(1) if $\Sigma = \{0, 1\}$, Find a counterexample to the following alleged theorem: $\forall x, y \in \Sigma^*$ we have

$$x^2 y = xyx$$

disproof:

$x^2 y = xxy$ $\quad$ $x = 0$, $y = 10$ $\Rightarrow$ $x^2 y = \cancel{xxy} = 0010$ $\neq$ $xyx = 0100$

$P(0) \wedge (\forall n \in \mathbb{N} . P(n) \Rightarrow P(n+1)) \Rightarrow \forall n \in \mathbb{N} . P(n)$

(2) Prove that $\sum_{i=0}^{n} i = \frac{n \cdot (n+1)}{2}$

page 17 _ L3 slides $\forall n \in \mathbb{N} . P(n)$

$P(n) := \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$

Base Case $n=0$: $\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$

Step Case $n = k+1$, $k \in \mathbb{N}$:

$\sum_{i=0}^{n} i = \sum_{i=0}^{k+1} i = (k+1) + \sum_{i=0}^{k} i \overset{I.H.}{=} (k+1) + \frac{k(k+1)}{2} =$

$(k+1)\left(1 + \frac{k}{2}\right) = (k+1)\left(\frac{k+2}{2}\right)$

re it! (3) prove that $\forall n \geq 4 . n^2 \leq 2^n \longrightarrow$ Page 21 $\Downarrow$ "another proof"

$P(n) := n^2 \leq 2^n$ $\forall n \geq 4 . P(n)$

proof by mathematical induction

Base Case $n=4$: $P(4) := 4^2 \leq 2^4 \Rightarrow 16 \leq 16$ True ✓

Step Case $n = k$: $P(k) := k^2 \leq 2^k$ $I.H._P$

$n = k+1$: $P(k+1) := (k+1)^2 \leq 2^{k+1}$

$(k+1)^2 = k^2 + 2k + 1 \overset{I.H._P}{\leq} 2^k + 2k + 1 \overset{?}{\leq} 2^k + 2^k = 2^{k+1}$

Theorem'

$Q(n) := 2n + 1 \leq 2^n$ Theorem': $\forall n \geq 3 . Q(n)$

Base Case $n = 3$: $P(3) := 2 \cdot 3 + 1 = 7 \leq 8$ ✓

Step Case $n = k$: $P(k) := 2k + 1 \leq 2^k$ $I.H._Q$

$n = k+1$: $P(k+1) := 2(k+1) + 1 \leq 2^{k+1}$

$2(k+1) + 1 = 2k + 3 = 2k + 1 + 2 \overset{I.H._Q}{\leq} 2^k + 2 \leq 2^k + 2^k = 2^{k+1}$ ✓

[for $k \geq 3$]

$P(0) \wedge (\forall n \in \mathbb{N} . (\forall i \in \mathbb{N} .\ i \leq n \Rightarrow P(i)) \Rightarrow P(n+1)) \Rightarrow$

$\forall n \in \mathbb{N} . P(n)$ (Complete induction)

(4) Suppose that we have stamps of 4kr and 3kr.

Show that any amount of postage over 5 kr

can be made with some combinations of these stamps.

$\forall n \geq 6 . P(n) \qquad P(n) := n = 3i + 4j$

Base Case $\quad n=6 : P(6) := 6 = 3.2 + 4.0$

$n=7 : P(7) := 7 = 3.1 + 4.1$

$n=8 : P(8) := 8 = 3.0 + 4.2$

Step Case $\quad n=k-2 : P(k-2) := k-2 = 3i+4j \quad$ I.H.

$n=k+1 : P(k+1) := k+1 = 3i'' + 4j'$

$k+1 = (k-2) + 3 \overset{\text{I.H.}}{=} 3i + 4j + 3 = 3(i+1) + 4j$ ✓

(5) Let us define by recursion the following function:

$0! = 1 \qquad (n+1)! = (n+1) \times n!$

Show that $n! \geq 2^n$ for $n \geq 4$.

Base Case $\quad P(4) := 4! \geq 2^4 \Rightarrow 24 \geq 16$ ✓

Step Case $\quad P(k) := k! \geq 2^k$

$P(k+1) := (k+1)! \geq 2^{k+1}$

$(k+1)! = k! \cdot (k+1) \overset{\text{I.H.}}{\geq} 2^k \cdot (k+1) \geq 2^k \cdot 2 = 2^{k+1}$

$k+1 \geq 2 \Rightarrow k \geq -1$ holds since $k \geq 4$

(6) Consider the following definitions for

$f, g : \mathbb{N} \to \mathbb{N}$:

$f(0) = 0$ $\qquad\qquad$ $g(0) = 0$

$f(1) = 1$ $\qquad\qquad$ $g(n+1) = 1 - g(n)$

$f(n+2) = f(n)$

(a) Compute $f(2), f(3), g(1), g(2), g(3)$.

(b) Prove that $\forall n \in \mathbb{N}.\ f(n) = g(n)$.

(a) $f(2) = f(0+2) = f(0) = 0$

$f(3) = f(1+2) = f(1) = 1$

$g(1) = g(0+1) = 1 - g(0) = 1$

$g(2) = g(1+1) = 1 - g(1) = 0$

$g(3) = g(2+1) = 1 - g(2) = 1$

by complete induction

(b) $P(n) := f(n) = g(n)$

Base Case $\quad n=0 : P(0) := f(0) = g(0) = 0$ ✓

Step Case $\quad n=k-1 : P(k-1) := f(k-1) = g(k-1)$

$n=k : P(k) := f(k) = g(k)$

$n=k+1 : P(k+1) := f(k+1) = g(k+1)$

$f(k+1) = f(k-1) = g(k-1) = 1 - g(k) = 1 - (1 - g(k+1)) =$

$g(k+1)$ ✓

(7) Consider the following definitions for $f, g, h : \mathbb{N} \to \mathbb{N}$:

$f(0) = 0 \quad g(0) = 0 \quad h(0) = 1$

$f(n+1) = 2 + f(n) \quad g(n+1) = 2h(n) \quad h(n+1) = g(n) + 2 - n$

(a) Compute $f(1), g(1), h(1), f(2), g(2), h(2), f(3), g(3), h(3)$.

(b) Prove that $\forall n \in \mathbb{N}.\ f(n) = g(n)$.

(a) $f(1) = f(0+1) = 2 + f(0) = 2$

$g(1) = 2h(0) = 2$

$h(1) = g(0) + 2 - 0 = 2$

$f(2) = 2 + f(1) = 4 \qquad f(3) = 2 + f(2) = 6$

$g(2) = 2 \cdot h(1) = 4 \qquad g(3) = 2 \cdot h(2) = 6$

$h(2) = g(1) + 2 - 1 = 3 \qquad h(3) = g(2) + 2 - 2 = 4$

(b)

$p(n) := f(n) = g(n) = 2n \wedge h(n) = n+1$

Base Case $n=0 : p(0) := f(0) = g(0) = 0 \wedge h(0) = 1 = 0+1$ ✓

step Case $n=k : p(k) := f(k) = g(k) = 2k \wedge h(k) = k+1$ I.H.

$n = k+1 : p(k+1) := f(k+1) = g(k+1) = 2(k+1) \wedge h(k+1) = k+2$

$h(k+1) = g(k) + 2 - k \overset{\text{I.H.1}}{=} 2k + 2 - k = k+2$ ✓

$f(k+1) = 2 + f(k) \overset{\text{I.H.1}}{=} 2 + 2k = 2(k+1) \overset{\text{I.H.2}}{=} 2 \cdot h(k) = g(k+1)$ ✓

(8) Let $\Sigma = \{0,1\}$. We define $\varphi : \Sigma^* \to \Sigma^*$ by recursion as follows:

$$\varphi(\varepsilon) = \varepsilon \qquad \varphi(0w) = 1\,\varphi(w) \qquad \varphi(1w) = 0\,\varphi(w)$$

(a) What are $\varphi(1011)$ and $\varphi(1101)$?

(b) Show by induction on w that $|\varphi(w)| = |w|$.

(a)

$$\varphi(1011) = 0\varphi(011) = 01\varphi(11) = 010\varphi(1) = 0100\varphi(\varepsilon) = 0100$$

$$\varphi(1101) = 0\varphi(101) = 00\varphi(01) = 001\varphi(1) = 0010\,\varphi(\varepsilon) = 0010$$

(b)

$$\forall w \in \Sigma^*.\ |\varphi(w)| = |w|$$

$$p(w) := |\varphi(w)| = |w|$$

Case ε: $\quad p(\varepsilon) := |\varphi(\varepsilon)| = |\varepsilon| = 0 \quad \checkmark$

Case $0w$: $\quad p(0w) := |\varphi(0w)| = |1\varphi(w)| = 1 + |\varphi(w)| \overset{\text{I.H.}}{=} 1 + |w| = |0w| \checkmark$

Case $1w$: $\quad p(1w) := |\varphi(1w)| = |0\varphi(w)| = 1 + |\varphi(w)| \overset{\text{I.H.}}{=} 1 + |w| = |1w| \checkmark$

(9) Let $\Sigma = \{0,1\}$. We define the reverse function on Σ^* by the equation

$$rev(\varepsilon) = \varepsilon \qquad rev(ax) = rev(x)\,a$$

(a) What are $rev(010)$ and $rev(10)$?

(b) Show by induction on y that we have

$$rev(yx) = rev(x)\,rev(y).$$

(c) Show by induction on $n \in \mathbb{N}$ that we have

$$rev(x^n) = (rev(x))^n$$

(a)

$rev(010) = rev(10)0 = rev(0)01 = rev(\varepsilon)010 = 010$

$rev(10) = rev(0)1 = rev(\varepsilon)01 = 01$

(b)

$\forall x, y \in \Sigma^*.\ rev(yx) = rev(x)\,rev(y)$

$p(y) := \forall x \in \Sigma^*.\ rev(yx) = rev(x)\,rev(y)$

Case ε $\quad p(\varepsilon) := \forall x \in \Sigma^*.\ rev(\varepsilon x) = rev(x)\varepsilon = rev(x)\,rev(\varepsilon)$ ✓

Case ay $\quad p(ay) := \forall x \in \Sigma^*.\ rev(ayx) = rev(yx)a \overset{I.H.}{=} rev(x)\,rev(y)\,a$

$= rev(x)\,rev(ay)$ ✓

(c)

$\forall n \in \mathbb{N}.\ \forall x \in \Sigma^*.\ rev(x^n) = (rev(x))^n$

$p(n) := \forall x \in \Sigma^*.\ rev(x^n) = (rev(x))^n$

Base Case $n=0$: $\quad P(0) := \forall x \in \Sigma^*.\ rev(x^0) = rev(\varepsilon) = \varepsilon = (rev(x))^0$ ✓

Step Case $\quad p(k) := \forall x \in \Sigma^*.\ rev(x^k) = (rev(x))^k$

$$p(k+1) := \forall x \in \Sigma^*.\ rev(x^{k+1}) = (rev(x))^{k+1}$$

[Remember: $w^0 = \varepsilon \qquad w^{n+1} = w w^n$]

$$rev(x^{k+1}) = rev(x x^k) = rev(x^k)\, rev(x) \overset{I.H.}{=}$$

$$(rev(x))^k\, rev(x) = (rev(x))^{k+1} \quad \checkmark$$

(10) A binary tree with information on the nodes is either a leaf with no information or a node containing some information and exactly two subtrees. You may assume that the information in the nodes being of any suitable type (string, natural numbers, ...).

(a) Give the inductive definition of the set 'Tree' of trees.

(b) Define the functions that count the number of leaves 'nl', the number of nodes that are not leaves 'nn', and the number of subtrees 'nt' of a tree.

(c) Prove by structural induction on the trees that $\forall t \in Tree \,.\, nl(t) = nn(t)+1$.

(d) Prove by structural induction on the trees that $\forall t \in Tree \,.\, 2\,nn(t) = nt(t)$.

(a) $\dfrac{}{() \in Tree\ A}$ $\qquad$ $\dfrac{x \in A \quad t_1, t_2 \in Tree\ A}{node(t_1, x, t_2) \in Tree\ A}$

(b) $nl(()) = 1$

$nl(node(t_1, x, t_2)) = nl(t_1) + nl(t_2)$

$nn(()) = 0$

$nn(node(t_1, x, t_2)) = nn(t_1) + nn(t_2) + 1$

$nt(()) = 0$

$nt(node(t_1, x, t_2)) = nt(t_1) + nt(t_2) + 2$

(c) $\forall t \in Tree\ A \,.\, nl(t) = nn(t) + 1$

$p(t) := nl(t) = nn(t) + 1$

Base Case $\quad t = () \quad p(()) := nl(()) = 1 = nn(()) + 1$ ✓

Step Case $\quad x \in A \wedge p(t_1) \wedge p(t_2) \Rightarrow p(node(t_1, x, t_2))$

$$nl(node(t_1, x, t_2)) = nl(t_1) + nl(t_2) \overset{\text{I.H.}}{=}$$

$$nn(t_1) + 1 + nn(t_2) + 1 = nn(node(t_1, x, t_2)) + 1$$ ✓

(d) $\quad \forall t \in \text{Tree } A \,.\, 2nn(t) = nt(t)$

$$p(t) := 2nn(t) = nt(t)$$

Base Case $\quad t = () \quad p(()) := 2nn(()) = 0 = nt(())$ ✓

Step Case $\quad x \in A \wedge p(t_1) \wedge p(t_2) \Rightarrow p(node(t_1, x, t_2))$

$$p(node(t_1, x, t_2)) = 2\, nn(node(t_1, x, t_2)) =$$

$$2\,[nn(t_1) + nn(t_2) + 1] = 2\,nn(t_1) + 2\,nn(t_2) + 2$$

$$= nt(t_1) + nt(t_2) + 2 = nt(node(t_1, x, t_2))$$ ✓