Lecture
Computability
(DIT312, DAT415)

Nils Anders Danielsson

2019-12-16

Today

- Repetition (mainly). Please interrupt if you want to discuss something in more detail.
- Course evaluation.

Models of computation

- Actual hardware or programming languages: Lots of (irrelevant?) details.
- In this course: Idealised models of computation.
- PRF, RF.
- X.
- Turing machines.

The Church-Turing thesis

- The thesis:
  Every effectively calculable function on the positive integers can be computed using a Turing machine.
- Widely believed to be true.
- Many models are Turing-complete.

Comparing sets' sizes

- Injections, surjections, bijections.
- Countable (injection to $\mathbb{N}$), uncountable.
- Diagonalisation.
- Not every function is computable.

Inductively defined sets

An inductively defined set:

$$\frac{}{\mathsf{nil} \in \mathit{List}\ A} \qquad \frac{x \in A \qquad xs \in \mathit{List}\ A}{\mathsf{cons}\ x\ xs \in \mathit{List}\ A}$$

Primitive recursion:

$$\begin{aligned}
&\mathit{listrec} \in B \to (A \to \mathit{List}\ A \to B \to B) \to \\
&\qquad\qquad \mathit{List}\ A \to B \\
&\mathit{listrec}\ n\ c\ \mathsf{nil} \qquad\qquad\quad = n \\
&\mathit{listrec}\ n\ c\ (\mathsf{cons}\ x\ xs) = c\ x\ xs\ (\mathit{listrec}\ n\ c\ xs)
\end{aligned}$$

Inductively defined sets

An inductively defined set:

$$\frac{}{\mathsf{nil} \in \mathit{List}\ A} \qquad \frac{x \in A \qquad xs \in \mathit{List}\ A}{\mathsf{cons}\ x\ xs \in \mathit{List}\ A}$$

Pattern (with recursive constructor arguments last):

$$\begin{array}{l}
drec \in \text{One assumption per constructor} \to D \to A \\
drec\ f_1 \ldots f_k\ (c_1\ x_1 \ldots x_{n_1}) = \\
\quad f_1\ x_1 \ldots x_{n_1}\ (drec\ f_1 \ldots f_k\ x_{i_1}) \ldots (drec\ f_1 \ldots f_k\ x_{n_1}) \\
\vdots \\
drec\ f_1 \ldots f_k\ (c_k\ x_1 \ldots x_{n_k}) = \\
\quad f_k\ x_1 \ldots x_{n_k}\ (drec\ f_1 \ldots f_k\ x_{i_k}) \ldots (drec\ f_1 \ldots f_k\ x_{n_k})
\end{array}$$

Inductively defined sets

An inductively defined set:

$$\frac{}{\mathsf{nil} \in \mathit{List}\ A} \qquad \frac{x \in A \qquad xs \in \mathit{List}\ A}{\mathsf{cons}\ x\ xs \in \mathit{List}\ A}$$

Structural induction (P: a predicate on $\mathit{List}\ A$):

$$\frac{\begin{array}{c} P\ \mathsf{nil} \\ \forall x \in A.\ \forall\ xs \in \mathit{List}\ A.\ P\ xs \Rightarrow P\ (\mathsf{cons}\ x\ xs) \end{array}}{\forall xs \in \mathit{List}\ A.\ P\ xs}$$

Quiz

Write down the "type" of one of the higher-order primitive recursion schemes for the following inductively defined set:

$$\frac{n \in \mathbb{N}}{\mathsf{leaf}\ n \in \mathit{Tree}} \qquad \frac{l, r \in \mathit{Tree}}{\mathsf{node}\ l\ r \in \mathit{Tree}}$$

PRF

Sketch:

$$f\,() = \mathsf{zero}$$

$$f\,(x) = \mathsf{suc}\ x$$

$$f\,(x_1, ..., x_k, ..., x_n) = x_k$$

$$f\,(x_1, ..., x_n) = g\,(h_1\ (x_1, ..., x_n), ..., h_k\ (x_1, ..., x_n))$$

$$f\,(x_1, ..., x_n, \mathsf{zero}) \ = g\,(x_1, ..., x_n)$$

$$f\,(x_1, ..., x_n, \mathsf{suc}\ x) = \\ \quad h\,(x_1, ..., x_n, f\,(x_1, ..., x_n, x), x)$$

PRF

- Abstract syntax (PRF_n).
- Denotational semantics:

$$[\![\_]\!] \in PRF_n \to (\mathbb{N}^n \to \mathbb{N})$$

- Big-step operational semantics:

$$f[\rho] \Downarrow n$$

PRF

- Strictly weaker than χ/Turing machines.
- Some χ-computable *total* functions are not PRF-computable, for instance the PRF semantics.

RF

- PRF $+$ minimisation.
- For $f \in \mathbb{N} \rightharpoonup \mathbb{N}$:
  f is RF-computable $\Leftrightarrow$
  f is χ-computable $\Leftrightarrow$
  f is Turing-computable.

X

$$
\begin{aligned}
e ::= {} & x \\
\mid \quad & (e_1 \ e_2) \\
\mid \quad & \lambda x.\, e \\
\mid \quad & \mathsf{C}(e_1, ..., e_n) \\
\mid \quad & \mathbf{case}\ e\ \mathbf{of}\ \{\, \mathsf{C}_1(x_1, ..., x_n) \rightarrow e_1; ... \} \\
\mid \quad & \mathbf{rec}\ x = e
\end{aligned}
$$

- Untyped, strict.
- $\mathbf{rec}\ x = e \ \approx\ \mathbf{let}\ x = e\ \mathbf{in}\ x$.

X

- Abstract syntax.
- Substitution of closed expressions.
- Big-step operational semantics, not total.
- The semantics as a partial function:

$$[\![\_]\!] \in CExp \rightharpoonup CExp$$

- Representation of inductively defined sets.

Representing expressions

Coding function:

$$
\begin{aligned}
&\ulcorner \_ \urcorner \in \mathit{Exp} \to \mathit{CExp} \\
&\ulcorner x \urcorner \qquad = \mathsf{Var}(\ulcorner x \urcorner) \\
&\ulcorner e_1\ e_2 \urcorner = \mathsf{Apply}(\ulcorner e_1 \urcorner, \ulcorner e_2 \urcorner) \\
&\ulcorner \lambda x.\, e \urcorner = \mathsf{Lambda}(\ulcorner x \urcorner, \ulcorner e \urcorner) \\
&\vdots
\end{aligned}
$$

Representing expressions

Coding function:

$$
\begin{aligned}
&\ulcorner \_ \urcorner \in \mathit{Exp} \to \mathit{CExp} \\
&\ulcorner \mathsf{var}\ x \urcorner &&= \mathsf{const}\ \ulcorner \mathsf{Var} \urcorner\ (\mathsf{cons}\ \ulcorner x \urcorner\ \mathsf{nil}) \\
&\ulcorner \mathsf{apply}\ e_1\ e_2 \urcorner &&= \mathsf{const}\ \ulcorner \mathsf{Apply} \urcorner \\
& &&\qquad (\mathsf{cons}\ \ulcorner e_1 \urcorner\ (\mathsf{cons}\ \ulcorner e_2 \urcorner\ \mathsf{nil})) \\
&\ulcorner \mathsf{lambda}\ x\ e \urcorner &&= \mathsf{const}\ \ulcorner \mathsf{Lambda} \urcorner \\
& &&\qquad (\mathsf{cons}\ \ulcorner x \urcorner\ (\mathsf{cons}\ \ulcorner e \urcorner\ \mathsf{nil})) \\
&\vdots
\end{aligned}
$$

Representing expressions

Coding function:

$$
\begin{aligned}
\ulcorner \_ \urcorner &\in \mathit{Exp} \to \mathit{CExp} \\
\ulcorner \mathsf{var}\ x \urcorner &= \mathsf{const}\ \ulcorner \mathsf{Var} \urcorner\ (\mathsf{cons}\ \ulcorner x \urcorner\ \mathsf{nil}) \\
\ulcorner \mathsf{apply}\ e_1\ e_2 \urcorner &= \mathsf{const}\ \ulcorner \mathsf{Apply} \urcorner \\
&\qquad (\mathsf{cons}\ \ulcorner e_1 \urcorner\ (\mathsf{cons}\ \ulcorner e_2 \urcorner\ \mathsf{nil})) \\
\ulcorner \mathsf{lambda}\ x\ e \urcorner &= \mathsf{const}\ \ulcorner \mathsf{Lambda} \urcorner \\
&\qquad (\mathsf{cons}\ \ulcorner x \urcorner\ (\mathsf{cons}\ \ulcorner e \urcorner\ \mathsf{nil})) \\
&\vdots
\end{aligned}
$$

Alternative “type”:

$$\ulcorner \_ \urcorner \in \mathit{Exp}\ A \to \mathit{CExp}\ (\mathit{Rep}\ A)$$

$\mathit{Rep}\ A$: Representations of programs of type A.

Computability

- $f \in A \rightharpoonup B$ is χ-computable if

$$\exists\, e \in \mathit{CExp}.\ \forall\, a \in A.\, [\![e \ulcorner a \urcorner]\!] = \ulcorner f\, a \urcorner.$$

- Use reasonable coding functions:
  - Injective.
  - Computable. But how is this defined?
- X-decidable: $f \in A \to \mathit{Bool}$.
- X-semi-decidable:
  If $f\, a =$ false then $[\![e \ulcorner a \urcorner]\!]$ is undefined.

Some computable partial functions

- The semantics $[\![\_]\!] \in \mathit{CExp} \rightharpoonup \mathit{CExp}$:

$$\forall\ e \in \mathit{CExp}.\ [\![\mathit{eval} \ulcorner e \urcorner]\!] = \ulcorner [\![e]\!] \urcorner.$$

- The coding function $\ulcorner \_ \urcorner \in \mathit{Exp} \to \mathit{CExp}$:

$$\forall\ e \in \mathit{Exp}.\ [\![\mathit{code} \ulcorner e \urcorner]\!] = \ulcorner \ulcorner e \urcorner \urcorner.$$

- The "Terminates in n steps?" function $\mathit{terminates\text{-}in} \in \mathit{CExp} \times \mathbb{N} \to \mathit{Bool}$:

$$\forall\ p \in \mathit{CExp} \times \mathbb{N}.$$
$$[\![\underline{\mathit{terminates\text{-}in}} \ulcorner p \urcorner]\!] = \ulcorner \mathit{terminates\text{-}in}\ p \urcorner.$$

Some non-computable functions

The halting problem with self-application,

$$\begin{aligned}
&\mathit{halts\text{-}self} \in \mathit{CExp} \to \mathit{Bool} \\
&\mathit{halts\text{-}self}\ p = \\
&\quad \mathbf{if}\ p\ \ulcorner p \urcorner\ \text{terminates}\ \mathbf{then}\ \text{true}\ \mathbf{else}\ \text{false},
\end{aligned}$$

can be reduced to the halting problem,

$$\begin{aligned}
&\mathit{halts} \in \mathit{CExp} \to \mathit{Bool} \\
&\mathit{halts}\ p = \mathbf{if}\ p\ \text{terminates}\ \mathbf{then}\ \text{true}\ \mathbf{else}\ \text{false}.
\end{aligned}$$

Some non-computable functions

Proof sketch:

- Assume that $\underline{halts}$ implements $halts$.
- Define $\underline{halts\text{-}self}$ in the following way:

$$\underline{halts\text{-}self} = \lambda\, p.\, \underline{halts}\ \mathsf{Apply}(p, code\ p)$$

- $\underline{halts\text{-}self}$ implements $halts\text{-}self$,

$$\begin{aligned} &\forall\ e \in\ CExp. \\ &\quad [\![\underline{halts\text{-}self} \ulcorner\, e\, \urcorner]\!] = \ulcorner\, halts\text{-}self\ e\, \urcorner, \end{aligned}$$

because $\mathsf{Apply}(\ulcorner\, e\, \urcorner, code\ \ulcorner\, e\, \urcorner) \Downarrow \ulcorner\, e\ \ulcorner\, e\, \urcorner \urcorner$.

Some non-computable functions

The halting problem can be reduced to:

- Semantic equality:

$$\begin{aligned}
&\mathit{equal} \in \mathit{CExp} \times \mathit{CExp} \to \mathit{Bool} \\
&\mathit{equal}\,(e_1, e_2) = \\
&\quad \mathbf{if}\ [\![e_1]\!] = [\![e_2]\!]\ \mathbf{then}\ \mathsf{true}\ \mathbf{else}\ \mathsf{false}
\end{aligned}$$

- Pointwise equality of elements in $\mathit{Fun} = \{\,(f, e) \mid f \in \mathbb{N} \to \mathit{Bool}, e \in \mathit{CExp},$ e implements $f\}$:

$$\begin{aligned}
&\mathit{pointwise\text{-}equal} \in \mathit{Fun} \times \mathit{Fun} \to \mathit{Bool} \\
&\mathit{pointwise\text{-}equal}\,((f, _), (g, _)) = \\
&\quad \mathbf{if}\ \forall\, n \in \mathbb{N}.\, f\, n = g\, n\ \mathbf{then}\ \mathsf{true}\ \mathbf{else}\ \mathsf{false}
\end{aligned}$$

Quiz

What is wrong with the following reduction of the halting problem to $\mathit{pointwise\text{-}equal}$?

$$\underline{\mathit{halts}} = \lambda\, p.\, \underline{\mathit{not}}\, (\underline{\mathit{pointwise\text{-}equal}}$$
$$\quad \mathsf{Lambda}(\ulcorner n \urcorner,$$
$$\qquad \mathsf{Apply}(\ulcorner \underline{\mathit{terminates\text{-}in}} \urcorner,$$
$$\qquad\quad \mathsf{Const}(\ulcorner \mathsf{Pair} \urcorner,$$
$$\qquad\qquad \mathsf{Cons}(p, \mathsf{Cons}(\mathsf{Var}(\ulcorner n \urcorner), \mathsf{Nil}())))))$$
$$\quad \ulcorner \lambda\, \_.\, \mathsf{False}() \urcorner)$$

Bonus question: How can the problem be fixed?

Some non-computable functions

The halting problem can be reduced to:

- An optimal optimiser:

$$\begin{array}{l} \mathit{optimise} \in \mathit{CExp} \rightarrow \mathit{CExp} \\ \mathit{optimise}\ e = \\ \quad \text{some optimally small expression with} \\ \quad \text{the same semantics as } e \end{array}$$

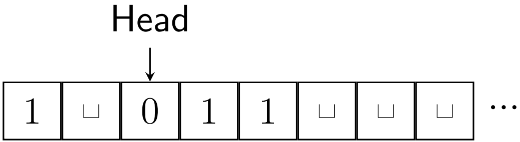
- Is a computable real number equal to zero?

$$\begin{array}{l} \mathit{is\text{-}zero} \in \mathit{Interval} \rightarrow \mathit{Bool} \\ \mathit{is\text{-}zero}\ x = \mathbf{if}\ [\![x]\!] = 0\ \mathbf{then}\ \mathsf{true}\ \mathbf{else}\ \mathsf{false} \end{array}$$

- Many other functions, see Rice's theorem.

Turing machines

- A tape with a head:



- A state.
- Rules.

Turing machines

- Abstract syntax.
- Small-step operational semantics.
- The semantics as a family of partial functions:

$$[\![\_]\!] \in \forall\, tm \in TM.\, List\, \Sigma_{tm} \rightharpoonup List\, \Gamma_{tm}$$

- Several variants:
  - Accepting states.
  - Possibility to stay put.
  - A tape without a left end.
  - Multiple tapes.
  - Only two symbols (plus ␣).

Turing-computability

- Representing inductively defined sets.
- Turing-computable partial functions.
- Turing-decidable languages.
- Turing-recognisable languages.

Some computable partial functions

- The semantics (uncurried):

  $$\{\,(tm, xs) \mid tm \in TM, xs \in List\,\Sigma_{tm}\,\} \rightharpoonup List\,\Gamma_{tm}$$

  Self-interpreter/universal TM.

  (The definition of computability can be generalised so that it applies to dependent partial functions.)
- The χ semantics.

Some non-computable functions

- The Post correspondence problem (seen as a function to $Bool$).
- Is a context-free grammar ambiguous?

Equivalence

- The Turing machine semantics is also χ-computable.
- Partial functions $f \in \mathbb{N} \rightharpoonup \mathbb{N}$ are Turing-computable iff they are χ-computable.

Some connections to other courses

- Time complexity: Different complexity classes.
- Grammars: The Chomsky hierarchy.

Current research in this area

A recent conference: CiE 2019.

Current research in this area

Connections in dependent type theory:

- Termination checking.
- Equality checking.
- Theorem proving.
- How can possibly non-terminating computations be represented in a terminating programming language?

Finally

- We have studied the concept of “computation”.
- How can “computation” be formalised?
  - To simplify our work: Idealised models.
  - The Church-Turing thesis.
- We have explored the limits of computation:
  - Programs that can run arbitrary programs.
  - A number of non-computable functions.

Good luck!