 \quad [u,v>=0,\ u+v<=1]$
- Set $\mathbf{t}(u,v)=\mathbf{r}(t)$, and solve!

$$\begin{pmatrix} | & | & | \\ -\mathbf{d} & \mathbf{v}_1-\mathbf{v}_0 & \mathbf{v}_2-\mathbf{v}_0 \\ | & | & | \end{pmatrix} \begin{pmatrix} t \\ u \\ v \end{pmatrix} = \begin{pmatrix} | \\ \mathbf{o}-\mathbf{v}_0 \\ | \end{pmatrix}$$

BONUS

Ray/Triangle (1)

$$\begin{pmatrix} | & | & | \\ -\mathbf{d} & \mathbf{v}_1-\mathbf{v}_0 & \mathbf{v}_2-\mathbf{v}_0 \\ | & | & | \end{pmatrix}\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \begin{pmatrix} | \\ \mathbf{o}-\mathbf{v}_0 \\ | \end{pmatrix}$$

- Solve for *t,u,v* using Cramer's rule for a system of *n* linear equations with *n* unknowns: $A\,x = b$

Cramer's rule:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix} \Rightarrow x = \frac{\begin{vmatrix} \boxed{\begin{matrix} j \\ k \\ l \end{matrix}} & \begin{matrix} b \\ e \\ h \end{matrix} & \begin{matrix} c \\ f \\ i \end{matrix} \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} \begin{matrix} a \\ d \\ g \end{matrix} & \boxed{\begin{matrix} j \\ k \\ l \end{matrix}} & \begin{matrix} c \\ f \\ i \end{matrix} \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}, \text{ and } z = \frac{\begin{vmatrix} \begin{matrix} a \\ d \\ g \end{matrix} & \begin{matrix} b \\ e \\ h \end{matrix} & \boxed{\begin{matrix} j \\ k \\ l \end{matrix}} \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}.$$

Simplify our equation system by setting:

$$\mathbf{e}_1 = \mathbf{v}_1 - \mathbf{v}_0 \qquad \mathbf{e}_2 = \mathbf{v}_2 - \mathbf{v}_0 \qquad \mathbf{s} = \mathbf{o} - \mathbf{v}_0$$

$$\Rightarrow \begin{pmatrix} | & | & | \\ -\mathbf{d} & \mathbf{e}_1 & \mathbf{e}_2 \\ | & | & | \end{pmatrix}\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \begin{pmatrix} | \\ \mathbf{s} \\ | \end{pmatrix}$$

Cramer's rule gives:

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \frac{1}{\det(-\mathbf{d},\mathbf{e}_1,\mathbf{e}_2)}\begin{pmatrix} \det(\mathbf{s},\mathbf{e}_1,\mathbf{e}_2) \\ \det(-\mathbf{d},\mathbf{s},\mathbf{e}_2) \\ \det(-\mathbf{d},\mathbf{e}_1,\mathbf{s}) \end{pmatrix}$$

BONUS

Ray/Triangle (2)

$$\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \frac{1}{\det(-\mathbf{d}, \mathbf{e}_1, \mathbf{e}_2)} \begin{pmatrix} \det(\mathbf{s}, \mathbf{e}_1, \mathbf{e}_2) \\ \det(-\mathbf{d}, \mathbf{s}, \mathbf{e}_2) \\ \det(-\mathbf{d}, \mathbf{e}_1, \mathbf{s}) \end{pmatrix}$$

- To compute determinant

Use this fact : $\det(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = -(\mathbf{a} \times \mathbf{c}) \cdot \mathbf{b}$

This gives: $$\begin{pmatrix} t \\ u \\ v \end{pmatrix} = \frac{1}{(\mathbf{d} \times \mathbf{e}_2) \cdot \mathbf{e}_1} \begin{pmatrix} (\mathbf{s} \times \mathbf{e}_1) \cdot \mathbf{e}_2 \\ (\mathbf{d} \times \mathbf{e}_2) \cdot \mathbf{s} \\ (\mathbf{s} \times \mathbf{e}_1) \cdot \mathbf{d} \end{pmatrix}$$

- Share factors to speed up computations:

$$\mathbf{p} = \mathbf{d} \times \mathbf{e}_2$$
$$a = \mathbf{p} \cdot \mathbf{e}_1$$
$$f = 1/a$$

- Compute as little as possible. Then test.

Compute $u = f(\mathbf{p} \cdot \mathbf{s})$

Then test valid bounds:

```
if (u<0 or u>1) exit;
```

$$\text{Plane:} \quad \pi : \mathbf{n} \cdot \mathbf{p} + d = 0$$

Point/Plane

- Insert a point $\mathbf{x}$ into plane equation:

$$f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x} + d$$

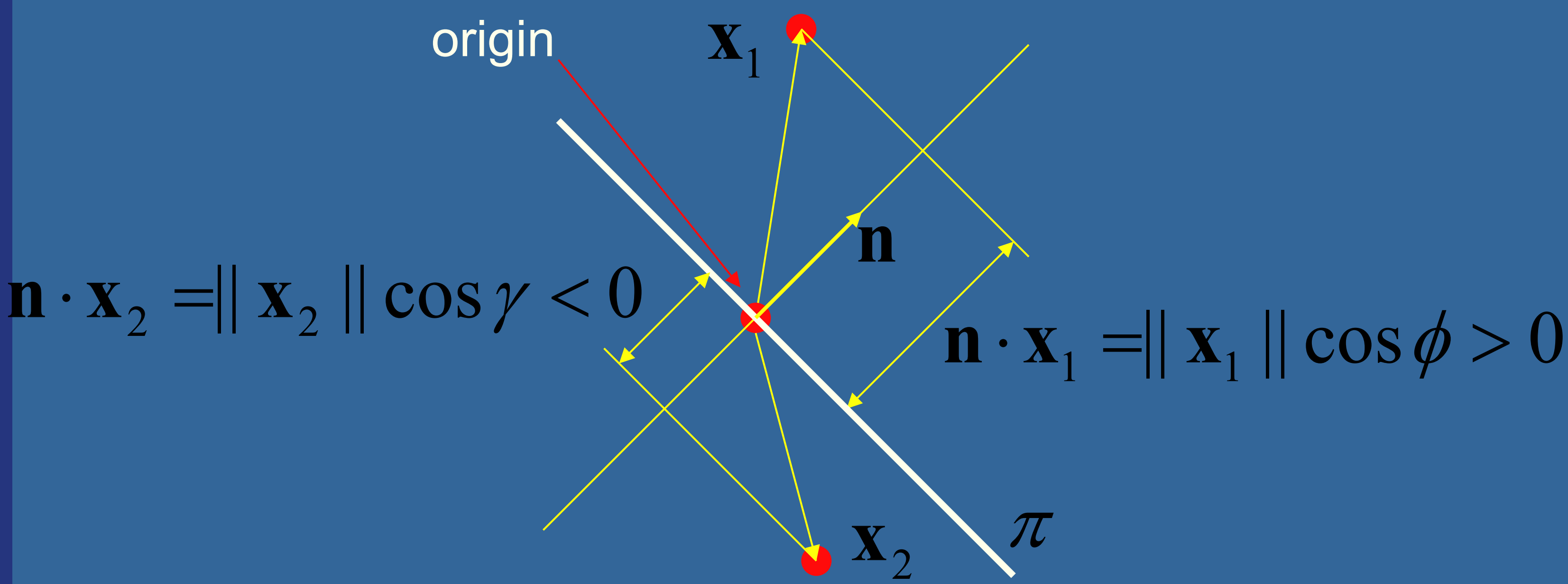
$$f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x} + d = 0 \quad \text{for } \mathbf{x}\text{'s on the plane}$$

Negative half space

$$f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x} + d < 0 \quad \text{for } \mathbf{x}\text{'s on one side of the plane}$$

Positive half space

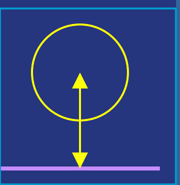
$$f(\mathbf{x}) = \mathbf{n} \cdot \mathbf{x} + d > 0 \quad \text{for } \mathbf{x}\text{'s on the other side}$$



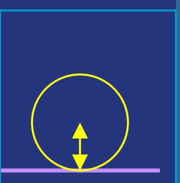
Sphere/Plane
Box/Plane

Plane: $\pi : \mathbf{n} \cdot \mathbf{p} + d = 0$
Sphere: $\mathbf{c}$ $\quad r$
$AABB$: $\mathbf{b}^{\min}$ $\quad \mathbf{b}^{\max}$

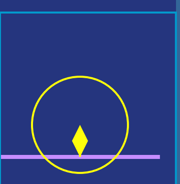
- Sphere: compute $f(\mathbf{c}) = \mathbf{n} \cdot \mathbf{c} + d$
- $f(\mathbf{c})$ is the signed distance ($\mathbf{n}$ normalized)



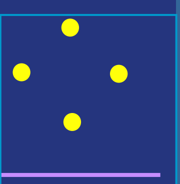
- $\text{abs}(f(\mathbf{c})) > \text{r}$ no collision



- $\text{abs}(f(\mathbf{c})) = \text{r}$ sphere touches the plane



- $\text{abs}(f(\mathbf{c})) < \text{r}$ sphere intersects plane

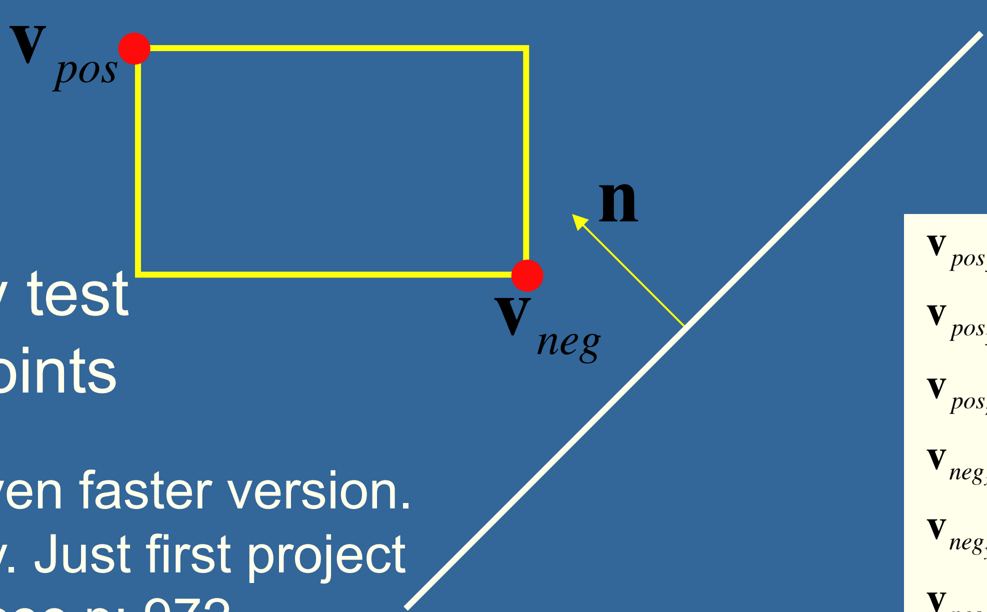


- Box: insert all 8 corners
- If all f's have the same sign, then all points are on the same side, and no collision

AABB/plane

$$\text{Plane}: \quad \pi : \mathbf{n}\cdot\mathbf{p} + d = 0$$
$$\text{Sphere}: \quad \mathbf{c} \quad r$$
$$\text{Box}: \quad \mathbf{b}^{\min} \quad \mathbf{b}^{\max}$$

- The smart way (shown in 2D)
- Find the two vertices that have the most positive and most negative value when tested againt the plane



Need only test the red points

$$\mathbf{v}_{pos_x} = (\mathbf{n}_x > 0)\,?\,\mathbf{b}_{\max_x} : \mathbf{b}_{\min_x}$$
$$\mathbf{v}_{pos_y} = (\mathbf{n}_y > 0)\,?\,\mathbf{b}_{\max_y} : \mathbf{b}_{\min_y}$$
$$\mathbf{v}_{pos_z} = (\mathbf{n}_z > 0)\,?\,\mathbf{b}_{\max z} : \mathbf{b}_{\min_z}$$
$$\mathbf{v}_{neg_x} = (\mathbf{n}_x < 0)\,?\,\mathbf{b}_{\max_x} : \mathbf{b}_{\min_x}$$
$$\mathbf{v}_{neg_y} = (\mathbf{n}_y < 0)\,?\,\mathbf{b}_{\max_y} : \mathbf{b}_{\min_y}$$
$$\mathbf{v}_{neg_z} = (\mathbf{n}_z < 0)\,?\,\mathbf{b}_{\max z} : \mathbf{b}_{\min_z}$$

See page 970 for even faster version.
OBB almost as easy. Just first project **n** on OBB's axes – see p: 972

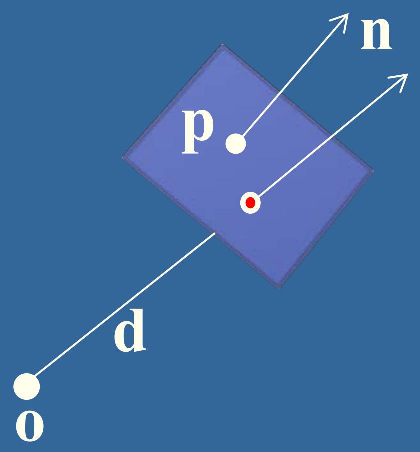
Ray/Plane Intersections

- Ray: $r(t)=\mathbf{o}+t\mathbf{d}$
- Plane: $\mathbf{n}\bullet\mathbf{x} + d = 0$; $(d=-\mathbf{n}\bullet\mathbf{p})$
- Set $\mathbf{x}=r(t)$:

$$\mathbf{n}\bullet(\mathbf{o}+t\mathbf{d}) + d = 0$$

$$\mathbf{n}\bullet\mathbf{o}+t(\mathbf{n}\bullet\mathbf{d}) + d = 0$$

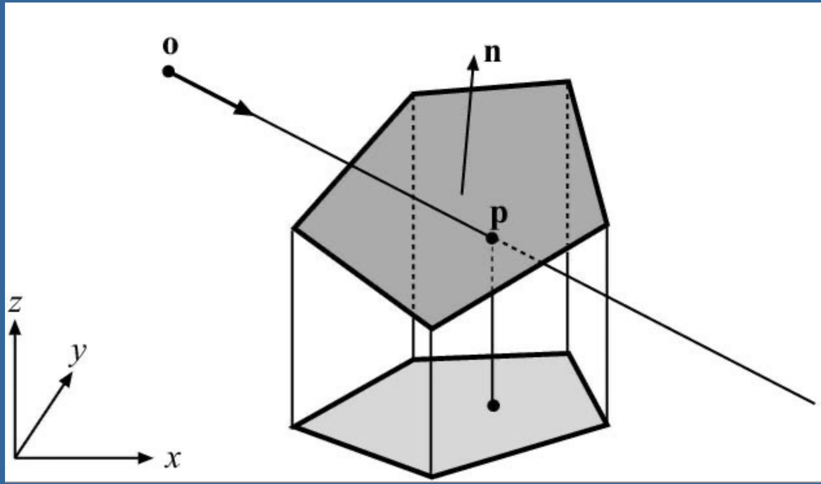
$$t = (-d - \mathbf{n}\bullet\mathbf{o}) / (\mathbf{n}\bullet\mathbf{d})$$

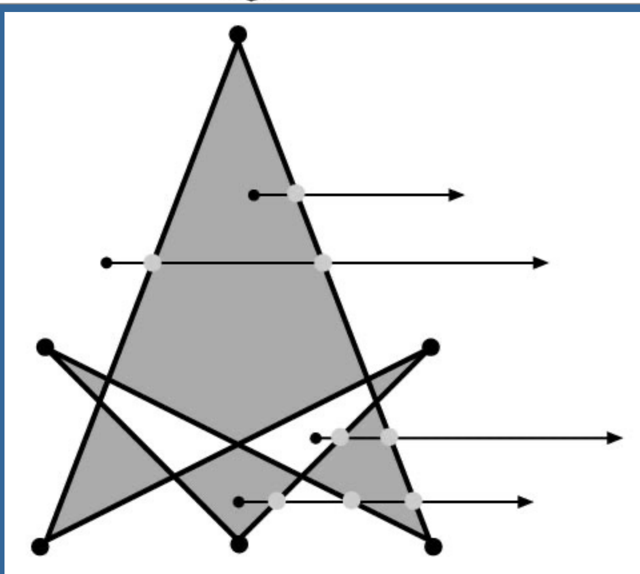



```
Vec3f rayPlaneIntersect(vec3f o,dir, n, d)
{
        float t=(-d-n.dot(o)) / (n.dot(dir));
        return o + dir*t;
}
```

Ray/Polygon: very briefly

- Intersect ray with polygon plane
- Project from 3D to 2D
- How?
- Find $\max(|n_x|,|n_y|,|n_z|)$
- Skip that coordinate!
- Then, count crossing in 2D



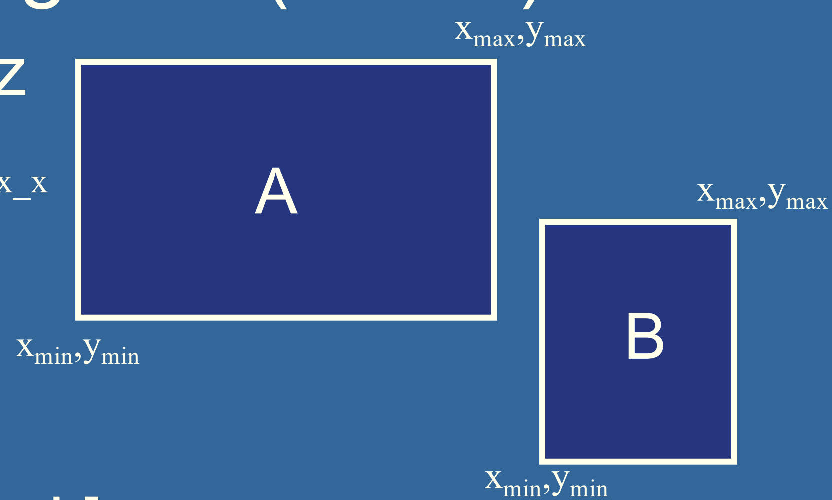


Volume/Volume tests

If $A_{min\_x} > B_{max\_x}$ or
$A_{min\_y} > B_{max\_y}$ or
$A_{min\_z} > B_{max\_z}$ or
$B_{min\_x} > A_{max\_x}$ or
$B_{min\_y} > A_{max\_y}$ or
$B_{min\_z} > A_{max\_z}$
return no_intersection
Else
return intersection.

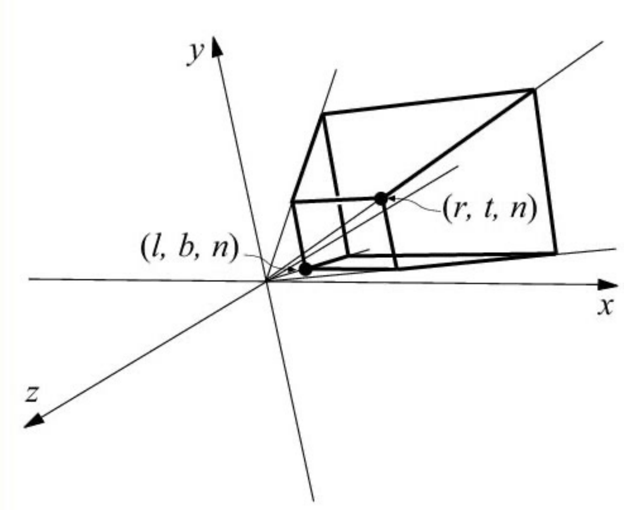
- Used in collision detection
- Sphere/sphere
  - Compute squared distance between sphere centers, and compare to $(r_1+r_2)^2$
- Axis-Aligned Bounding Box (AABB)
  - Test in 1D for x,y, and z

If $B_{min\_x} > A_{max\_x}$ or $A_{min\_x} > B_{max\_x}$
=> no intersection.
… same with y,z …



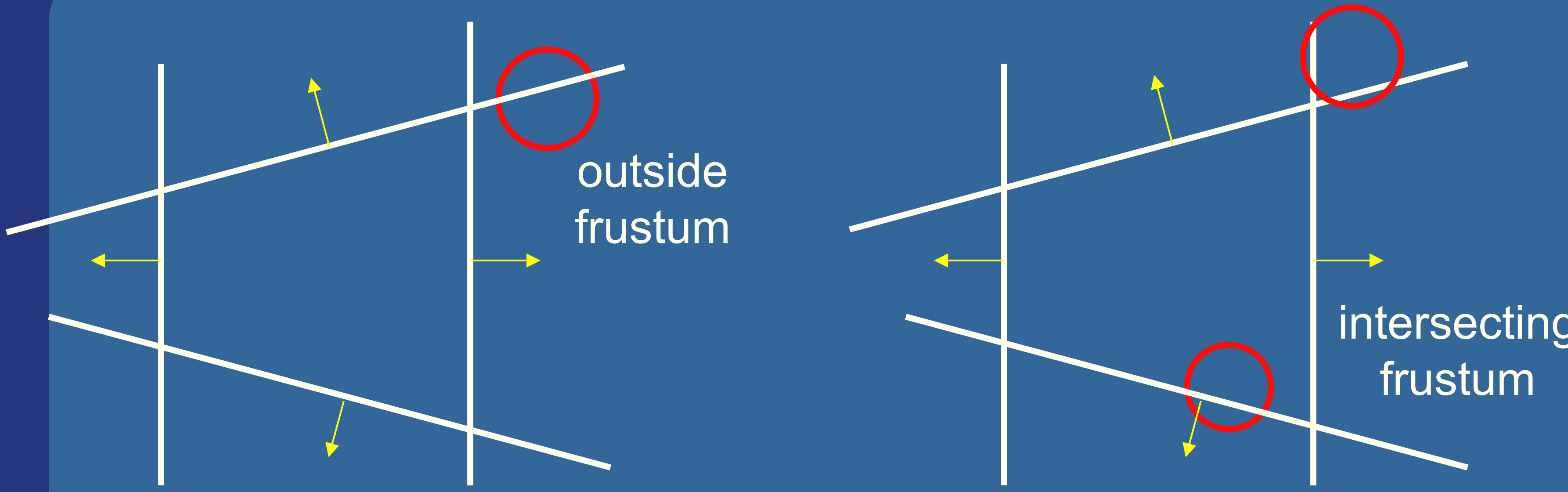
- Oriented Bounding boxes
  - Use SAT [details in book]

View frustum testing



- View frustum is 6 planes:
- Near, far, right, left, top,
- Create planes from projection matrix
  - Let all positive half spaces be outside frustum
  - Not dealt with here -- p. 983-984.
- Sphere/frustum common approach:
  - Test sphere against each of the 6 frustum planes:
    - If outside the plane => no intersection
    - If intersecting the plane or inside, continue
  - If not outside after all six planes, then conservatively concider sphere as inside or intersecting
- Example follows…

View frustum testing example



- Not exact test, but not incorrect
  - A sphere that is reported to be inside, can be outside
  - Not vice versa
- Similarly for boxes

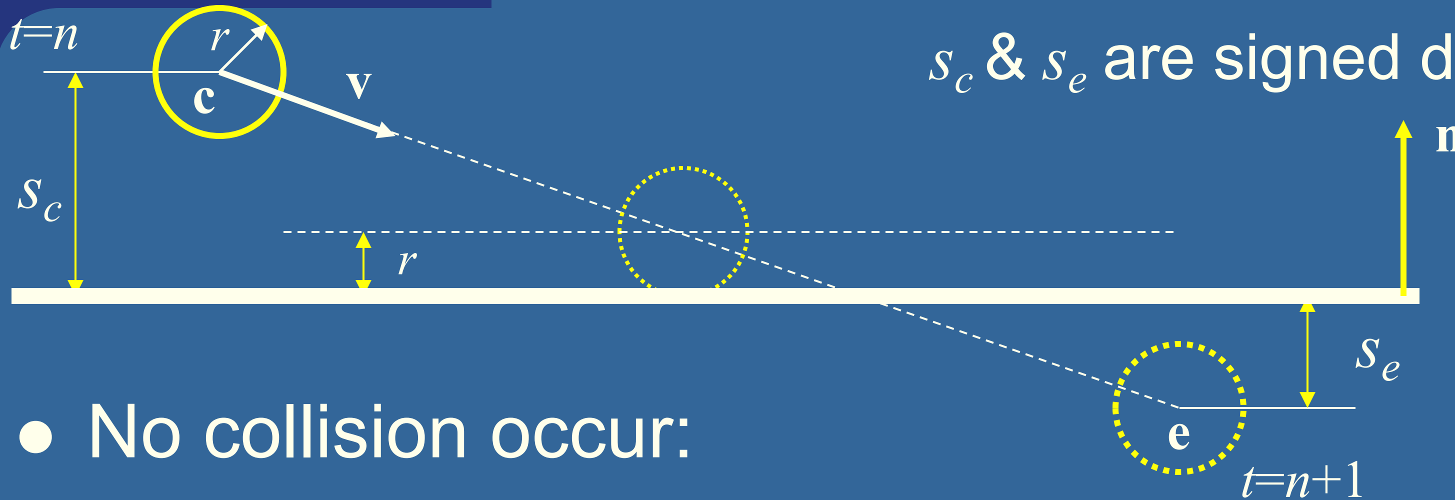
Dynamic Intersection Testing
[In book: 620-628]

- Testing is often done every rendered frame, i.e., at discrete time intervals
- Therefore, you can get ”quantum effects”

- Dynamic testing deals with this
- Is more expensive
- Deals with a time interval: time between two frames

Dynamic intersection testing Sphere/Plane

BONUS



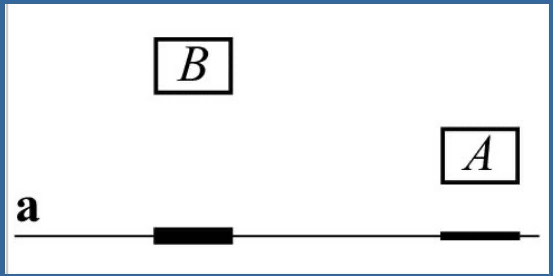
s_c & s_e are signed distances

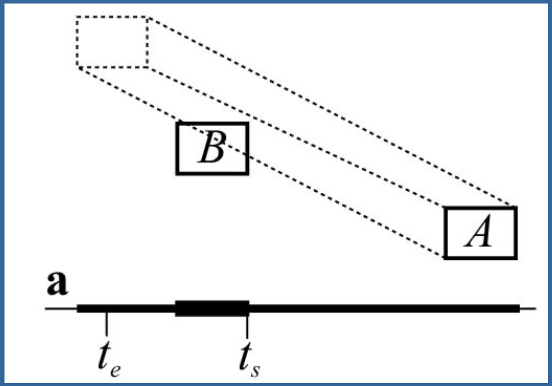
- No collision occur:
  - If they are on the same side of the plane ($s_c s_e > 0$)
    - and: $|s_c| > r$ and $|s_e| > r$
- Otherwise, sphere can move $|s_c|$-r
- Time of collision: $$t_{cd} = n + \frac{s_c - r}{s_c - s_e}$$ s_e is signed distance
- Response: reflect $\mathbf{v}$ around $\mathbf{n}$, and move: $(1\text{-}t_{cd})\mathbf{r}$ ($\mathbf{r}$=refl vector)

BONUS

Dynamic Separating Axis Theorem

- SAT: tests one axis at a time for overlap





- Same with DSAT, but:
  - Use a relative system where B is fixed
    - i.e., compute A's relative motion to B.
  - Need to adjust A's projection on the axis so that the interval moves on the axis as well
- Need to test same axes as with SAT
- Same criteria for overlap/disjoint:
  - If no overlap on axis => disjoint
  - If overlap on all axes => objects overlap

Exercises

- Create a function (by writing code on paper) that tests for intersection between:
  - two spheres
  - a ray and a sphere
  - view frustum and a sphere

What you need to know

- Analytic test:
  - Be able to compute ray vs sphere or other similar formula
  - Ray/triangle, ray/plane
  - Point/plane, Sphere/plane, box/plane
  - Know equations for ray, sphere, cylinder, plane, triangle
- Geometrical tests
  - Ray/box with slab-test
  - Ray/polygon (3D->2D)
  - AABB/AABB
  - View frustum vs spheres/AABB:s/BVHs.
  - Separating Axis Theorem (SAT)
- Know what a dynamic test is