Formal Methods for Software Development

Reasoning about Programs with Loops and Method Calls

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Program Logic Calculus – Repetition

Calculus realises symbolic interpreter:

- decomposition of complex statements into simpler ones
- simple assignment to update
- update captures accumulated effect (abbr. w. $\mathcal{U}$)
- control flow branching induces proof splitting
- application of update computes weakest precondition

$$\frac{\Gamma' \Longrightarrow \{\mathcal{U}'\}\phi \qquad \ldots}{\begin{array}{l}\ldots \qquad\qquad \ldots \\ \textit{'branch1'}\quad \Gamma, \{\mathcal{U}\}(\mathtt{j} > 10) \Longrightarrow \{\mathcal{U}\}\langle\{\mathtt{ok=true;}\}\ldots\rangle\phi \quad \textit{'branch1'} \quad \Gamma, \mathtt{j}+1 \\ \textit{'branch2'}\quad \Gamma, \{\mathcal{U}\}\neg(\mathtt{j} > 10) \Longrightarrow \{\mathcal{U}\}\langle\ldots\rangle\phi \quad \textit{'branch2'} \quad \Gamma, \neg(\mathtt{j}+1 > 10) \Longrightarrow\end{array}}$$

$$\frac{\Gamma \Longrightarrow \{\mathtt{t} := \mathtt{j} \| \mathtt{j} := \mathtt{j}+1 \| \mathtt{i} := \mathtt{j}\}\{\mathcal{U}\}\{\mathcal{U}\}\langle\mathtt{if(j>10)\{ok}}}{\ldots}$$

$$\frac{\Gamma \Longrightarrow \{\mathtt{t} := \mathtt{j}\}\langle\mathtt{j=j+1;i=t;if(j>10)\{ok=true;}}{\Gamma \Longrightarrow \langle\mathtt{t=j;j=j+1;i=t;if(j>10)\{ok=true;\}}}$$

$$\frac{}{\Gamma \Longrightarrow \langle\mathtt{i=j++;if(j>10)\{ok=true;\}...}\rangle}$$

Method Call: Example

```
\javaSource "src/";

\programVariables{
 Person p;
 int j;
}

\problem {
  (\forall int i;
    (!p=null ->
      ({j := i}\<{p.setAge(j);}\>(p.age = i))))
}
```

Method Calls

Method Call with actual parameters $arg_0, \ldots, arg_n$

$$\langle \mathtt{o.m}(arg_0, \ldots, arg_n);\ \omega \rangle \phi$$

assume $\mathtt{m}$ declared as $\mathbf{void}\ \mathtt{m}(\tau_0\ \mathtt{p_0}, \ldots, \tau_n\ \mathtt{p_n})$

Actions of rule methodCall

1. Declare new local variables $\mathtt{p\#i}$, initialize them with actual parameter: $\tau_i\ \mathtt{p\#i} = arg_i;$
2. Look-up implementing class C of $\mathtt{m}$;
   split proof if implementation cannot be uniquely determined.
3. Replace method call with implementation invocation
   $\mathtt{o.m(p\#0, \ldots, p\#n)@}C$

Method Calls Cont'd

After executing the initialisers: τ_i p#i =arg_i; apply:

Method Body Expand

Rule methodBodyExpand (simplified)

$$\frac{\Gamma \Longrightarrow \langle \texttt{method-frame(source=m(}\tau_0,\ldots,\tau_n\texttt{)@C, this=o):\{}body\texttt{\}}\ \omega\rangle\phi, \Delta}{\Gamma \Longrightarrow \langle \texttt{o.m(p\#0,...,p\#n)@C;}\ \omega\rangle\phi, \Delta}$$

1. Replaces method invocation by method frame with method body
2. Renames p_i in body to p#i

Method frames:
Required in proof to represent call stack

Demo

```
methods/instanceMethodInlineSimple.key
methods/inlineDynamicDispatch.key
```

Localisation of Fields and Method Implementations

JAVA has complex rules for localisation of fields and method implementations

- Overloading
- Late binding (dynamic dispatch)
- Scoping (class vs. instance)
- Visibility (private, protected, public)

Proof split into cases if implementation not statically determined

Object initialization

JAVA has complex rules for object initialization

- Chain of constructor calls until Object
- Implicit calls to `super()`
- Visibility issues
- Initialization sequence

Coding of initialization rules in methods `<createObject>()`, `<init>()`,...
which are then symbolically executed

Limitations of Method Inlining: methodBodyExpand

- Source code might be unavailable
  - API method implementation vendor-specific
  - Source code often unavailable for commercial APIs
- Method is invoked multiple times in a program
  - Avoid multiple symbolic execution of identical code
- Cannot handle unbounded recursion
- Not modular:
  Changing a method requires re-verification of all callers

Use method contract instead of method implementation:

1. Show that requires clause is satisfied
2. Continue after method call:
   - assume ensures clause
   - forget prestate values of modifiable locations

Method Contract Rule: Normal Behavior Case

Warning: Simplified version

```
/*@ public normal_behavior
  @ requires preNormal;
  @ ensures postNormal;
  @ assignable mod;
  @*/ // implementation contract of m()
```

$$\frac{\Gamma \Longrightarrow \mathcal{U}\mathcal{F}(\texttt{preNormal}), \Delta \quad \text{(precondition)} \qquad \Gamma \Longrightarrow \mathcal{U}\mathcal{V}_{\texttt{mod}}(\mathcal{F}(\texttt{postNormal}) \rightarrow \langle \pi\ \omega \rangle \phi), \Delta \quad \text{(normal)}}{\Gamma \Longrightarrow \mathcal{U}\langle \pi\ \texttt{result} = \texttt{m}(\texttt{a}_1, \ldots, \texttt{a}_\texttt{n}); \omega \rangle \phi, \Delta}$$

- π are openings of try blocks and method frames
- $\mathcal{F}(\cdot)$: translation from JML to Java DL
- $\mathcal{V}_{\texttt{mod}}$: anonymising update,
  forgetting prevalues of modifiable locations

JML Method Contracts Revisited

```
/*@ public normal_behavior
  @ requires preNormal;
  @ ensures postNormal;
  @ assignable mod;
  @*/
T m(T1 a1, ..., Tn an) { ... }
```

Implicit Preconditions and Postconditions

- The object referenced by **this** is not **null**: **this**!=**null** (precondition only; **this** cannot be changed by method)
- The heap is wellformed: `wellFormed(heap)` (precondition only)
- Invariant for 'this': `\invariant_for(this)`

Method Contract Rule: Normal Behavior Case

Warning: Simplified version

```
/*@ public normal_behavior
  @ requires preNormal;
  @ ensures postNormal;
  @ assignable mod;
  @*/ // implementation contract of m()
```

$$\frac{\Gamma \Longrightarrow \mathcal{U}\mathcal{F}(\texttt{preNormal}), \Delta \quad \text{(precondition)} \qquad \Gamma \Longrightarrow \mathcal{U}\mathcal{V}_{\texttt{mod}}(\mathcal{F}(\texttt{postNormal}) \rightarrow \langle \pi\ \omega \rangle \phi), \Delta \quad \text{(normal)}}{\Gamma \Longrightarrow \mathcal{U}\langle \pi\ \texttt{result} = \texttt{m}(\texttt{a}_1, \ldots, \texttt{a}_\texttt{n}); \omega \rangle \phi, \Delta}$$

- π are openings of try blocks and method frames
- $\mathcal{F}(\cdot)$: translation from JML to Java DL
- $\mathcal{V}_{\texttt{mod}}$: anonymising update,
  forgetting prevalues of modifiable locations

Keeping the Context

- Want to keep part of prestate $\mathcal{U}$ that is unmodified by called method
- **`assignable`** clause of contract tells what can possibly be modified

```
@ assignable mod;
```

- How to erase all values of **`assignable`** locations in state $\mathcal{U}$?
- Anonymising updates $\mathcal{V}$ erase information about modified locations

Anonymising Heap Locations

Define anonymising function $\texttt{anon}$: Heap $\times$ LocSet $\times$ Heap $\rightarrow$ Heap

The resulting heap $\texttt{anon}(...)$ coincides with the first heap on all locations except for those specified in the location set. Those locations attain the value specified by the second heap.

Definition:

$$\texttt{select}(\texttt{anon}(h1, locs, h2), o, f) = \begin{cases} \texttt{select}(h2, o, f) & \text{if } (o, f) \in locs \\ \texttt{select}(h1, o, f) & \text{otherwise} \end{cases}$$

Usage:

$$\mathcal{V}_{mod} = \{\texttt{heap} := \texttt{anon}(\texttt{heap}, locs_{mod}, \texttt{h}_{an})\}$$

where $\texttt{h}_{an}$ a new (not yet used) constant of type Heap

Effect: After $\mathcal{V}_{mod}$, modfied locations have unknown values

Anonymising Heap Locations: Example

```
@ assignable o.a, this.*;
```

To erase all knowledge about the values of the locations of the assignable expression:

- Anonymise the current heap on the designated locations:

$$\texttt{anon}(\texttt{heap}, \{(\texttt{o}, \texttt{a})\} \cup \texttt{allFields}(\texttt{this}), \texttt{h}_{an})$$

- Make that anonymised current heap the new current heap.

$$\mathcal{V}_{mod} = \{\texttt{heap} := \texttt{anon}(\texttt{heap}, \{(\texttt{o}, \texttt{a})\} \cup \texttt{allFields}(\texttt{this}), \texttt{h}_{an})\}$$

Method Contract Rule: Exceptional Behavior Case

Warning: Simplified version

```
/*@ public exceptional_behavior
  @ requires preExc;
  @ signals (Exception exc) postExc;
  @ assignable mod;
  @*/
```

$$\frac{\Gamma \Longrightarrow \mathcal{U}\mathcal{F}(\texttt{preExc}), \Delta \quad \text{(precondition)} \qquad \Gamma \Longrightarrow \mathcal{U}\mathcal{V}_{mod}((\mathcal{F}(\texttt{postExc}) \land \texttt{exc} \neq \textbf{null}) \rightarrow \langle \pi\, \textbf{throw}\ \texttt{exc};\ \omega\rangle\phi), \Delta \quad \text{(exceptional)}}{\Gamma \Longrightarrow \mathcal{U}\langle \pi\, \texttt{result} = \texttt{m}(\texttt{a}_1, \ldots, \texttt{a}_n); \omega\rangle\phi, \Delta}$$

- π are openings of try blocks and method frames
- $\mathcal{F}(\cdot)$: translation from JML to Java DL
- $\mathcal{V}_{mod}$: anonymising update

Method Contract Rule – Combined

(background only, no need to remember)

KeY uses actually one rule for both kinds of cases.

Therefore translation of postcondition ϕ_{post} as follows (simplified):

$$\begin{aligned} \phi_{post\_n} &\equiv \mathcal{F}(\texttt{\textbackslash old(normalPre)}) \land \mathcal{F}(\texttt{normalPost}) \\ \phi_{post\_e} &\equiv \mathcal{F}(\texttt{\textbackslash old(excPre)}) \land \mathcal{F}(\texttt{excPost}) \end{aligned}$$

$$\frac{\begin{array}{l} \Gamma \Longrightarrow \mathcal{U}(\mathcal{F}(\texttt{normalPre}) \lor \mathcal{F}(\texttt{excPre})), \Delta \quad \text{(precondition)} \\ \Gamma \Longrightarrow \mathcal{U}\mathcal{V}_{mod_{normal}}(\phi_{post\_n} \rightarrow \langle \pi\, \omega \rangle \phi), \Delta \quad \text{(normal)} \\ \Gamma \Longrightarrow \mathcal{U}\mathcal{V}_{mod_{exc}}((\phi_{post\_e} \land \texttt{exc} \neq \texttt{null}) \\ \qquad\qquad \rightarrow \langle \pi\, \texttt{throw exc; } \omega \rangle \phi), \Delta \quad \text{(exceptional)} \end{array}}{\Gamma \Longrightarrow \mathcal{U}\langle \pi\, \texttt{result} = \texttt{m}(\texttt{a}_1, \ldots, \texttt{a}_n); \ \omega \rangle \phi, \Delta}$$

- $\mathcal{F}(\cdot)$: translation to Java DL
- $\mathcal{V}_{mod}$: anonymising update

Method Contract Rule: Example

```
class Person {
 private /*@ spec_public @*/ int age;
 /*@ public normal_behavior
   @ requires age < 29;
   @ ensures age == \old(age) + 1;
   @ assignable age;
   @ also
   @ public exceptional_behavior
   @ requires age >= 29;
   @ signals_only ForeverYoungException;
   @ assignable \nothing;
   @//allows object creation (not \strictly_nothing)
   @*/
 public void birthday() {
   if (age >= 29) throw new ForeverYoungException();
   age++;
 } }
```

Method Contract Rule: Example Cont'd

Demo

`methods/useContractForBirthday.key`

- Prove without contracts
  - Method treatment: Expand
- Prove with contracts (until method contract application)
  - Method treatment: Contract
- Prove used contracts
  - Method treatment: Expand
  - Select contracts for `birthday()` in `src/Person.java`
  - Prove both specification cases

Verification of Loops

Symbolic execution of loops: unwind

$$\text{unwindLoop}\ \frac{\Gamma \Longrightarrow \mathcal{U}[\pi\ \texttt{if(b)\{p; while(b) p\}}\ \omega]\phi, \Delta}{\Gamma \Longrightarrow \mathcal{U}[\pi\ \texttt{while(b) p}\ \omega]\phi, \Delta}$$

How to handle a loop with...

- 0 iterations? Unwind 1×
- 10 iterations? Unwind 11×
- 10000 iterations? Unwind 10001×
- an unknown number of iterations?

We need an invariant rule (or some form of induction)

Loop Invariants

Idea behind loop invariants

- A formula *Inv* whose validity is preserved by loop body whenever the loop guard is true
- Consequence: if *Inv* was valid at start of the loop, then it still holds after arbitrarily many loop iterations
- In particular, if the loop terminates, then *Inv* holds afterwards
- Challenge: construct *Inv* such that, *together with loop exit condition*, it implies postcondition of loop

Basic Invariant Rule

$$\text{loopInvariant}\ \frac{\begin{array}{cc} \Gamma \Longrightarrow \mathcal{U}\,\mathit{Inv}, \Delta & \text{(valid when entering loop)} \\ \mathit{Inv},\ b = \texttt{TRUE} \Longrightarrow [\mathrm{p}]\mathit{Inv} & \text{(preserved by p)} \\ \mathit{Inv},\ b = \texttt{FALSE} \Longrightarrow [\pi\ \omega]\phi & \text{(assumed after exit)} \end{array}}{\Gamma \Longrightarrow \mathcal{U}[\pi\ \texttt{while(b)}\ \mathrm{p}\ \omega]\phi, \Delta}$$

How to Derive Loop Invariants Systematically?

Example (Active statement of symbolic execution is loop)

```
    n >= 0 & wellFormed(heap)
->  {i := 0}
      \[{ while (i < n) {
              i = i + 1;
            }
        }\] i = n
```

Look at desired postcondition `i = n`

What, in addition to negated guard `i >= n`, is needed? `i <= n`

Is `i <= n` preserved by loop body?
Does it hold when entering loop?

Yes! We have found a suitable loop invariant!

Demo `loops/simple.key` (auto after inv)

Obtaining Invariants by Strengthening

Example (Slightly changed problem)

```
    n >= 0 & n = m & wellFormed(heap)
->  {i := 0}
      \[{ while (i < n) {
              i = i + 1;
          }
        }\] i = m
```

Look at desired postcondition `i = m`

What, in addition to negated guard `i >= n`, is needed?

`i <= n & n = m`

Is `i <= n & n = m` preserved by loop body?
Does it hold when entering loop?

Yes! We have found a suitable loop invariant!

Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

```
x = x0 & y = y0 & y0 >= 0 & wellFormed(heap) ==>
\[{
  while (y > 0) {
    x = x + 1;
    y = y - 1;
  }
}\] (x = x0 + y0)
```

Finding the invariant

First attempt: use postcondition `x = x0 + y0`

- Not true at start whenever `y0 > 0`
- Not preserved by loop, because `x` is increased

Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

```
x = x0 & y = y0 & y0 >= 0 & wellFormed(heap) ==>
\[{
  while (y > 0) {
    x = x + 1;
    y = y - 1;
  }
}\] (x = x0 + y0)
```

Finding the invariant

What stays invariant?

- The sum of `x` and `y`: `x + y = x0 + y0` "Generalization"
- Can help to think of "δ" between `x` and `x0 + y0`

Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

```
x = x0 & y = y0 & y0 >= 0 & wellFormed(heap) ==>
\[{
  while (y > 0) {
    x = x + 1;
    y = y - 1;
  }
}\] (x = x0 + y0)
```

Checking the invariant

Is `x + y = x0 + y0` a good invariant?

- Holds in the beginning and is preserved by loop
- But postcondition not implied by `x + y = x0 + y0` and exit condition `y <= 0`

Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

```
x = x0 & y = y0 & y0 >= 0 & wellFormed(heap) ==>
\[{
  while (y > 0) {
    x = x + 1;
    y = y - 1;
  }
}\] (x = x0 + y0)
```

Strenghtening the invariant

Postcondition holds if `y = 0`

- Add `y >= 0` to invariant: `x + y = x0 + y0 & y >= 0`

Demo `loops/simple3.key`

Basic Loop Invariant: Context Loss

Problems with the Basic Invariant Rule

$$\text{loopInvariant}\ \frac{\begin{array}{rl} \Gamma \Longrightarrow \mathcal{U}\mathit{Inv}, \Delta & \text{(initially valid)} \\ \mathit{Inv},\ b = \texttt{TRUE} \Longrightarrow [\mathrm{p}]\mathit{Inv} & \text{(preserved)} \\ \mathit{Inv},\ b = \texttt{FALSE} \Longrightarrow [\pi\ \omega]\phi & \text{(use case)} \end{array}}{\Gamma \Longrightarrow \mathcal{U}[\pi\ \texttt{while(b)}\ \mathrm{p}\ \omega]\phi, \Delta}$$

- Context Γ, Δ, $\mathcal{U}$ must be omitted in 2nd and 3rd premise:

  $\Gamma, \neg\Delta$ cannot be assumed for arbitrary iterations or at loop exit

  2nd premise State after some loop iterations is not $\mathcal{U}$

  3rd premise State at loop exit is not $\mathcal{U}$
- Context contains preconditions and class invariants
- Only way to propagate context: add to loop invariant *Inv*

Example

Precondition: $\texttt{a} \neq \textbf{null}$ & *ClassInv*

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
    i++;
}
```

Postcondition: $\forall\, \textbf{int}\ x;\ (0 \leq x\ \&\ x < \texttt{a.length} \rightarrow \texttt{a}[x] = 1)$

Loop invariant: $0 \leq \texttt{i}\ \ \&\ \ \texttt{i} \leq \texttt{a.length}$
$\&\ \ \forall\, \textbf{int}\ x;\ (0 \leq x\ \&\ x < \texttt{i} \rightarrow \texttt{a}[x] = 1)$
$\&\ \ \texttt{a} \neq \textbf{null}$
$\&\ \ \textit{ClassInv}$

Keeping the Context (As In Method Contract Rule)

- Want to keep part of the context that is not modified by loop
- **assignable** clauses for loops tell what can possibly be modified

```
@ assignable i, a[*];
```

- How to erase all values of **assignable** locations?
- Anonymising updates $\mathcal{V}$ erase information about modified locations

Anonymising JAVA Locations

```
@ assignable i, a[*];
```

To erase all knowledge about these assignable locations:

- introduce a new (not yet used) constant of type `int`, e.g., `c`
- introduce a new (not yet used) constant of type Heap, e.g., $\mathtt{h}_{an}$
  - anonymise the current heap: $\mathtt{anon}(\mathtt{heap}, \mathtt{allFields}(\mathtt{a}), \mathtt{h}_{an})$
- compute anonymizing update for assignable locations

$$\mathcal{V} = \{\mathtt{i} := \mathtt{c} \parallel \mathtt{heap} := \mathtt{anon}(\mathtt{heap}, \mathtt{allFields}(\mathtt{a}), \mathtt{h}_{an})\}$$

For local program variables (e.g., `i`) KeY computes assignable clause automatically

Loop Invariants Cont'd

Improved Invariant Rule

$$\frac{\begin{array}{rl} \Gamma \Longrightarrow \mathcal{U}\mathit{Inv}, \Delta & \text{(initially valid)} \\ \Gamma \Longrightarrow \mathcal{U}\mathcal{V}(\mathit{Inv} \ \&\ b = \texttt{TRUE} \rightarrow [\mathrm{p}]\mathit{Inv}), \Delta & \text{(preserved)} \\ \Gamma \Longrightarrow \mathcal{U}\mathcal{V}(\mathit{Inv} \ \&\ b = \texttt{FALSE} \rightarrow [\pi\ \omega]\phi), \Delta & \text{(use case)} \end{array}}{\Gamma \Longrightarrow \mathcal{U}[\pi\ \texttt{while(b)}\ \mathrm{p}\ \omega]\phi, \Delta}$$

- Context is kept as far as possible:
  $\mathcal{V}$ erases only information in locations assignable in the loop
- Invariant *Inv* does not need to include unmodified locations
- For **`assignable \everything`** (the default):
  - $\mathtt{heap} := \mathtt{anon}(\mathtt{heap}, \mathtt{allLocs}, \mathtt{h}_{an})$ wipes out **all** heap information
  - Equivalent to basic invariant rule
  - Avoid this! Always give a specific **`assignable`** clause

Example with Improved Invariant Rule

Precondition: $\texttt{a} \neq \textbf{null}$ & *ClassInv*

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
    i++;
}
```

Postcondition: $\forall\, \textbf{int}\ x;\ (0 \leq x\ \&\ x < \texttt{a.length} \rightarrow \texttt{a}[x] = 1)$

Loop invariant: $0 \leq \texttt{i}\ \ \&\ \ \texttt{i} \leq \texttt{a.length}$
$\&\ \ \forall\, \textbf{int}\ x;\ (0 \leq x\ \&\ x < \texttt{i} \rightarrow \texttt{a}[x] = 1)$

Example in JML/JAVA – Loop.java

Demo

```
public int[] a;
/*@ public normal_behavior
  @  ensures (\forall int x; 0<=x && x<a.length; a[x]==1);
  @  diverges true;
  @*/
public void m() {
  int i = 0;
  /*@ loop_invariant
    @  0 <= i && i <= a.length &&
    @  (\forall int x; 0<=x && x<i; a[x]==1);
    @ assignable a[*];
    @*/
  while(i < a.length) {
    a[i] = 1;
    i++;
  }
}
```

Example from an earlier Lecture

$$\forall\ \texttt{int}\ x;$$
$$(x = \texttt{n} \wedge x >= 0 \rightarrow$$
$$[\ \texttt{i = 0; r = 0;}$$
$$\texttt{while (i<n) \{ i = i + 1; r = r + i;\}}$$
$$\texttt{r=r+r-n;}$$
$$]\,(\texttt{r} = x * x)$$

How can we prove that the above formula is valid
(i.e., satisfied in all states)?

Needed Invariant:

```
@ loop_invariant
@    i>=0  && i <= n && 2*r == i*(i + 1);
@ assignable \nothing; // no heap locations changed
```

Demo Loop2.java

Hints

Proving `assignable`

- Invariant rule above assumes that **`assignable`** is correct
  E.g., possible to prove nonsense with incorrect
  `assignable \nothing;`
- Invariant rule of KeY generates proof obligation that ensures correctness of **`assignable`**
  This proof obligation is part of 'Body Preserves Invariant' branch

Setting in the KeY Prover when proving loops w. given invariant

- Loop treatment: Invariant
- Quantifier treatment: No Splits with Progs
- If program contains `*`, `/`: Arithmetic treatment: DefOps
- Is search limit high enough (time out, rule apps.)?
- To prove only partial correctness, add **`diverges true;`**

Total Correctness

Is the sequent

$$\Longrightarrow [\texttt{i = -1; while (true)\{\}}]\texttt{i} = 4711$$

provable?

Yes, e.g.,

```
@ loop_invariant true;
@ assignable \nothing;
```

With this, correctness of non-terminating loop is provable:

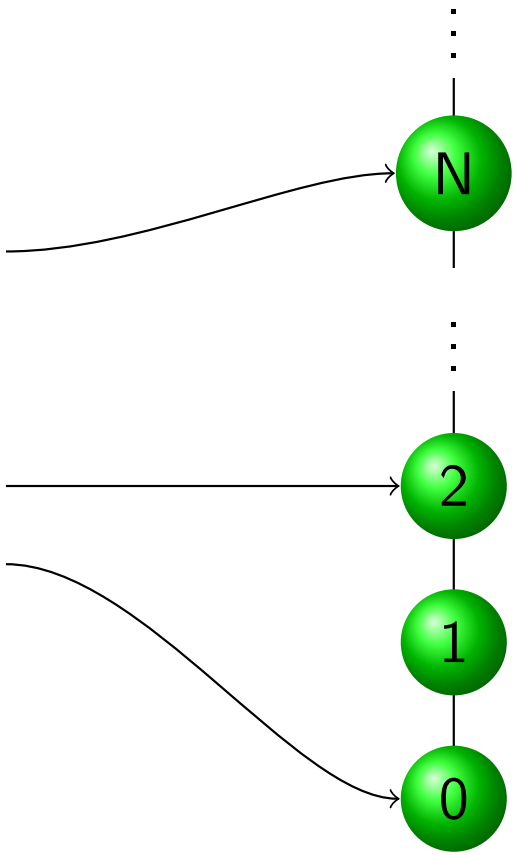
- Invariant trivially initially valid and preserved:
  Initial Case and Preserved Case close immediately
- Negated loop condition is `false`: Use case close immediately

But need a method to prove termination of loops

Mapping Loop Execution to Well-Founded Order

```
while (b) {
  body
}
```

```
if (b) { body }1
⋮
if (b) { body }17
if (b) { body }18
```



Need to find expression getting smaller wrt $\mathbb{N}$ in each iteration

Such an expression is called a decreasing term or variant

Total Correctness: Decreasing Term (Variant)

Find a decreasing integer term v (called variant)

Add the following premisses to the invariant rule:

- $v \geq 0$ is initially valid
- $v \geq 0$ is preserved by the loop body
- v is *strictly* decreased by the loop body

Proving termination in JML/JAVA

- Remove `diverges true;` from contract
- Add `decreasing v;` to loop invariant
- KeY creates suitable invariant rule and PO (with $\langle \ldots \rangle \phi$)

Example (The `array` loop)

```
@ decreasing  a.length - i;
```

Files:

- `LoopT.java`
- `Loop2T.java`

Final Example: Computing the GCD (see 16.3.8 [KeYbook])

```
public class Gcd {
 /*@ public normal_behavior
   @ requires _small>=0 && _big>=_small;
   @ ensures _big!=0 ==>
   @  (_big % \result == 0 && _small % \result == 0 &&
   @    (\forall int x; x>0 && _big % x == 0
   @       && _small % x == 0; \result % x == 0));
   @ assignable \nothing;
   @*/
 private static int gcdHelp(int _big, int _small) {
   int big = _big; int small = _small;
   while (small != 0) {
     final int t = big % small;
     big = small;
     small = t;
   }
   return big;
 }
}
```

Computing the GCD: Method Specification

```
public class Gcd {
 /*@ public normal_behavior
   @ requires _small>=0 && _big>=_small;
   @ ensures _big!=0 ==>
   @ (_big % \result == 0 && _small % \result == 0 &&
   @   (\forall int x; x>0 && _big % x == 0
   @     && _small % x == 0; \result % x == 0));
   @ assignable \nothing;
   @*/
 private static int gcdHelp(int _big, int _small) {...}
```

requires normalization assumptions on method parameters (both non-negative and `_big` $\geq$ `_small`)

ensures if `_big` positive, then

- the return value `\result` is a divisor of both arguments
- all other divisors `x` of the arguments are also dividers of `\result` and thus smaller or equal to `\result`

Computing the GCD: Specify the Loop Body

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Which locations are changed (at most)?

```
@ assignable \nothing; // no heap locations changed
```

What is the variant?

```
@ decreases small;
```

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Possible for `big` to become 0 in a loop iteration? No.
- Adding `big>0` to loop invariant? No. Not initially valid.
- Weaker condition necessary: `big==0 ==> _big==0`

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Weaker condition necessary: `big==0 ==> _big==0`
- What does the loop preserve? The set of dividers!
  All common dividers of `_big`, `_small` are also dividers of `big`, `small`

```
(\forall int x; x > 0;
                (_big%x == 0 && _small%x == 0)
                <==>
                (big%x == 0 && small%x == 0));
```

Computing the GCD: Final Specification

```
int big = _big; int small = _small;
/*@ loop_invariant small >= 0 && big >= small &&
  @   (big == 0 ==> _big == 0) &&
  @   (\forall int x; x > 0; (_big % x == 0 && _small % x == 0)
  @                          <==>
  @                          (big % x == 0 && small % x == 0));
  @ decreases small;
  @ assignable \nothing;
  @*/
  while (small != 0) {
    final int t = big % small;
    big = small;
    small = t;
  }
  return big; // assigned to \result
```

Why does `big` divides `_small` and `_big` follow from the loop invariant?

If `big` is positive, one can instantiate `x` with it, and use `small == 0`

Computing the GCD: Demo

Demo `loops/Gcd.java`

1. Show `Gcd.java` and `gcd(a,b)`
2. Ensure that "DefOps" and "Contract" is selected, $\geq$ 10,000 steps
3. Proof contract of `gcd()`, using contract of `gcdHelp()`
4. Note KeY check sign in parentheses:
   - 4.1 Click "Proof Management"
   - 4.2 Choose tab "By Proof"
   - 4.3 Select proof of `gcd()`
   - 4.4 Select used method contract of `gcdHelp()`
   - 4.5 Click "Start Proof"
5. After finishing proof obligations of `gcdHelp()` parentheses are gone

Some Hints On Finding Invariants

General Advice

- Invariants must be developed, they don't come out of thin air!
- Be as systematic in deriving invariants as when debugging a program

Some Hints On Finding Invariants, Cont'd

Technical Hints

- Good starting point: desired postcondition (of the loop!)
  - What, in addition to negated loop guard, is needed for it to hold?
- If the invariant candidate is not preserved by the loop body:
  - Can you add stuff from the precondition?
  - Does it need strengthening?
  - Try to express the relation between partial and final result
- Simulate a few loop body executions to discover invariant patterns
- If the invariant is not initially valid:
  - Can it be weakened such that the postcondition still follows?
  - Did you forget an assumption in the requires clause?
- Several "rounds" of weakening/strengthening might be required
- Use the KeY tool to iteratively try invariants:
  - Loop treatment: None
  - apply **Loop Invariant** $\rightarrow$ **Enter Loop Specification**
  - After each change of invariant make sure all cases are ok
  - If not, prue and retry

Understanding Unclosed Proofs (see also p.528ff [KeYbook])

Reasons why a proof may not close

- Buggy or incomplete specification
- Bug in program
- Maximal number of steps reached: restart or increase # of steps
- Automatic proof search fails: apply some rules manually

Understanding open proof goals

- Follow the control flow from the proof root to the open goal
- Branch labels give useful hints
- Identify unprovable part of post condition or invariant
- Sequent remains always in "pre-state"
  Constraints on program variables refer to value at start of program
  (exception: formula is behind update or modality)
- NB: $\Gamma \Longrightarrow o = \mathbf{null}, \Delta$ is equivalent to $\Gamma, o \neq \mathbf{null} \Longrightarrow \Delta$

Literature for this Lecture

KeYbook *W. Ahrendt*, B. Beckert, R. Bubel, R. Hähnle, P. Schmitt, M. Ulbrich, editors.
Deductive Software Verification - The KeY Book
Vol 10001 of *LNCS*, Springer, 2016
(E-book at `link.springer.com`)

- *W. Ahrendt*, S. Grebing, Using the KeY Prover
  Chapter 15 in [KeYbook], p.528ff + Section 15.3 (also for Lab2)
- B. Beckert, R. Hähnle, M. Hentschel, P.H. Schmitt,
  Formal Verification with KeY: A Tutorial
  Chapter 16 in [KeYbook], except Section 16.6

further reading:

- B. Beckert, V. Klebanov, B. Weiß, Dynamic Logic for Java
  Chapter 3 in [KeYbook], Section 3.7

Master's Thesis Projects in Formal Methods

see Formal Methods Master Theses on the web (cklick here).

Thank You