Data structures

Binary, leftist, and skew heaps

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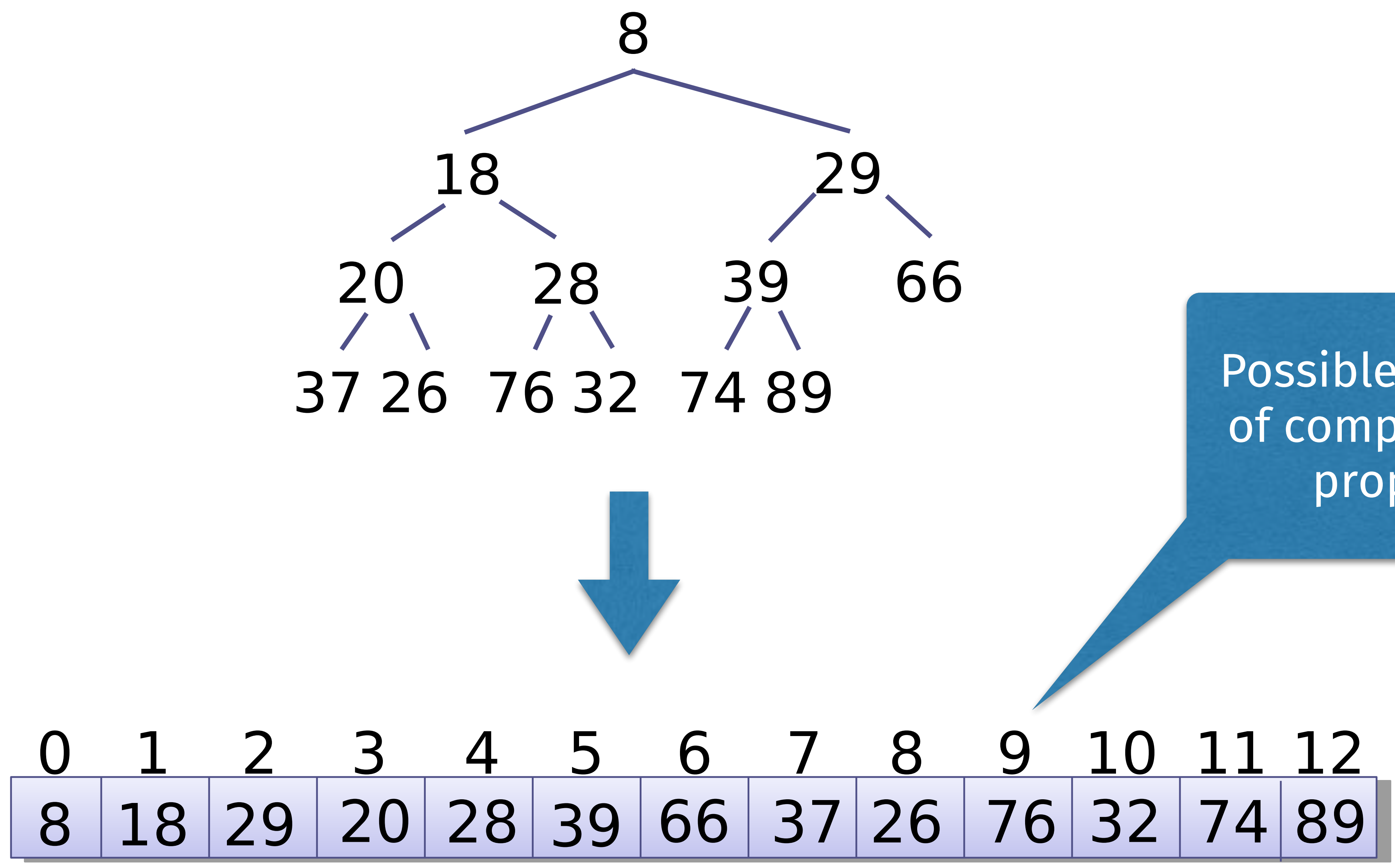
Why completeness?

- There are a couple of reasons why we choose to have a complete tree:
  - It makes sure the tree is balanced
  - When we insert a new element, it means there is only one place the element can go – this is one less design decision we have to make

- **There is a third reason which trumps the first two, but that will have to wait for next time!**

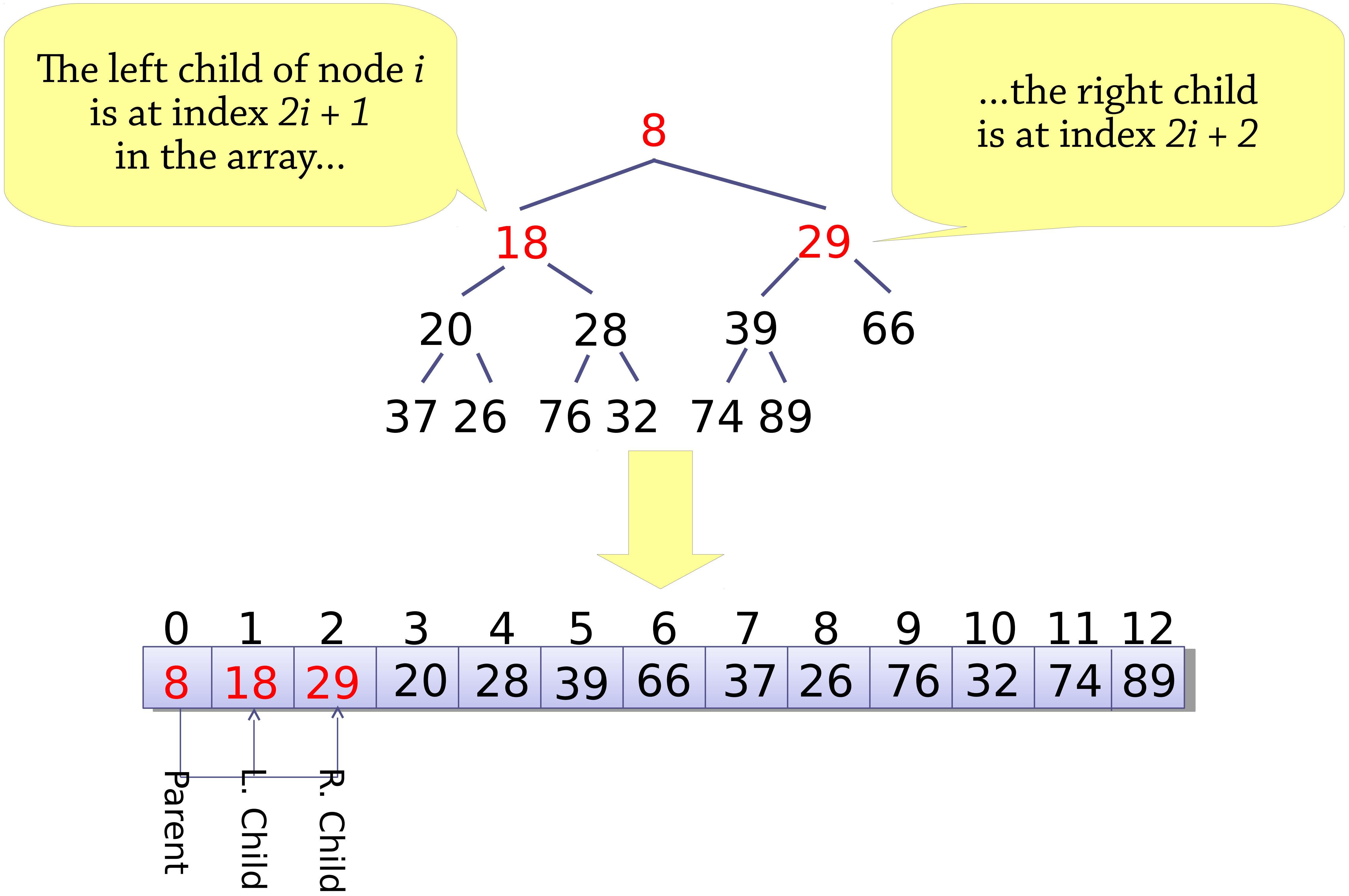
Binary heaps are arrays!

- A binary heap is really implemented using an array!

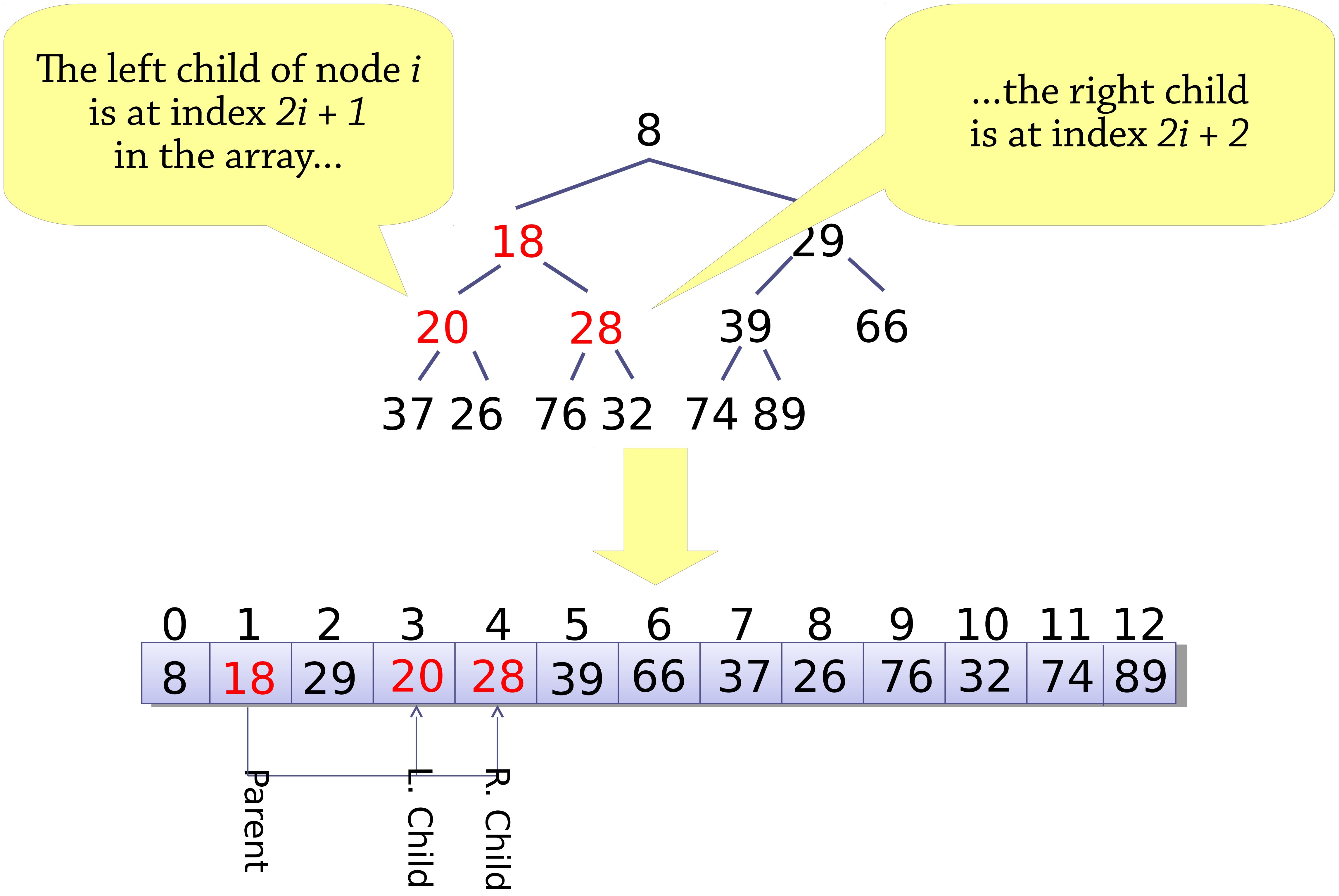


Possible because of completeness property

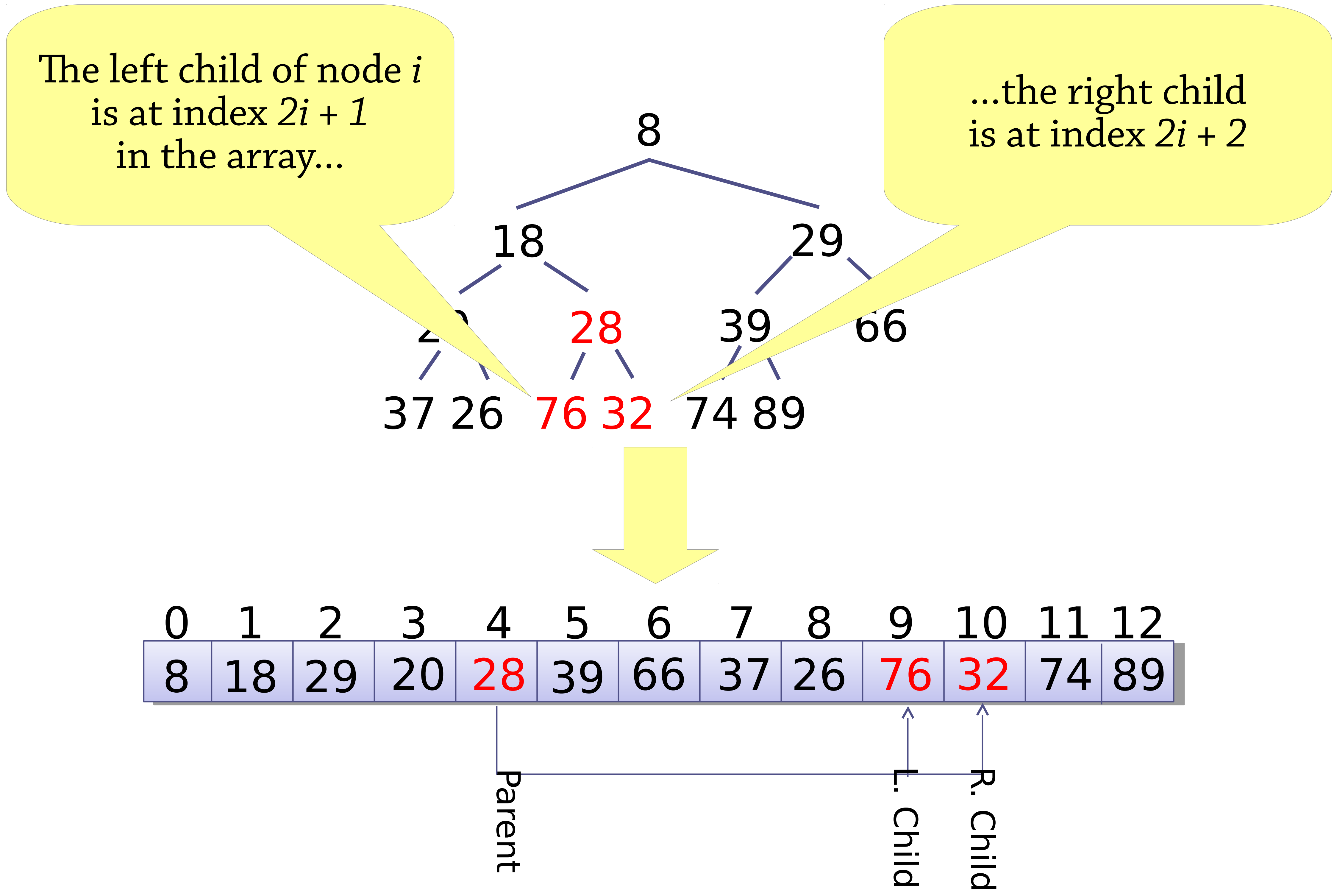
Child positions



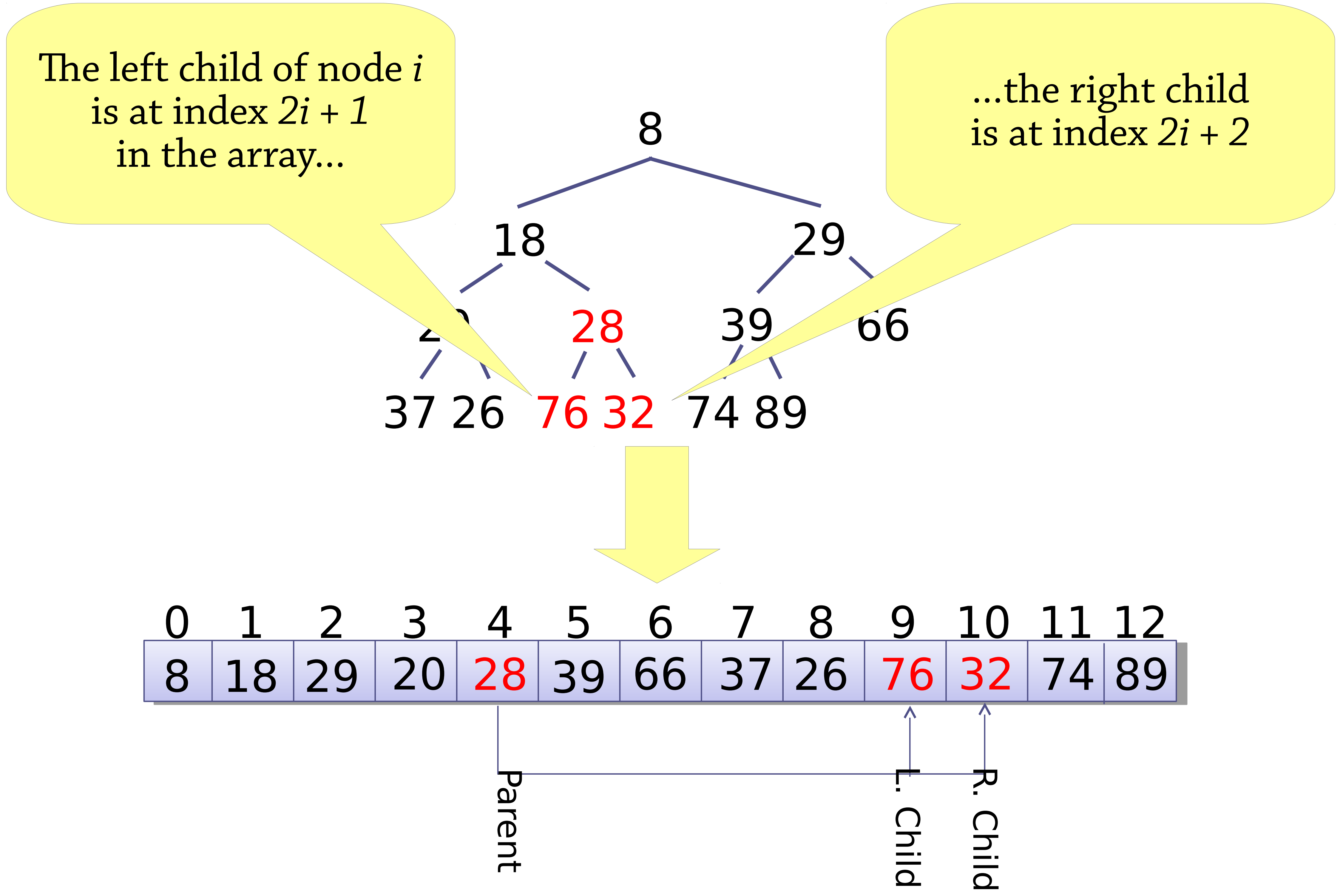
Child positions



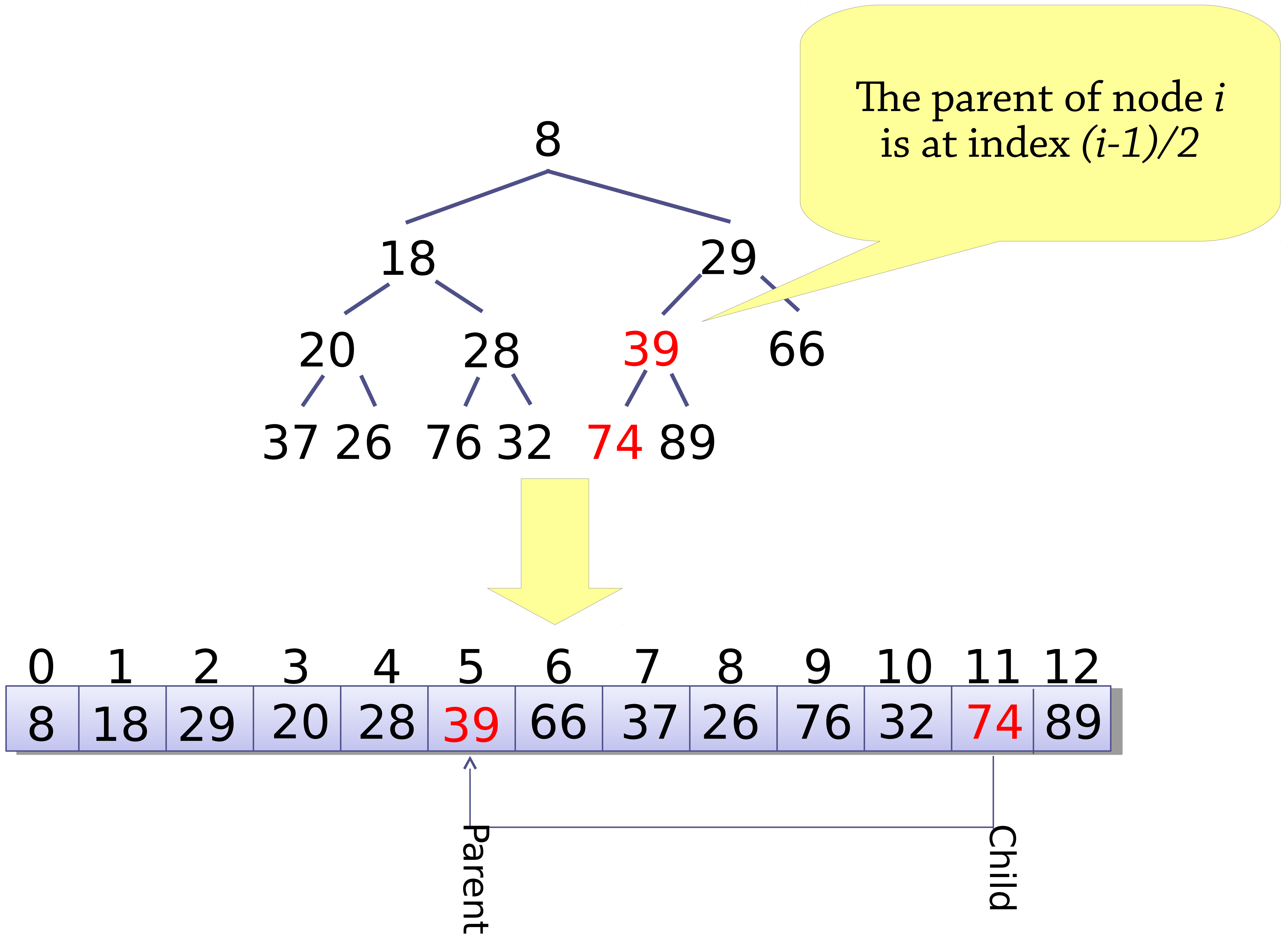
Child positions



Child positions



Parent position

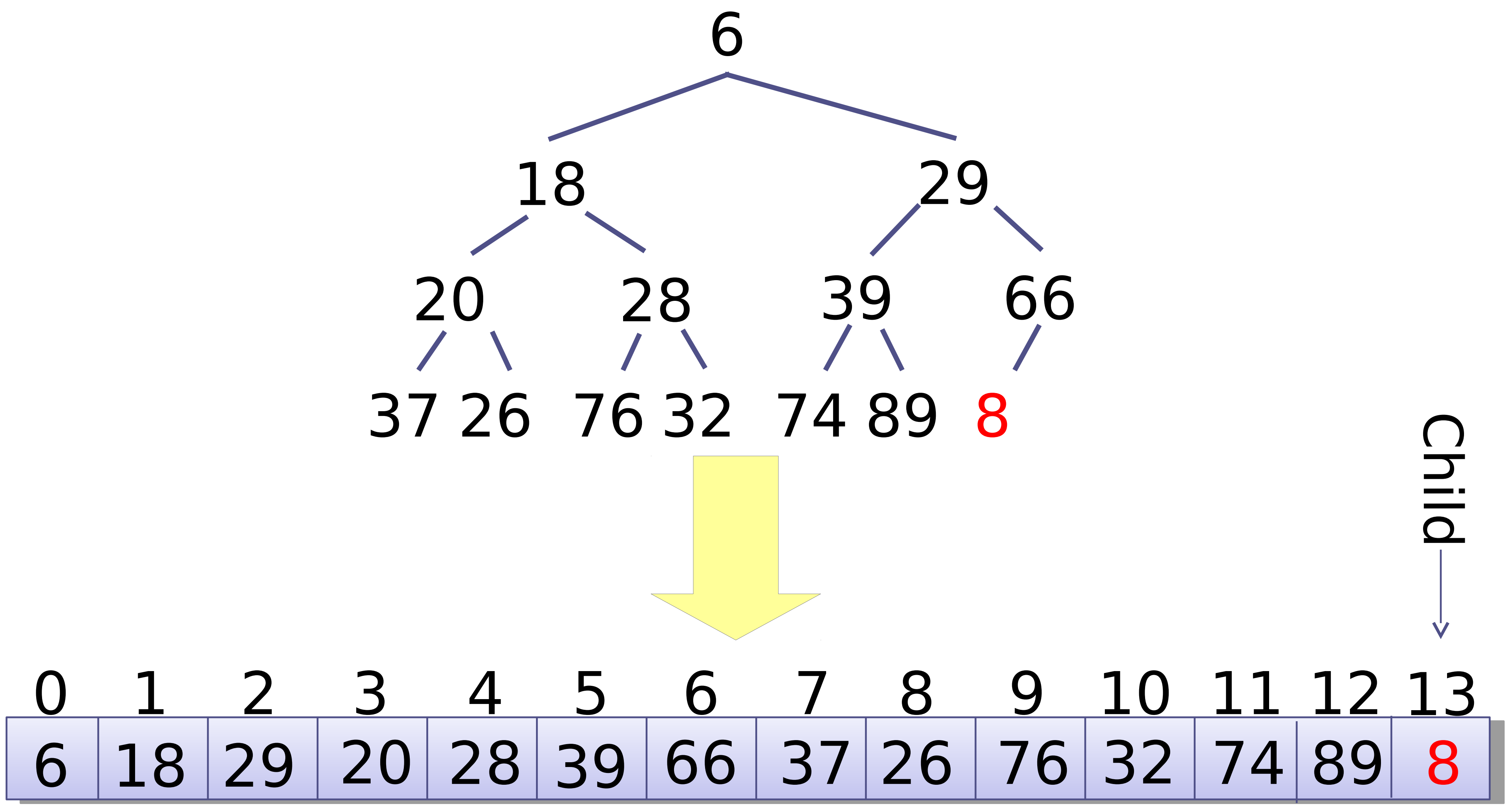


Reminder: inserting into a binary heap

- To insert an element into a binary heap:
  - Add the new element at the end of the heap
  - Sift the element up: while the element is less than its parent, swap it with its parent
- We can do exactly the same thing for a binary heap represented as an array!

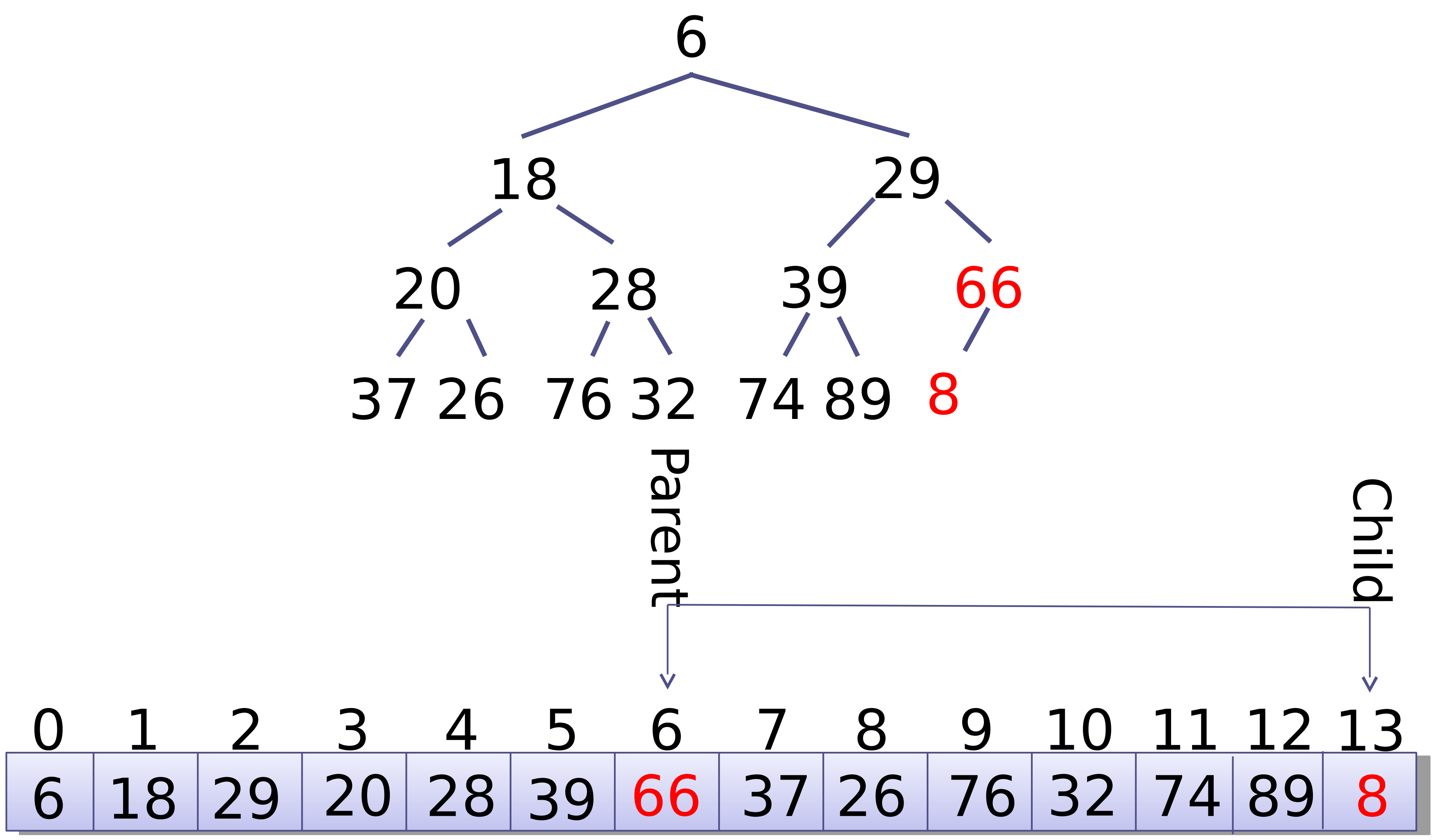
Inserting into a binary heap

- Step 1: add the new element to the end of the array, set child to its index



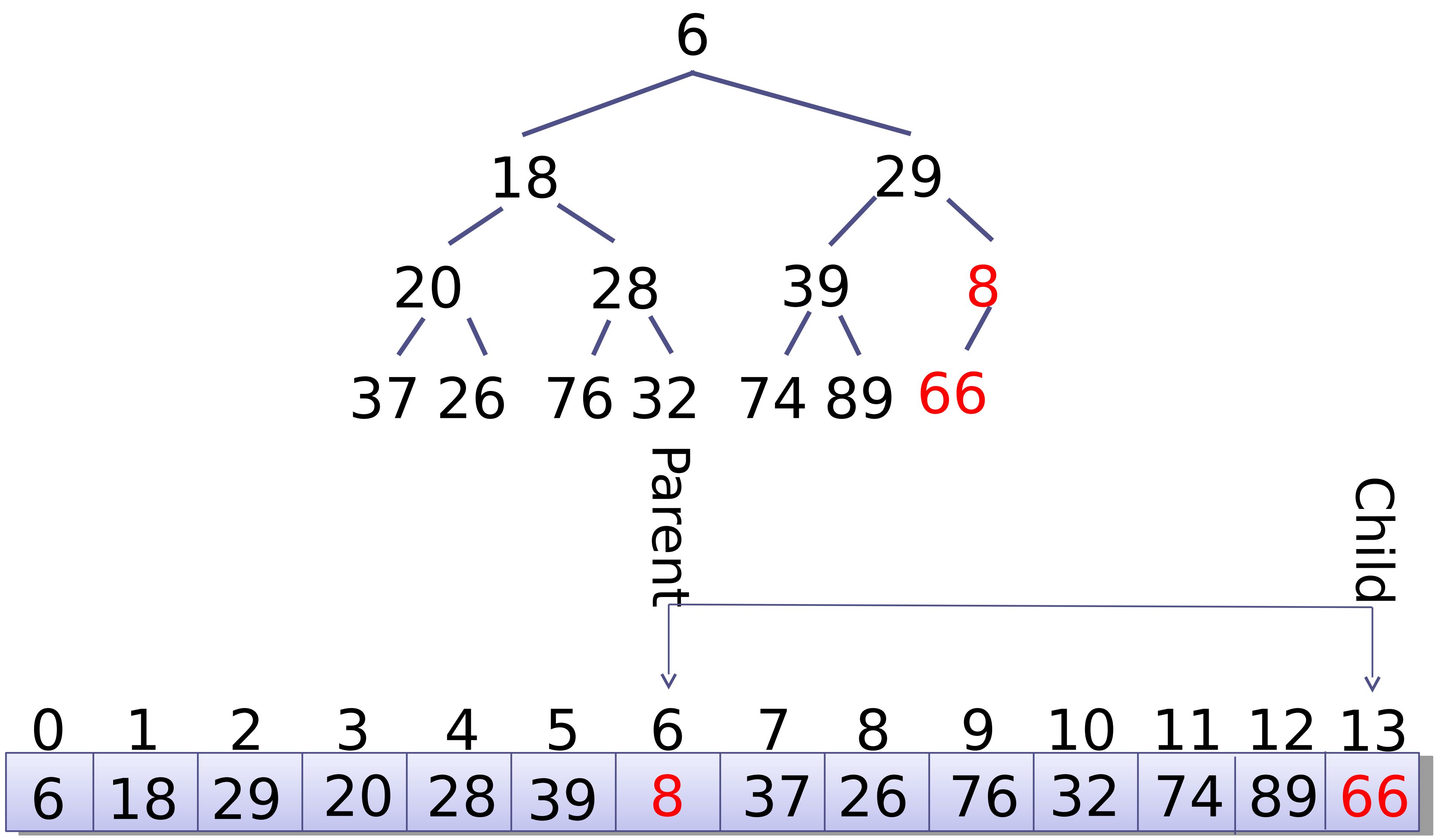
Inserting into a binary heap

- Step 2: compute `parent = (child-1)/2`



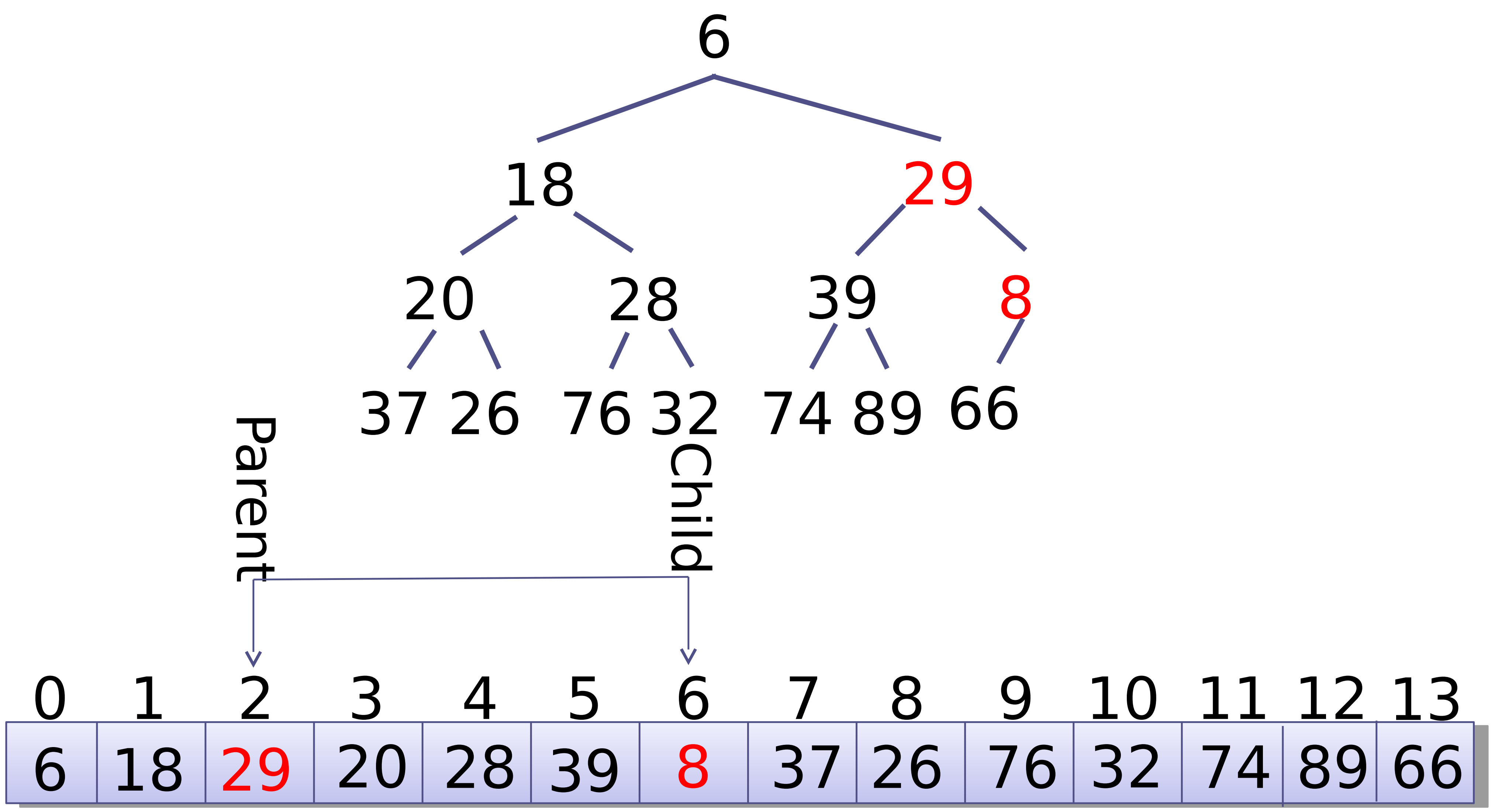
Inserting into a binary heap

- Step 3: `if array[parent] > array[child]`, swap them



Inserting into a binary heap

- Step 4: set `child = parent`, `parent = (child – 1) / 2`, and repeat



Inserting into a binary heap

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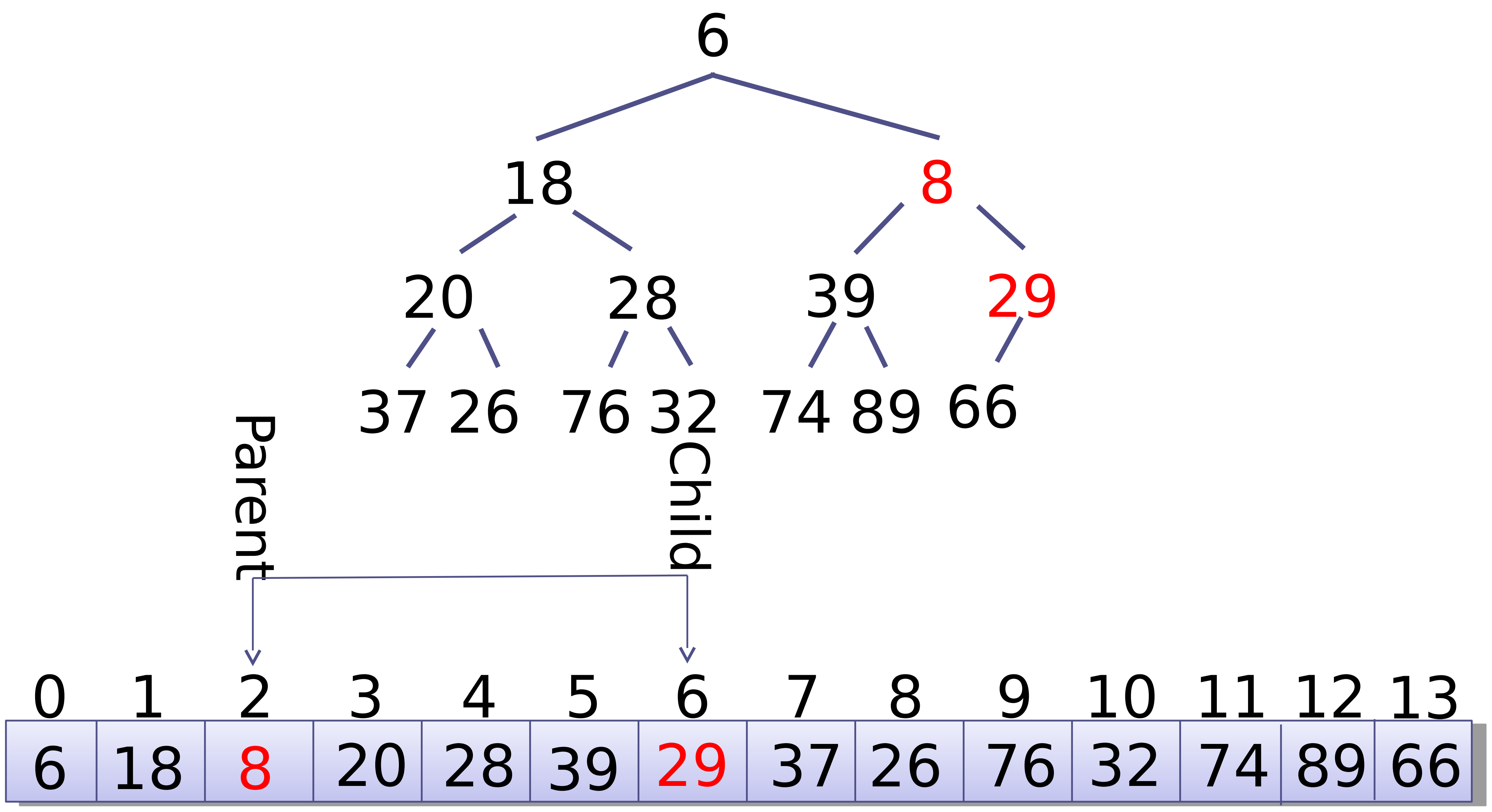


Binary heaps as arrays

- Binary heaps are “morally” trees
  - This is how we view them when we design the heap algorithms
- But we implement the tree as an array
  - The actual implementation translates these tree concepts to use arrays
- In the rest of the lecture, we will only show the heaps as trees
  - But you should have the “array view” in your head when looking at these trees

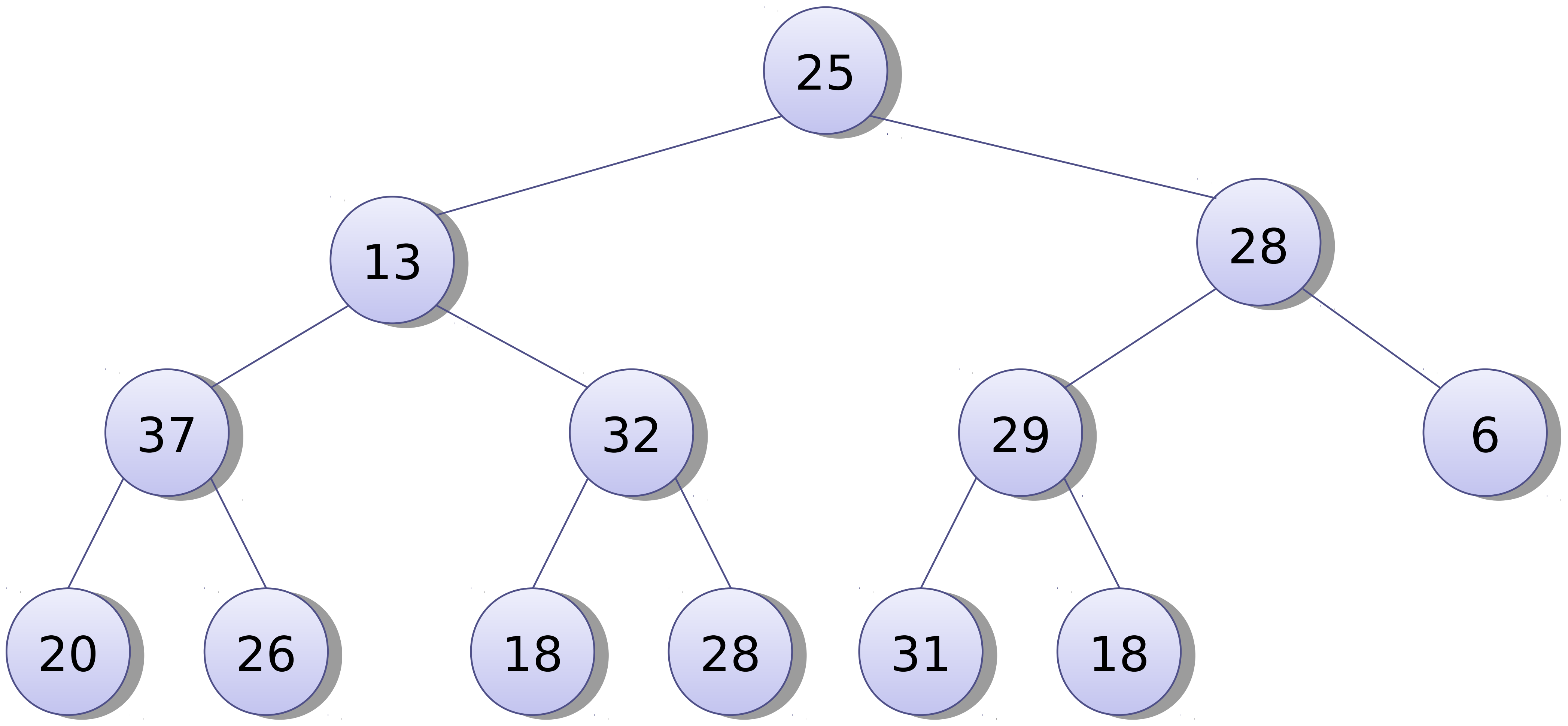
Building a heap

- One more operation, build heap
  - Takes an arbitrary array and makes it into a heap
  - In-place: moves the elements around to make the heap property hold

- Idea:
  - Heap property: each element must be less than its children If a node breaks this property, we can fix it by sifting down
  - So simply looping through the array and sifting down each element in turn ought to fix the invariant
  - But when we sift an element down, its children must already have the heap property (otherwise the sifting doesn't work)
  - To ensure this, loop through the array in reverse

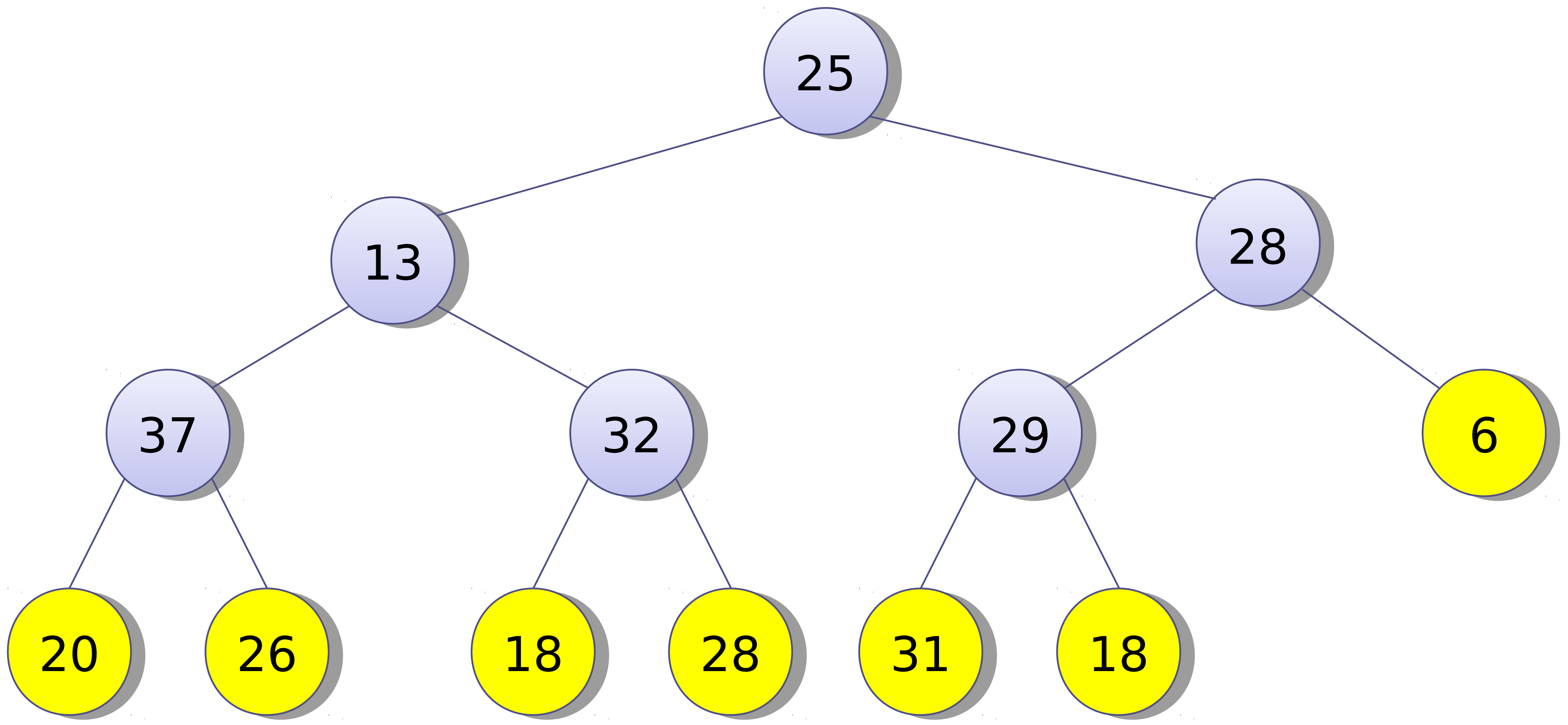
Building a heap

- Go through elements in reverse order, sifting each down



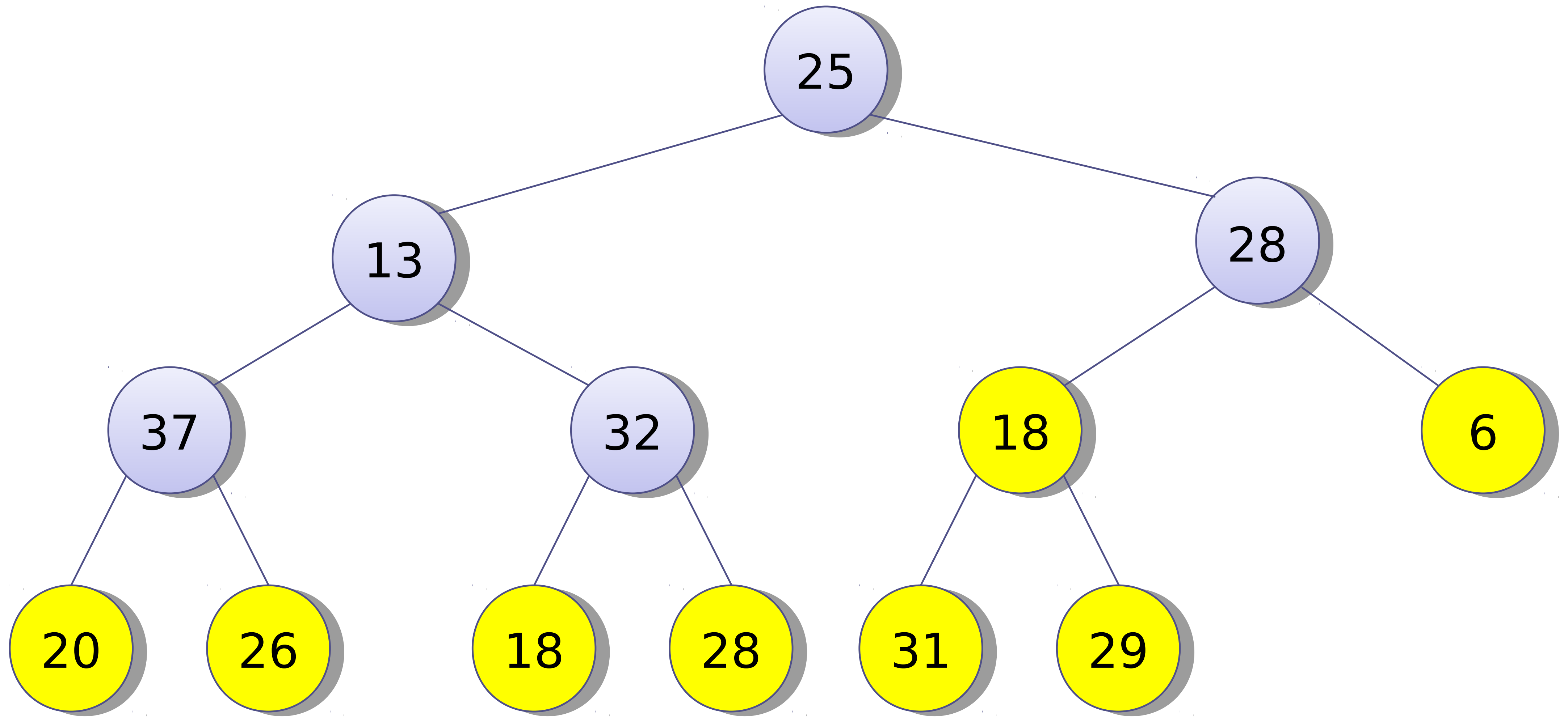
Building a heap

- Leaves never need sifting down!



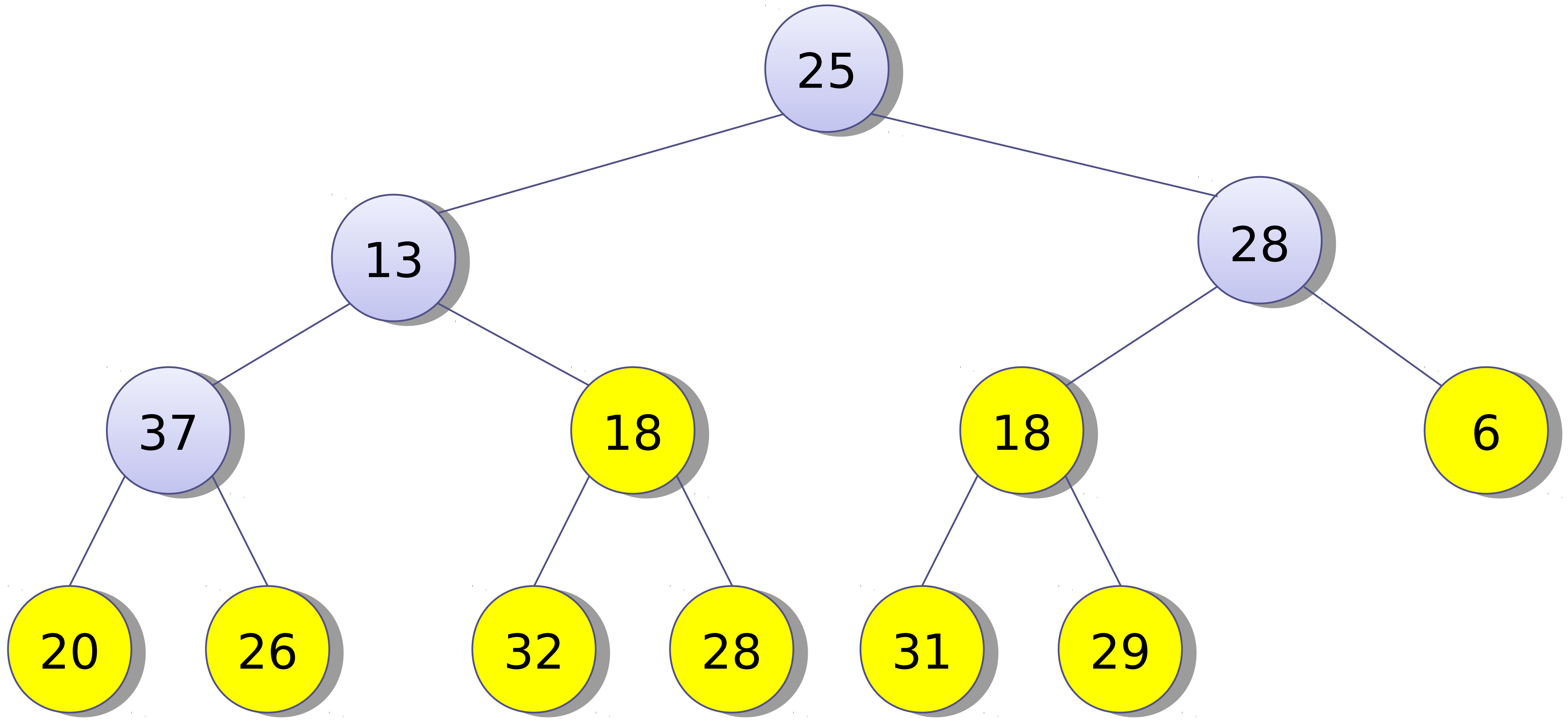
Building a heap

- 29 is greater than 18 so needs swapping



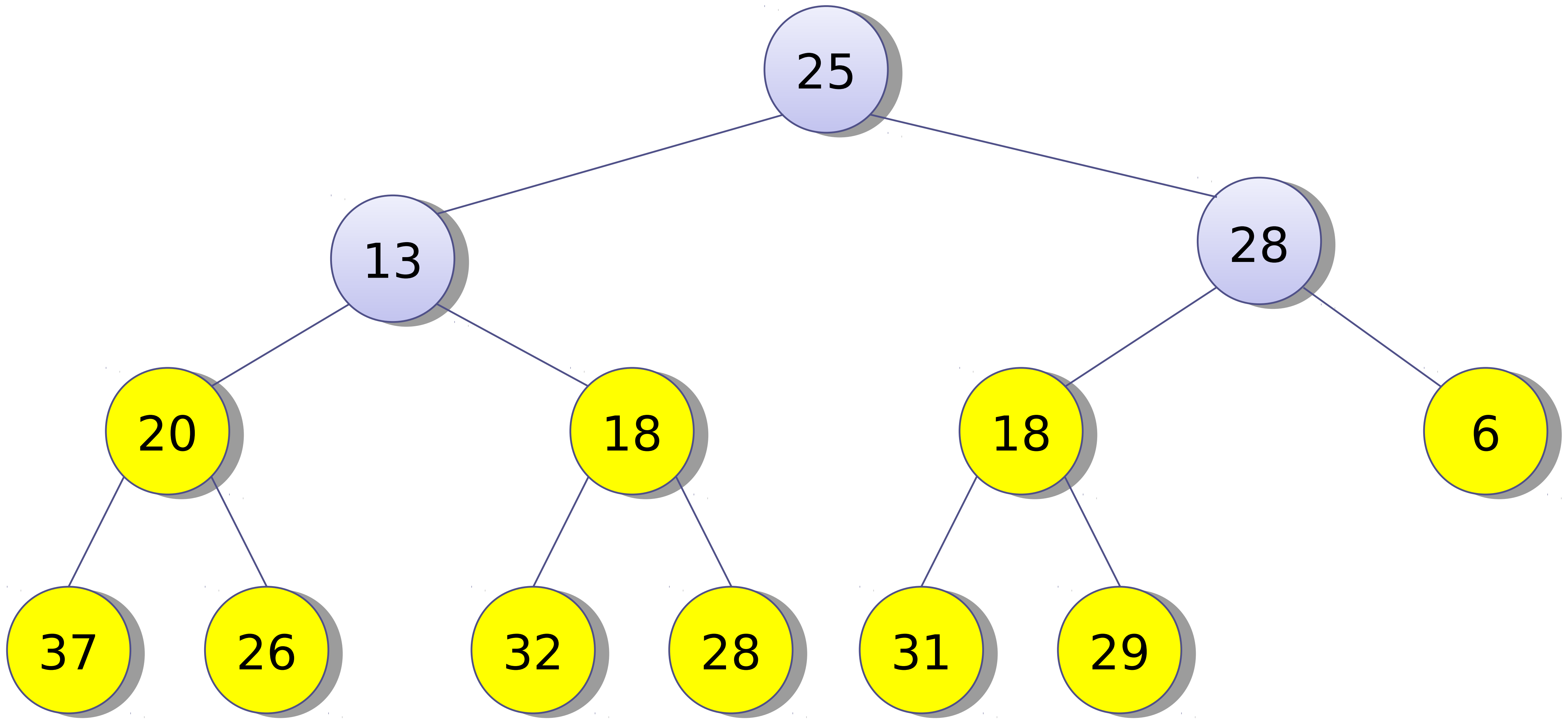
Building a heap

- 32 is greater than 18 so needs swapping



Building a heap

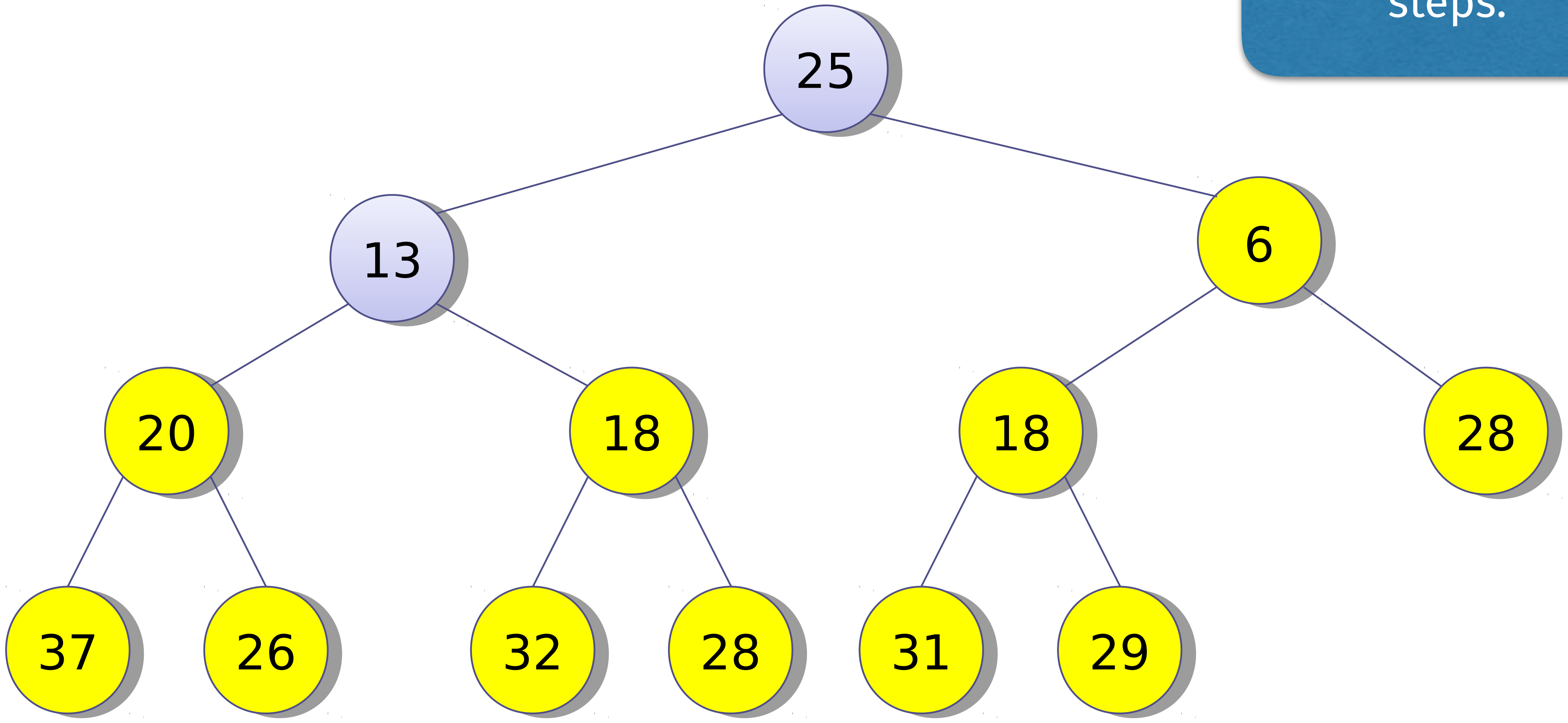
- 37 is greater than 20 so needs swapping



Building a heap

- 28 is greater than 6 so needs swapping

Your turn! Think about the next steps.



Build heap complexity

- You would expect $O(n \log n)$ complexity:
  - n “sift down” operations
  - each sift down has $O(\log n)$ complexity
- Actually, it's $O(n)$!
  - (Rough reason: sifting down is most expensive for elements near the root of the tree, but the vast majority of elements are near the leaves)

Heapsort

- To sort a list using a heap:
  - start with an empty heap
  - add all the list elements in turn
  - repeatedly find and remove the smallest element from the heap, and add it to the result list
- (this is a kind of selection sort)
- However, this algorithm is not in-place. Heapsort uses the same idea, but without allocating any extra memory.

Heapsort, in-place

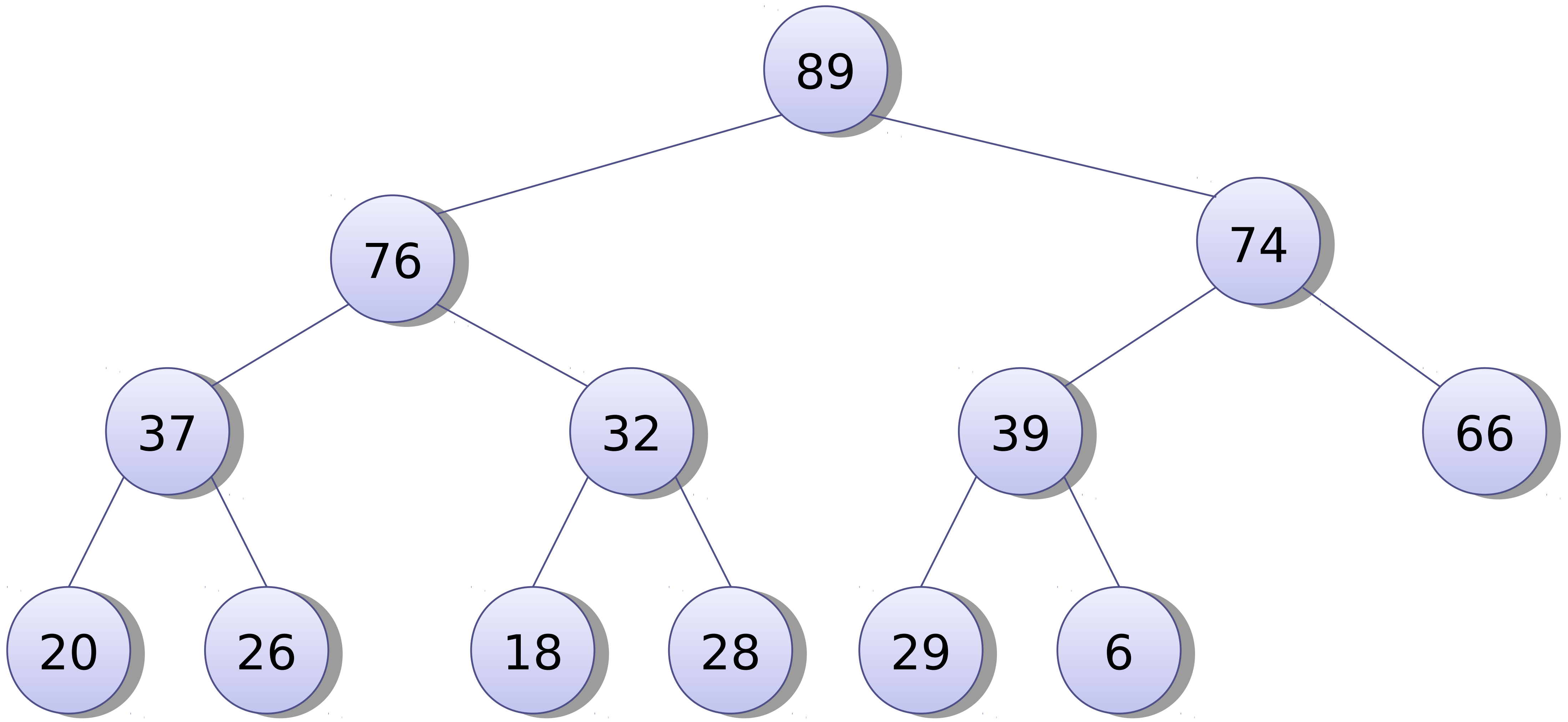
- We are going to repeatedly remove the *largest* value from the array and put it in the right place
  - using a so-called *max heap*, a heap where you can find and delete the *maximum* element instead of the minimum
- We'll divide the array into two parts
  - The first part will be a heap
  - The rest will contain the values we've removed
- (This division is the same idea we used for in-place selection and insertion sort)

Heapsort, in-place

- First turn the array into a heap
- Then just repeatedly delete the maximum element!
  Remember the deletion algorithm:
  - Swap the maximum (first) element with the last element of the heap
  - Reduce the size of the heap by 1
    (deletes the first element, breaks the invariant)
  - Sift the first element down
    (repairs the invariant)
- The *swap* actually puts the maximum element in the right place in the array

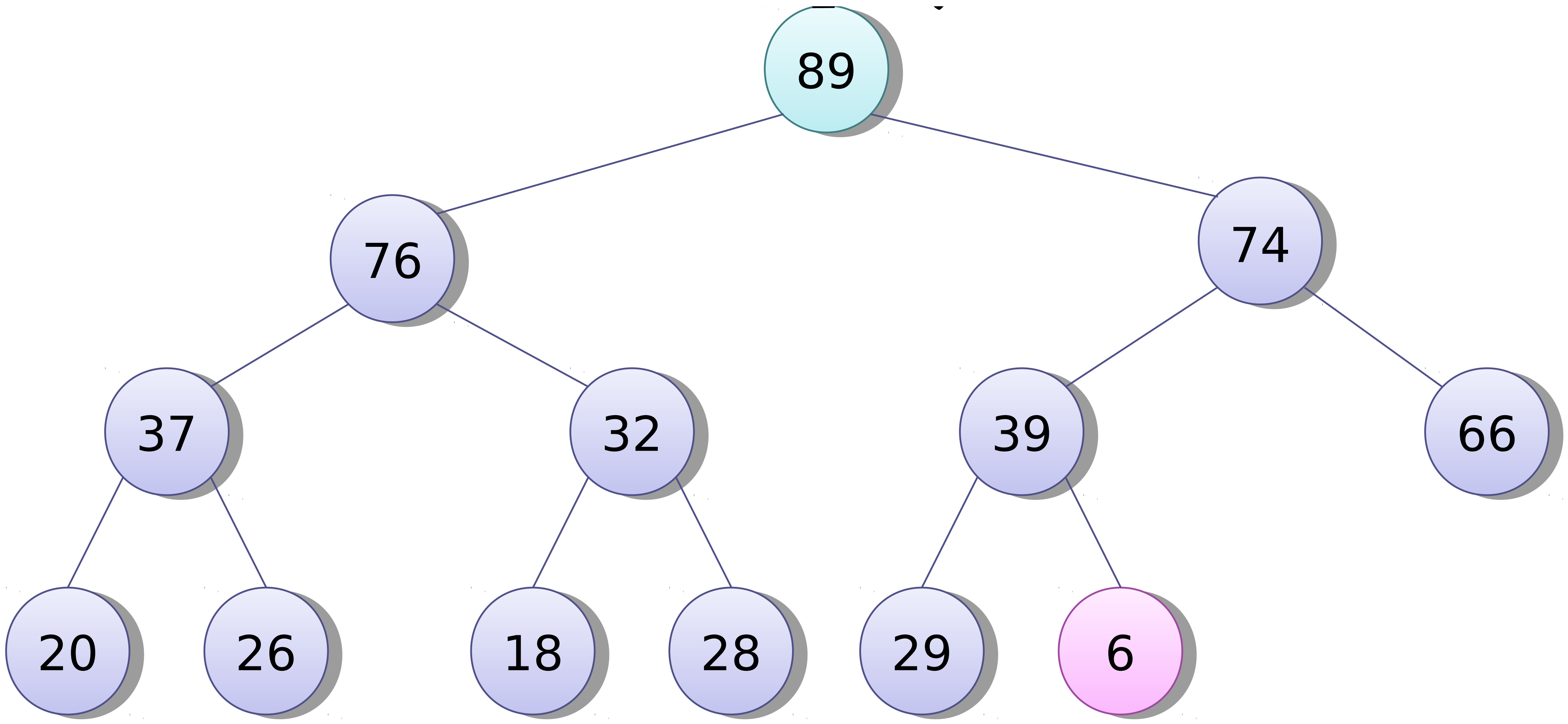
Trace of heapsort

- First build a heap (not shown)



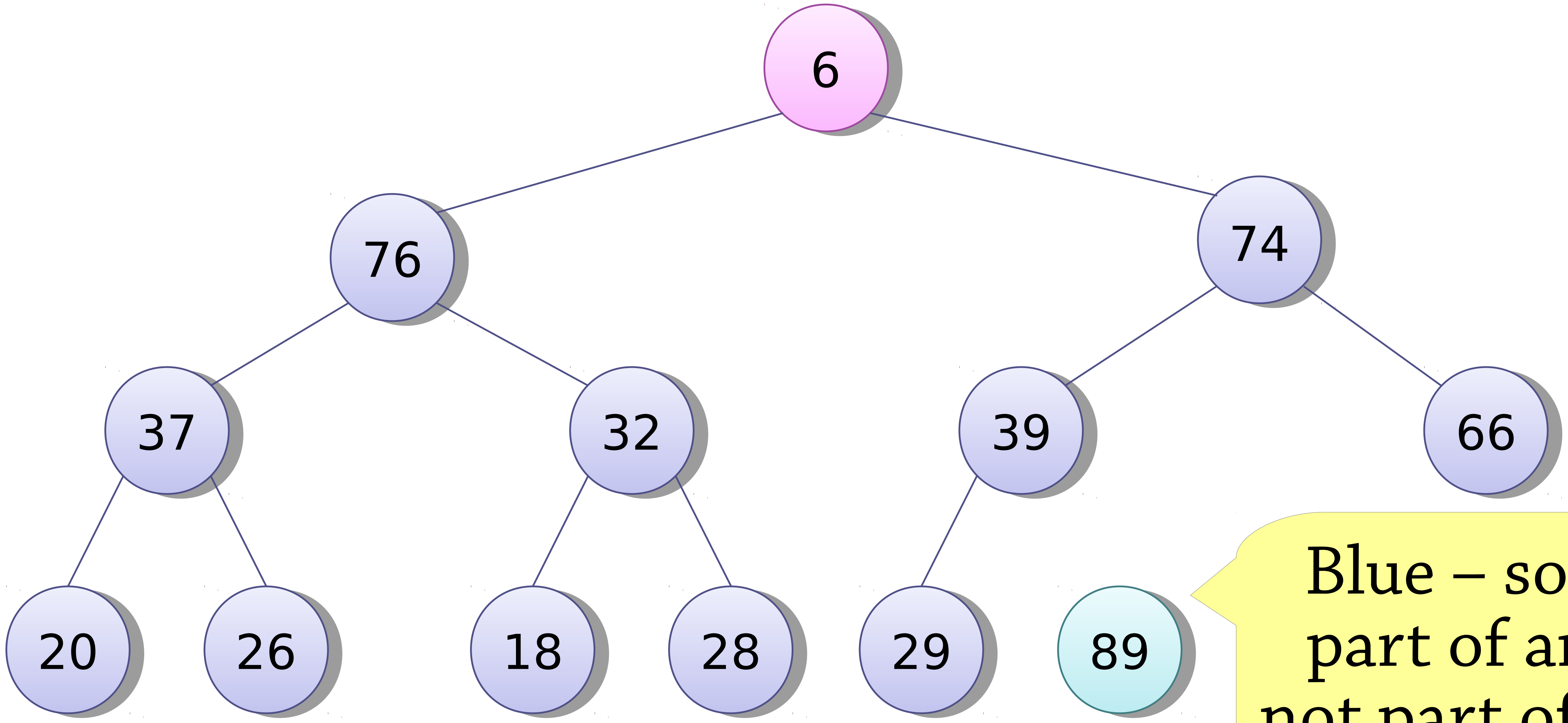
Trace of heapsort

- Step 1: swap maximum and last element; decrease size of heap by 1



Trace of heapsort

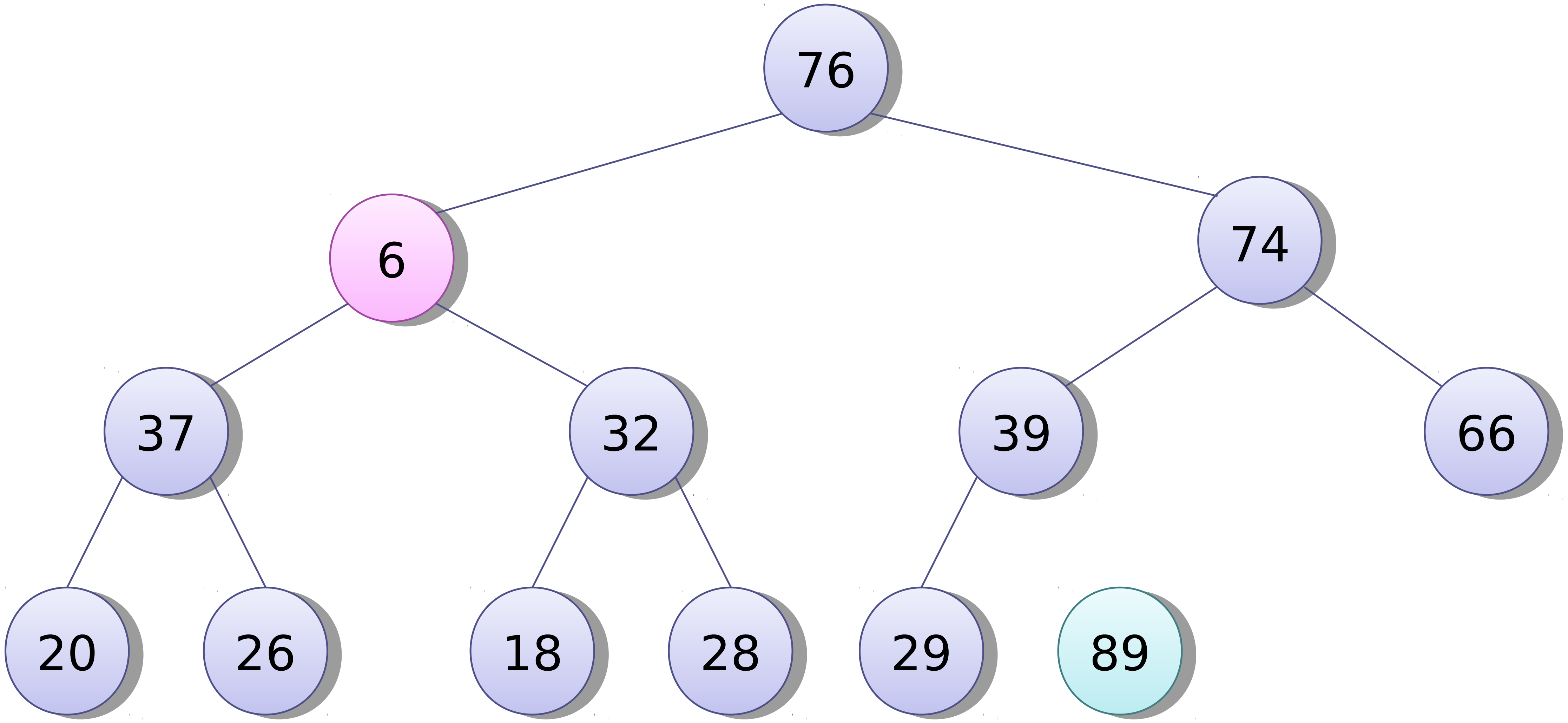
- Step 2: sift first element down



Blue – sorted part of array, not part of heap

Trace of heapsort

- Step 2: sift first element down



Trace of heapsort

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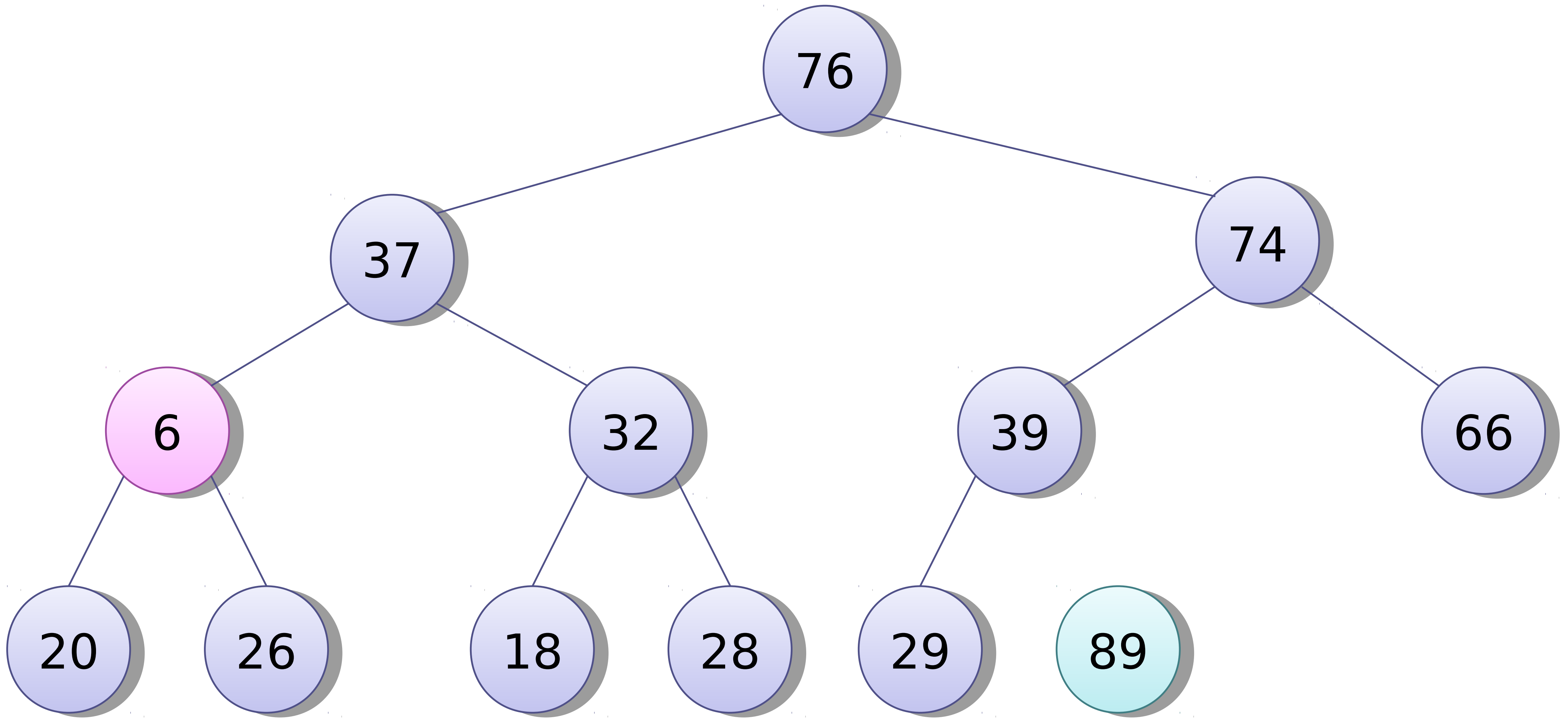
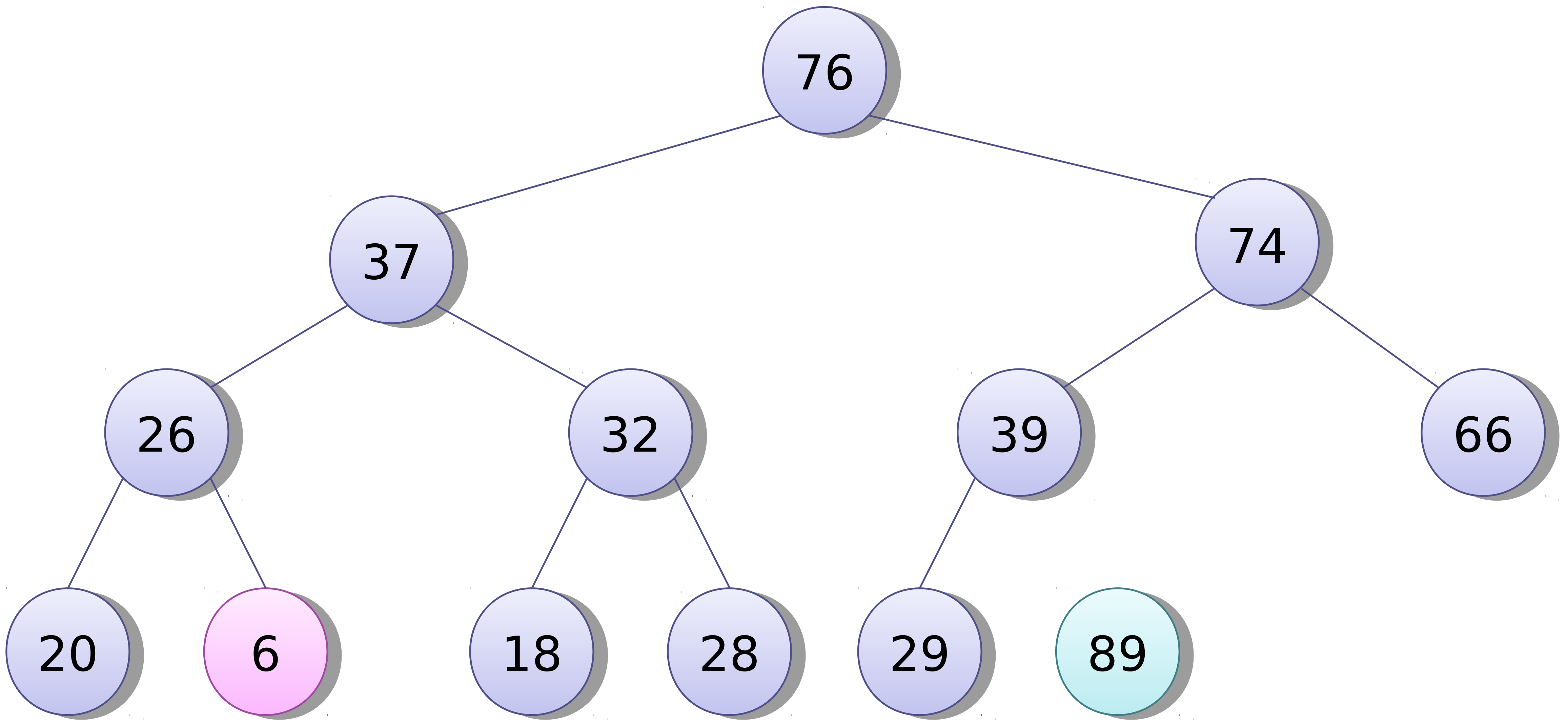


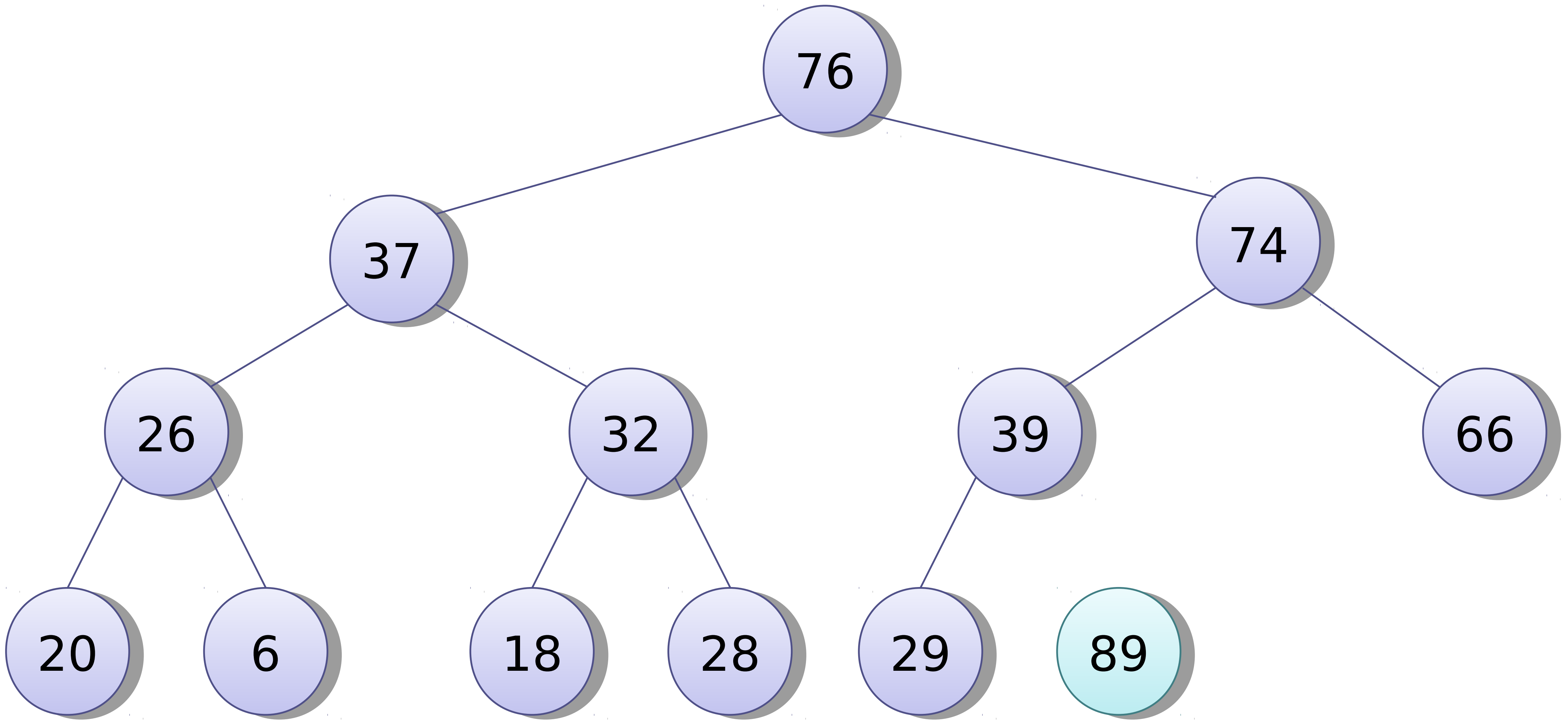
Trace of heapsort

- Step 2: sift first element down



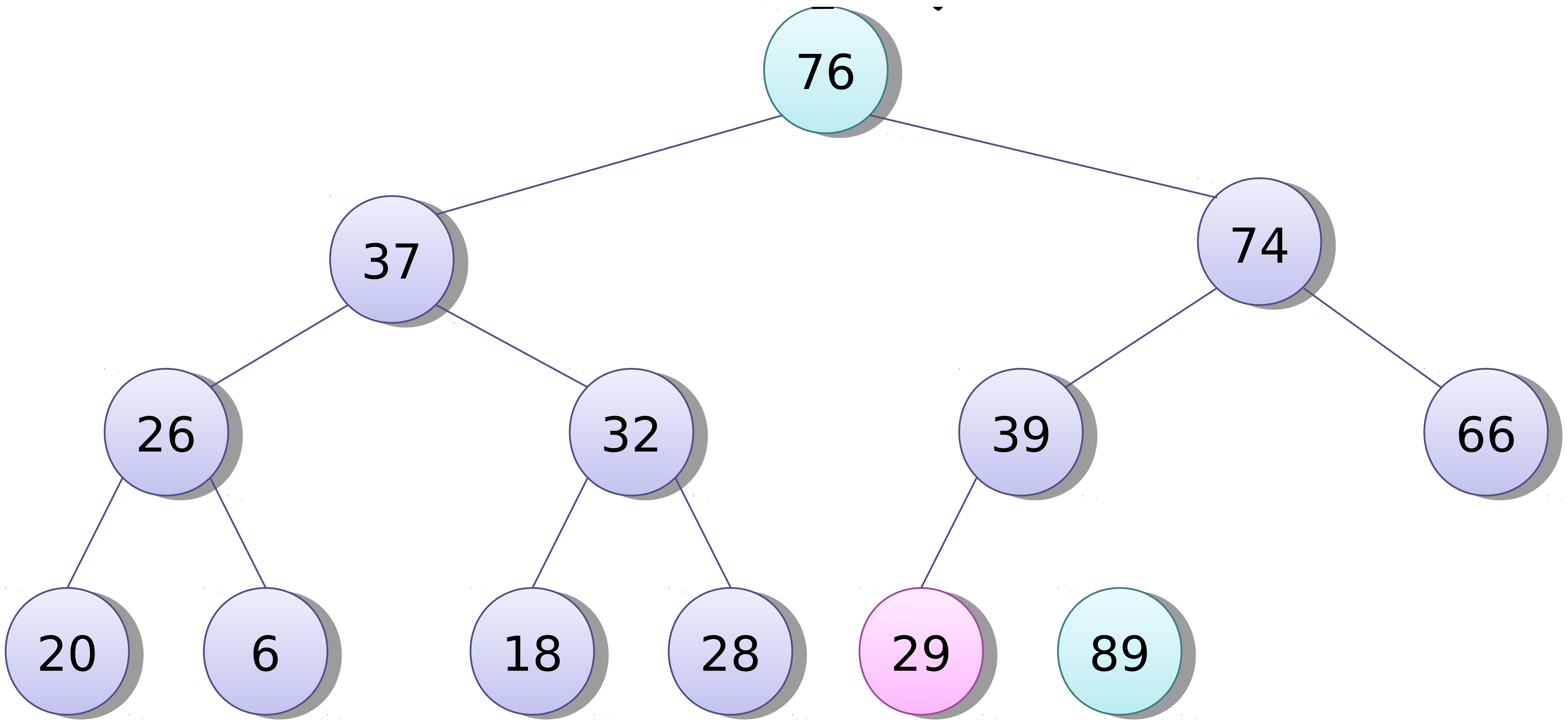
Trace of heapsort

- We now have the biggest element at the end of the array, and a heap that's one element smaller!



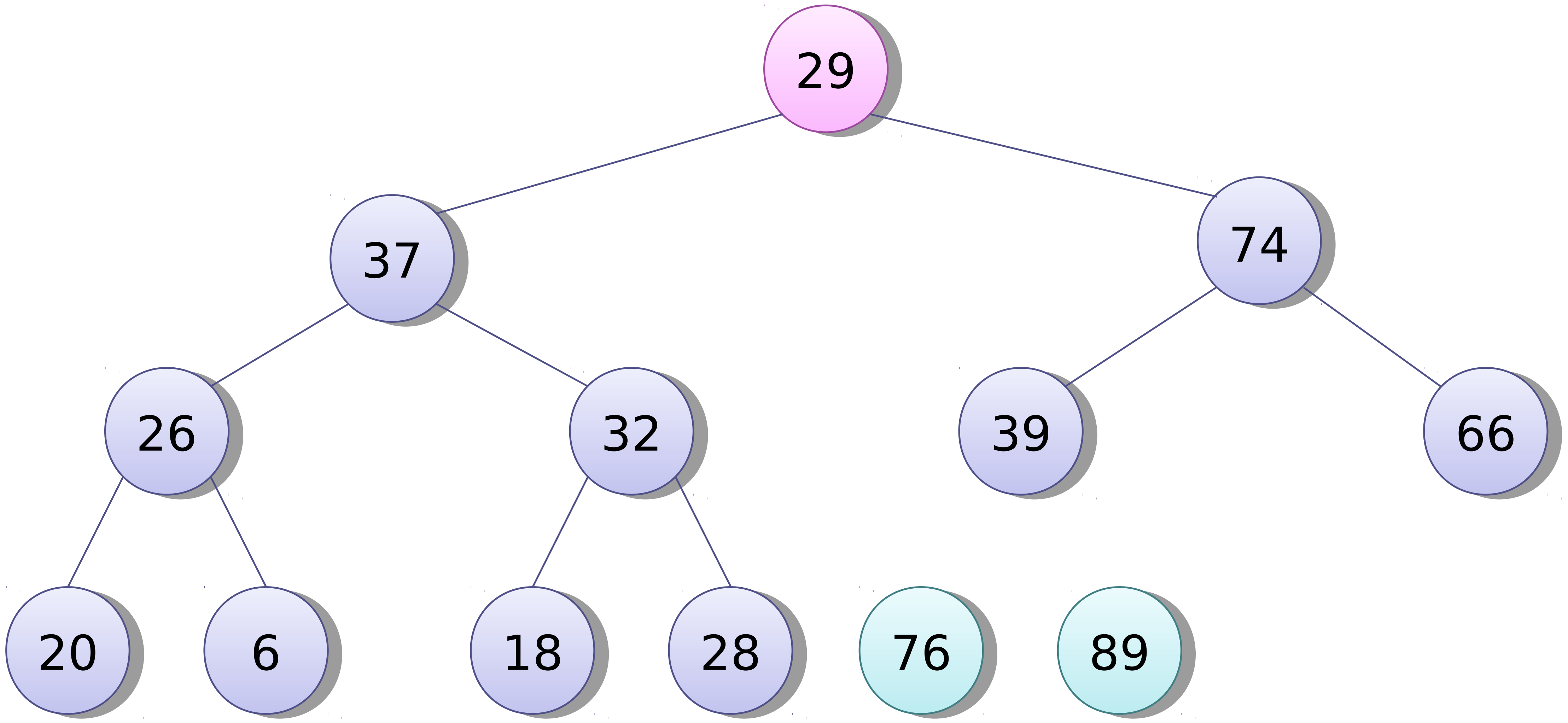
Trace of heapsort

- Step 1: swap maximum and last element; decrease size of heap by 1



Trace of heapsort

- Step 2: sift first element down



Trace of heapsort

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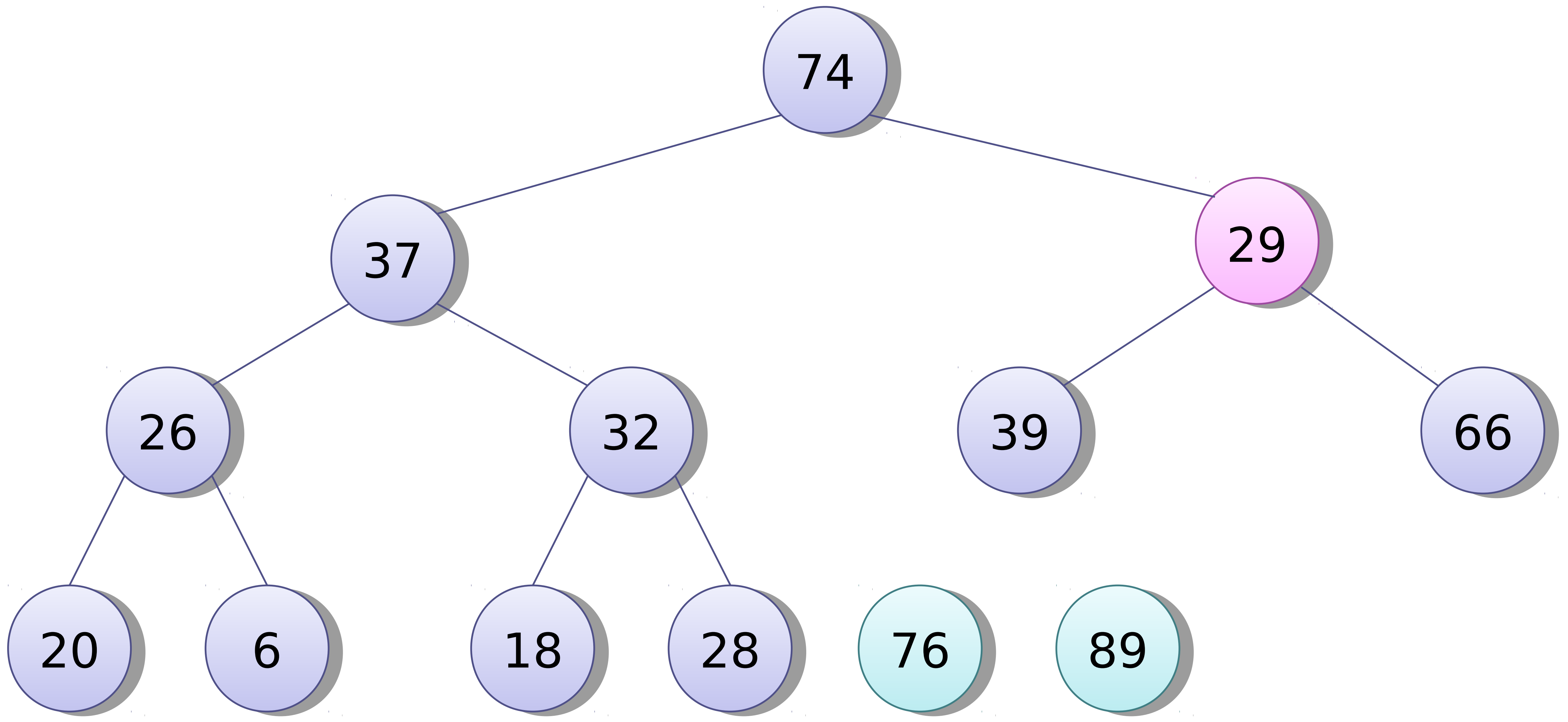

Trace of heapsort

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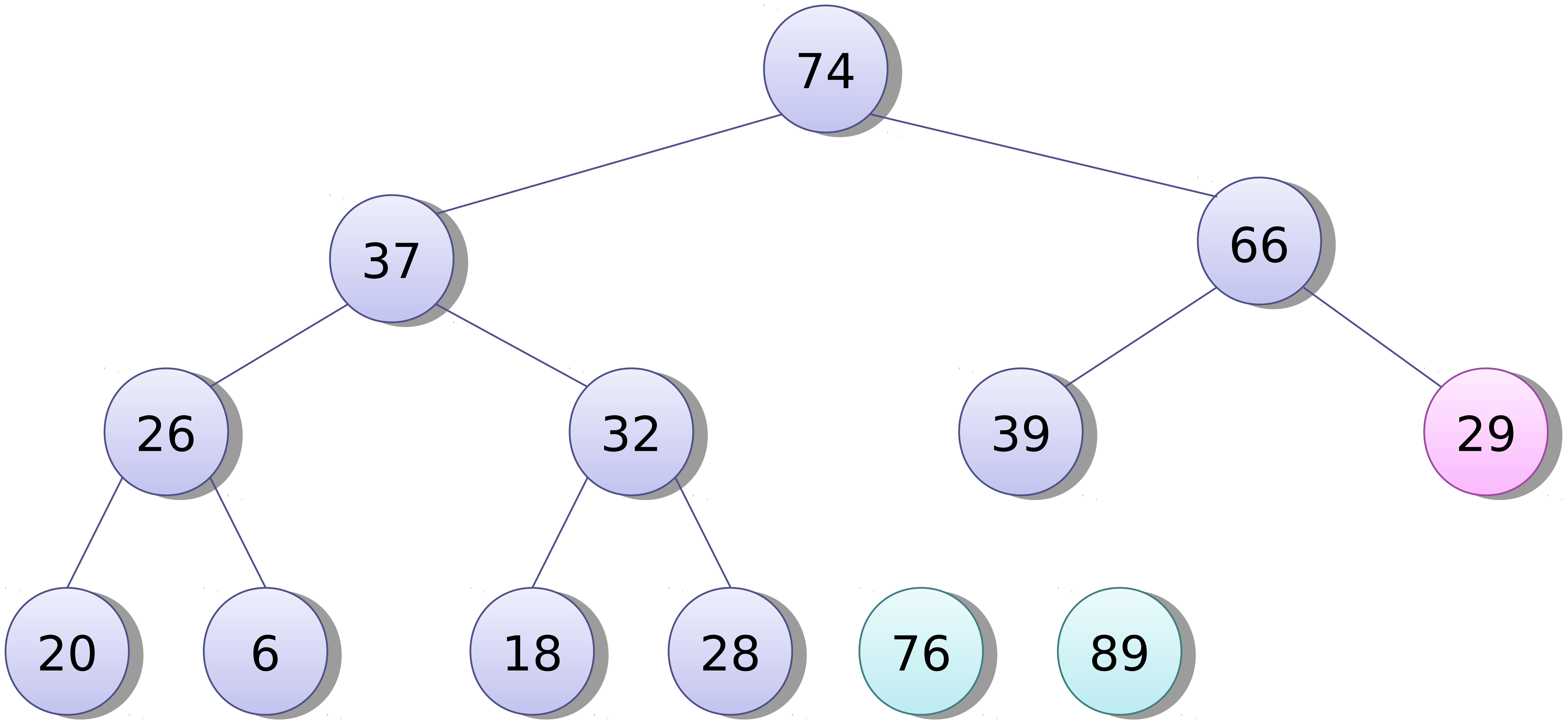

Trace of heapsort

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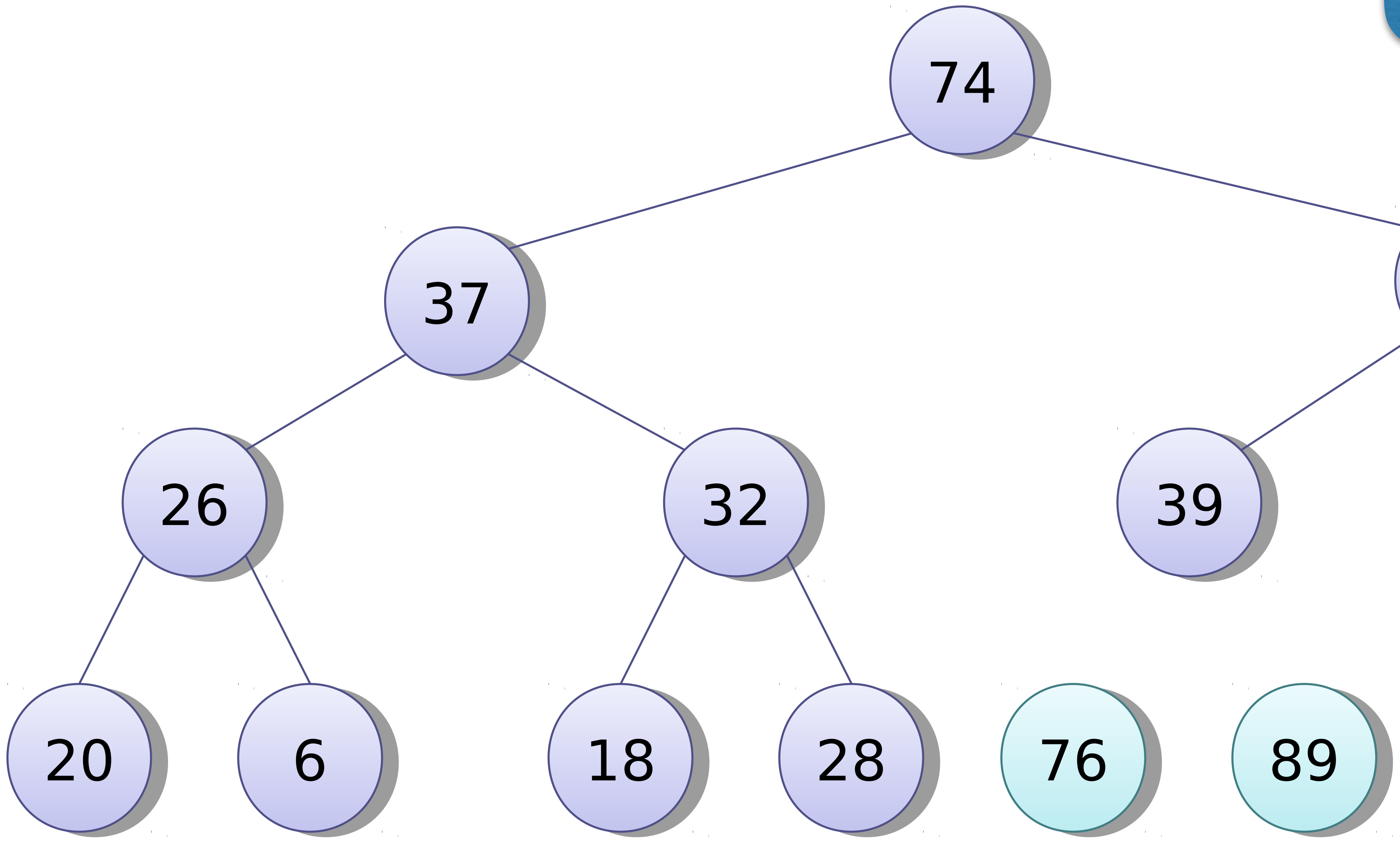

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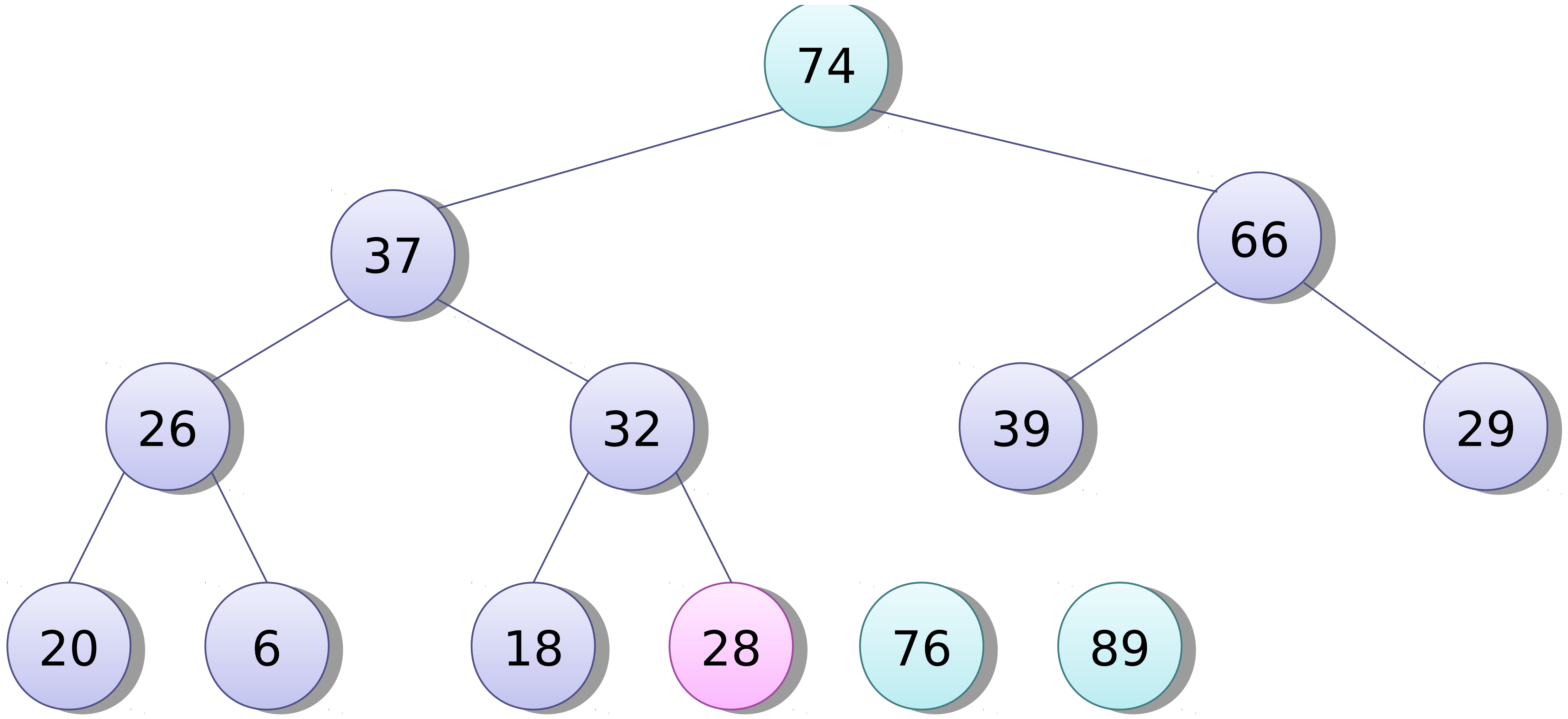

Trace of heapsort

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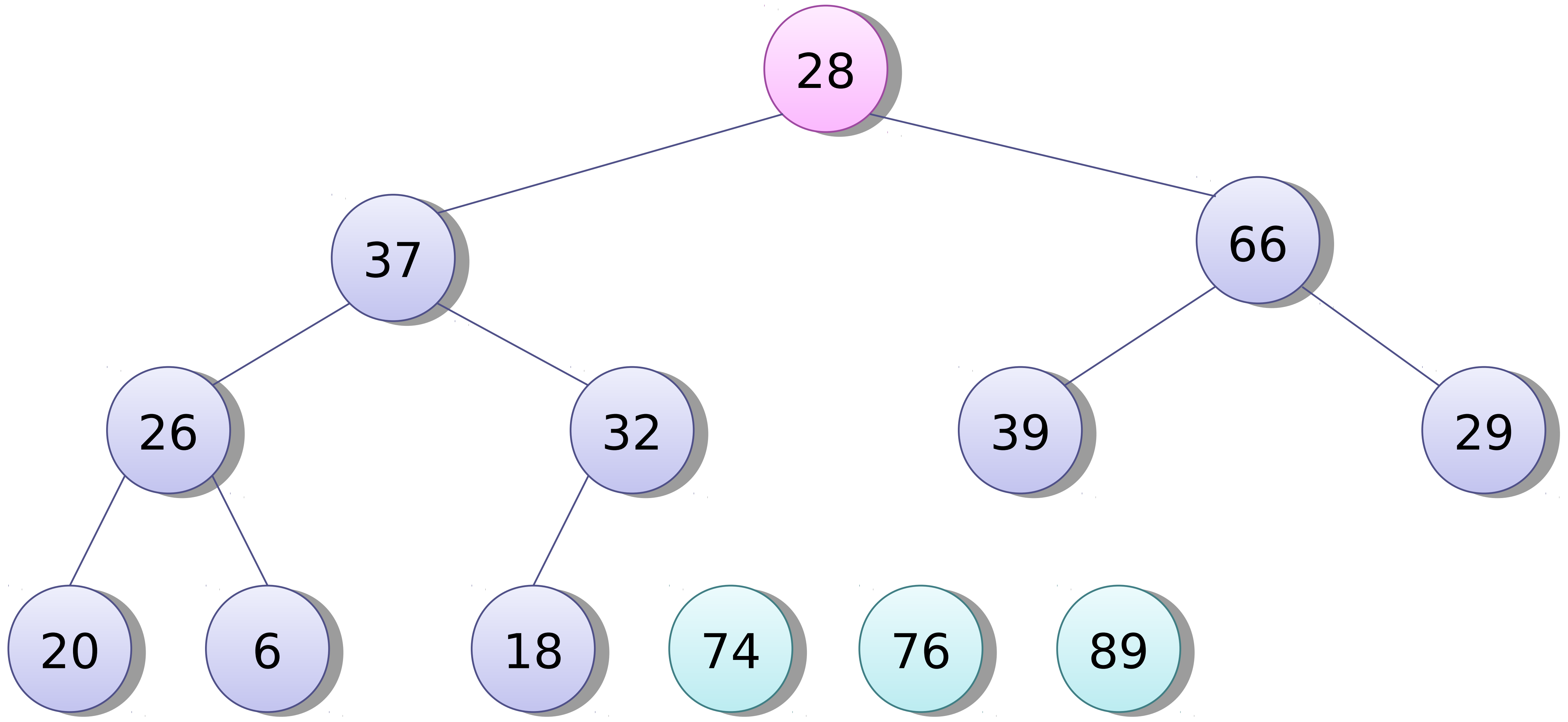

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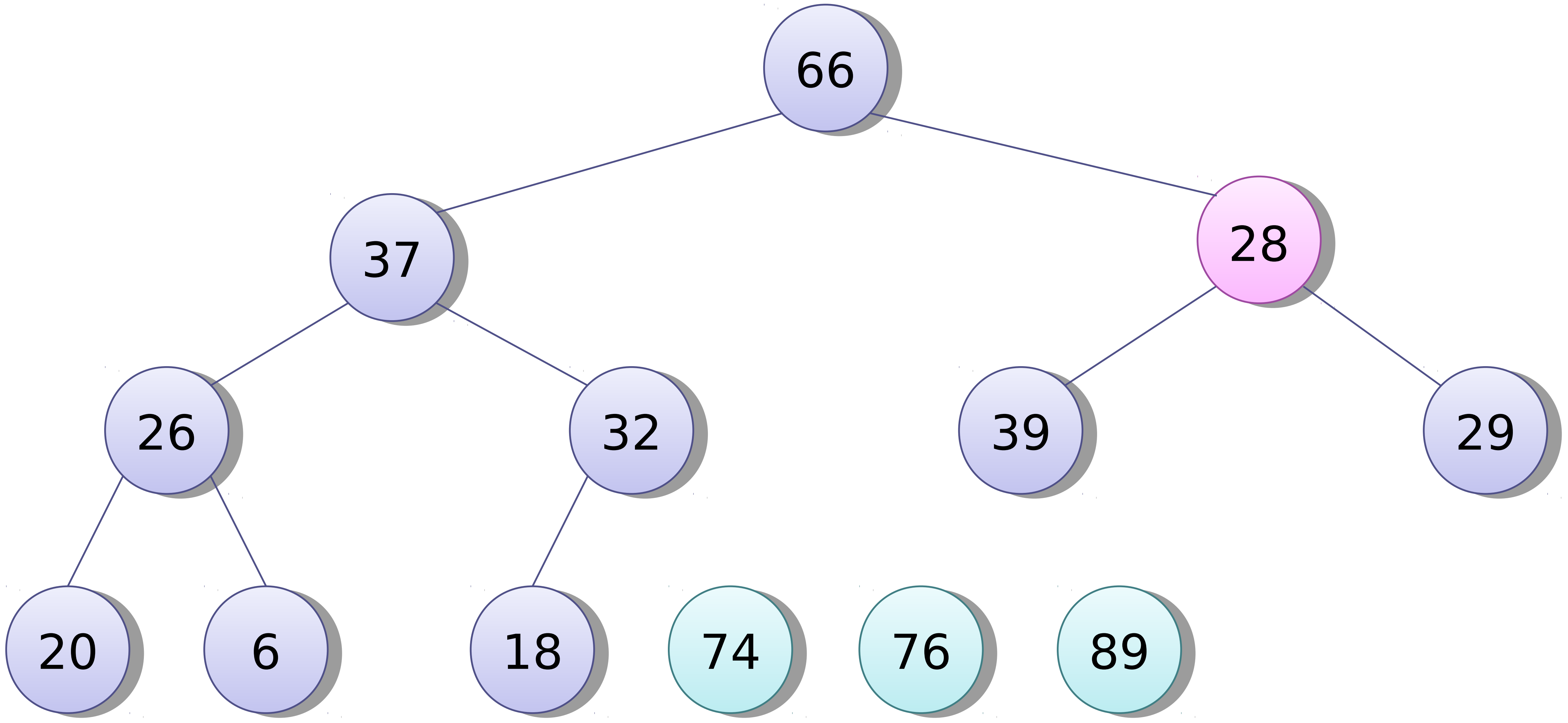

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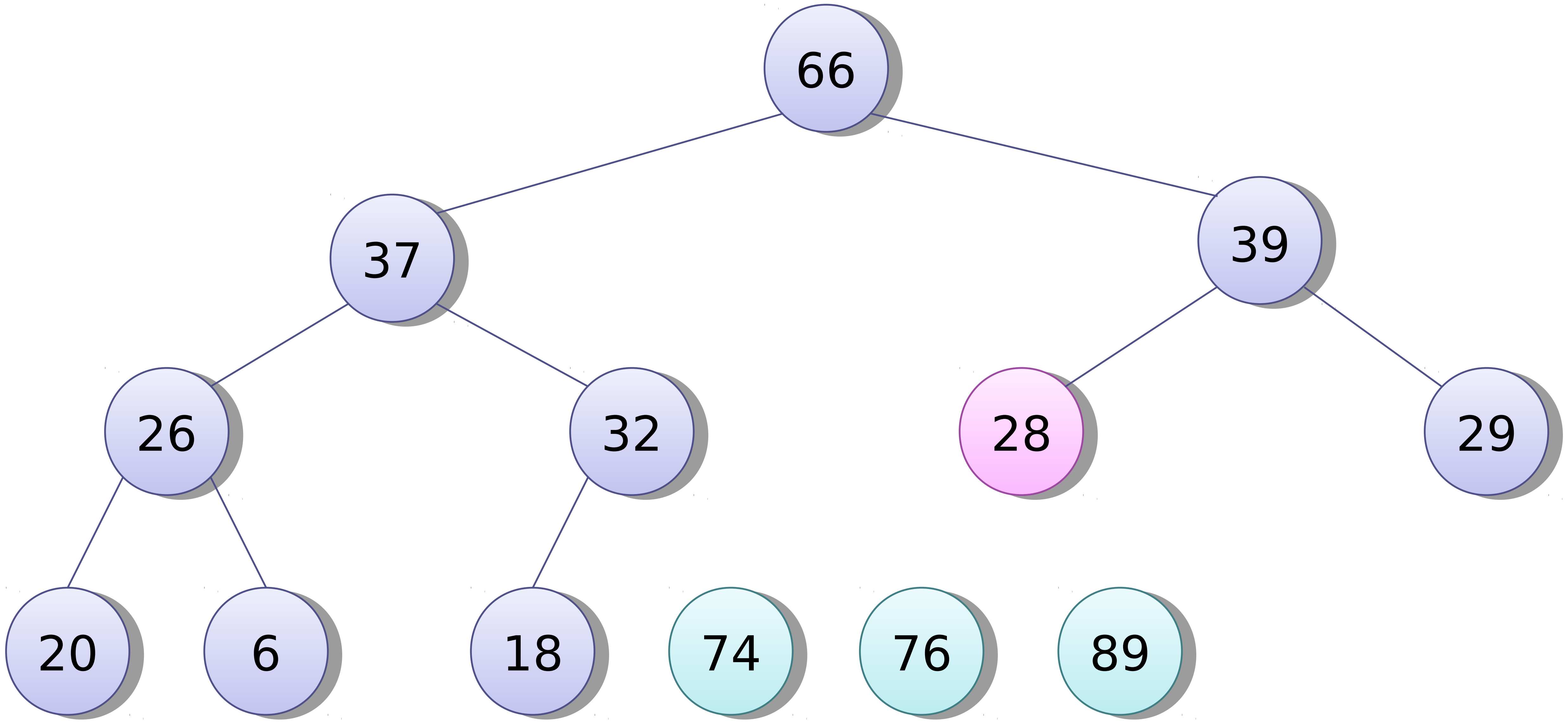
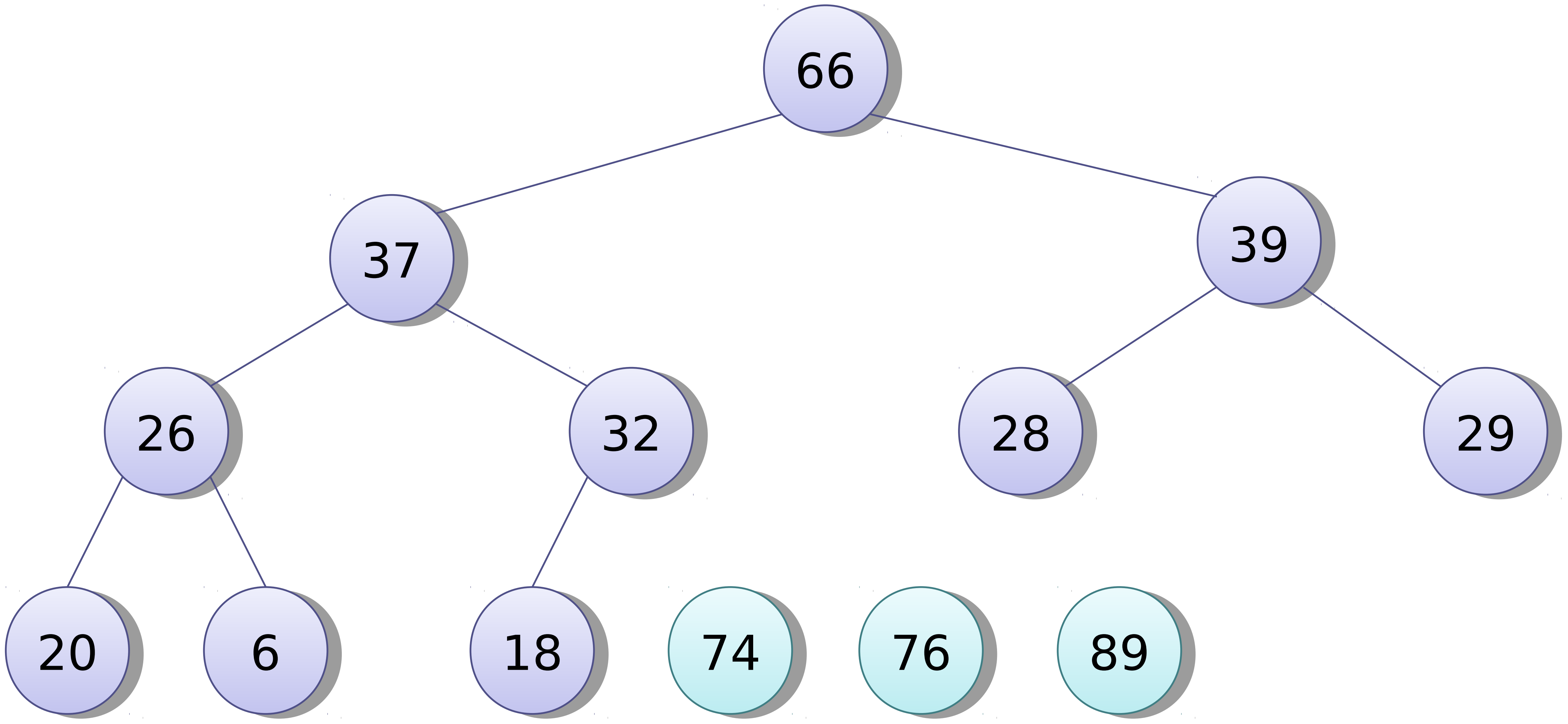


Trace of heapsort



Complexity of heapsort

- Building the heap takes $O(n)$ time
- We delete the maximum element n times, each deletion taking $O(\log n)$ time
- Hence the total complexity is $O(n \log n)$

Warning

- Our formulas for finding children and parents in the array assume 0-based arrays
- Others, for some reason, use 1-based arrays
- In a heap implemented using a 1-based array:
  - the left child of index `i` is index `2i`
  - the right child is index `2i+1`
  - the parent is index `i/2`
- Be careful when doing the lab!

Summary of binary heaps

- Binary heaps: a complete binary tree with the heap property, represented as an array
  - insert: $O(\log n)$
  - find minimum: $O(1)$
  - delete minimum: $O(\log n)$
- Heapsort: build a max heap, repeatedly remove last element and place at end of array
  - Can be done in-place, $O(n \log n)$
- In fact, heaps were originally invented for heapsort!

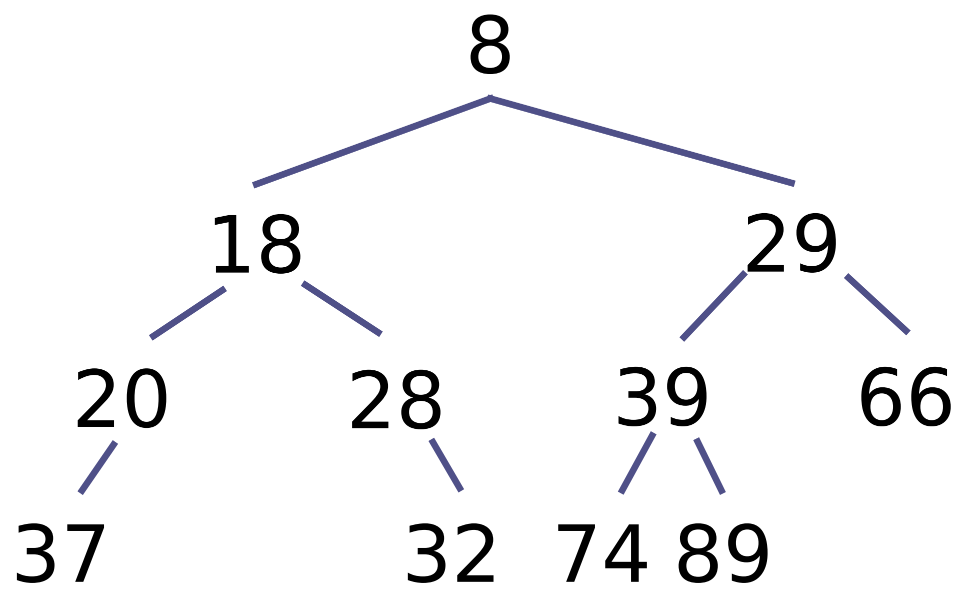
Leftist heaps

Merging two heaps

- Another operation we might want to do is *merge* two heaps
  - Build a new heap with the contents of both heaps
  - e.g., merging a heap containing 1, 2, 8, 9, 10 and a heap containing 3, 4, 5, 6, 7 gives a heap containing 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
- For our earlier naive priority queues:
  - An unsorted array: concatenate the arrays
  - A sorted array: merge the arrays (as in mergesort)
- For binary heaps:
  - Takes $O(n)$ time because you need to at least copy the contents of one heap to the other
  - Can't combine two arrays in less than $O(n)$ time!

Merging two heaps

- Go back to our idea of a binary tree with the heap property:

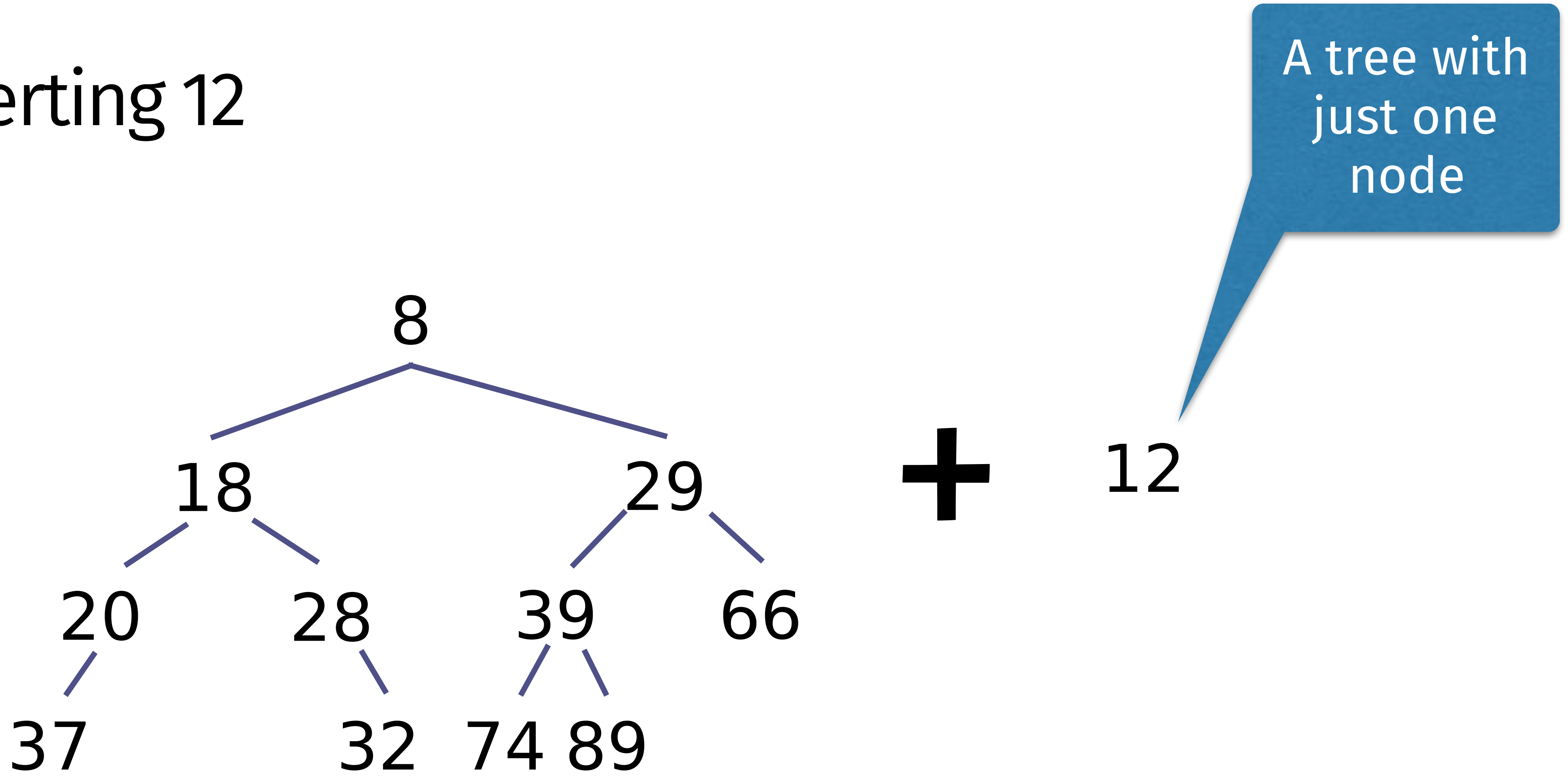


- If we can merge two of these trees, we can implement insertion and delete minimum!

- (We'll see how to implement merge later)

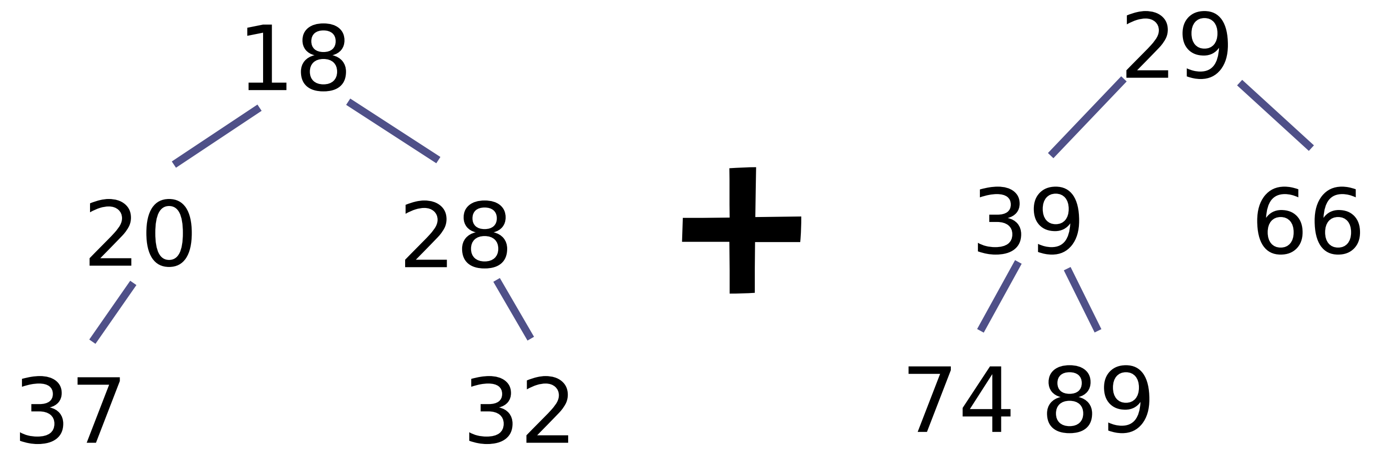
Insertion

- To insert a single element:
  - build a heap containing just that one element
  - merge it into the existing heap!
- E.g., inserting 12



Delete minimum

- To delete the minimum element:
  - take the left and right branches of the tree
  - these contain every element except the smallest
  - merge them!
- E.g., deleting 8 from the previous heap

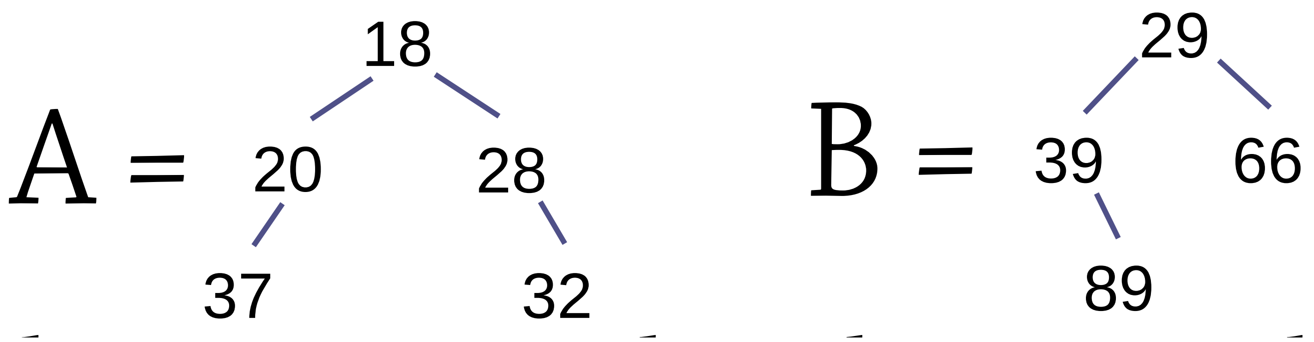


Heaps based on merging

- If we can take trees with the heap property, and implement merging with $O(\log n)$ complexity, we get a priority queue with:
  - $O(1)$ find minimum
  - $O(\log n)$ insertion (by merging)
  - $O(\log n)$ delete minimum (by merging)
  - plus this useful merge operation itself
- There are lots of heaps based on this idea:
  - skew heaps, Fibonacci heaps, binomial heaps
- We will study two: leftist and skew heaps

Naive merging

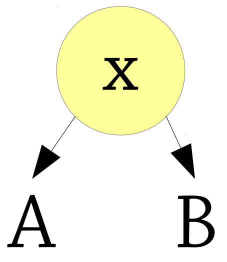
- How to merge these two heaps?



- Idea: root of resulting heap must be 18
- Take heap A, it has the smallest root
- Pick one of its children and recursively merge B into that child
- Which child should we pick? Let's pick the right child for no particular reason

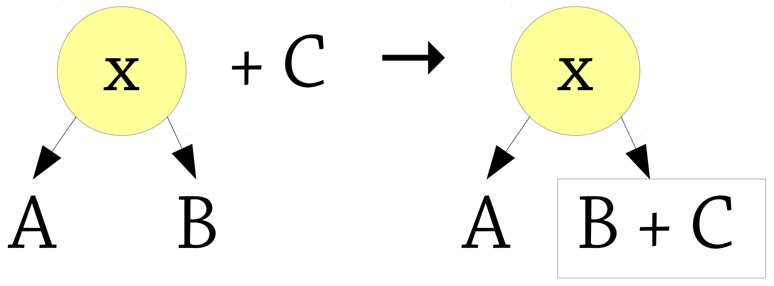
Naive merging

- To merge two non-empty heaps: pick the heap with the smallest root:



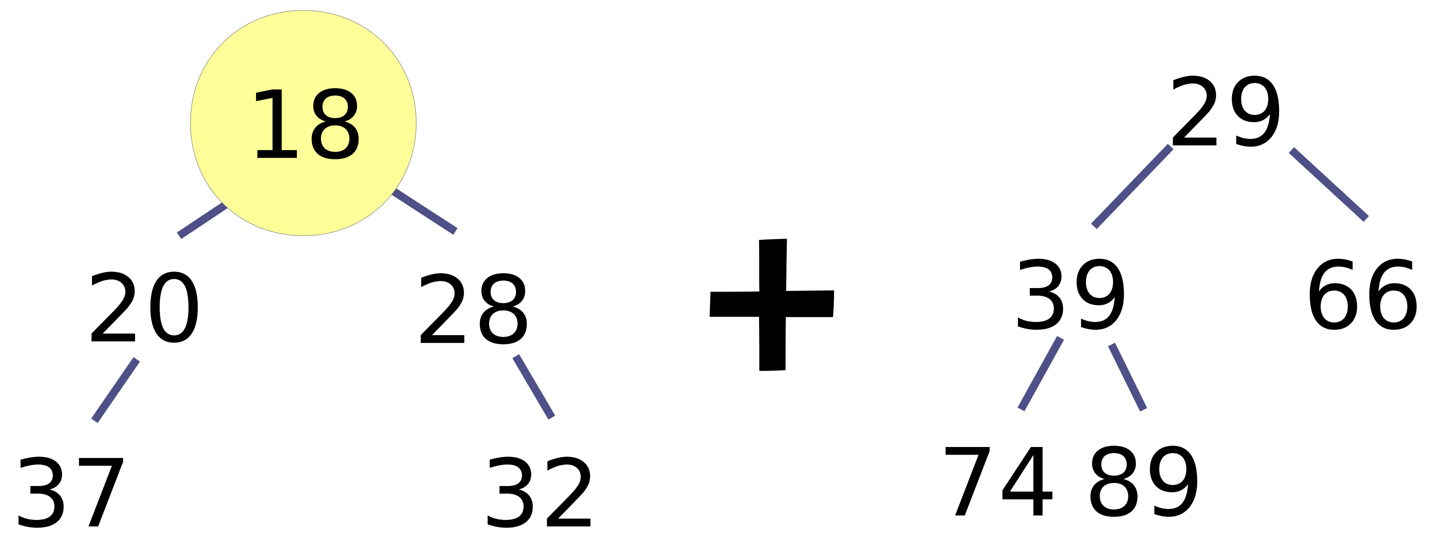
- Let C be the other heap

- Recursively merge B and C!



Naive merging

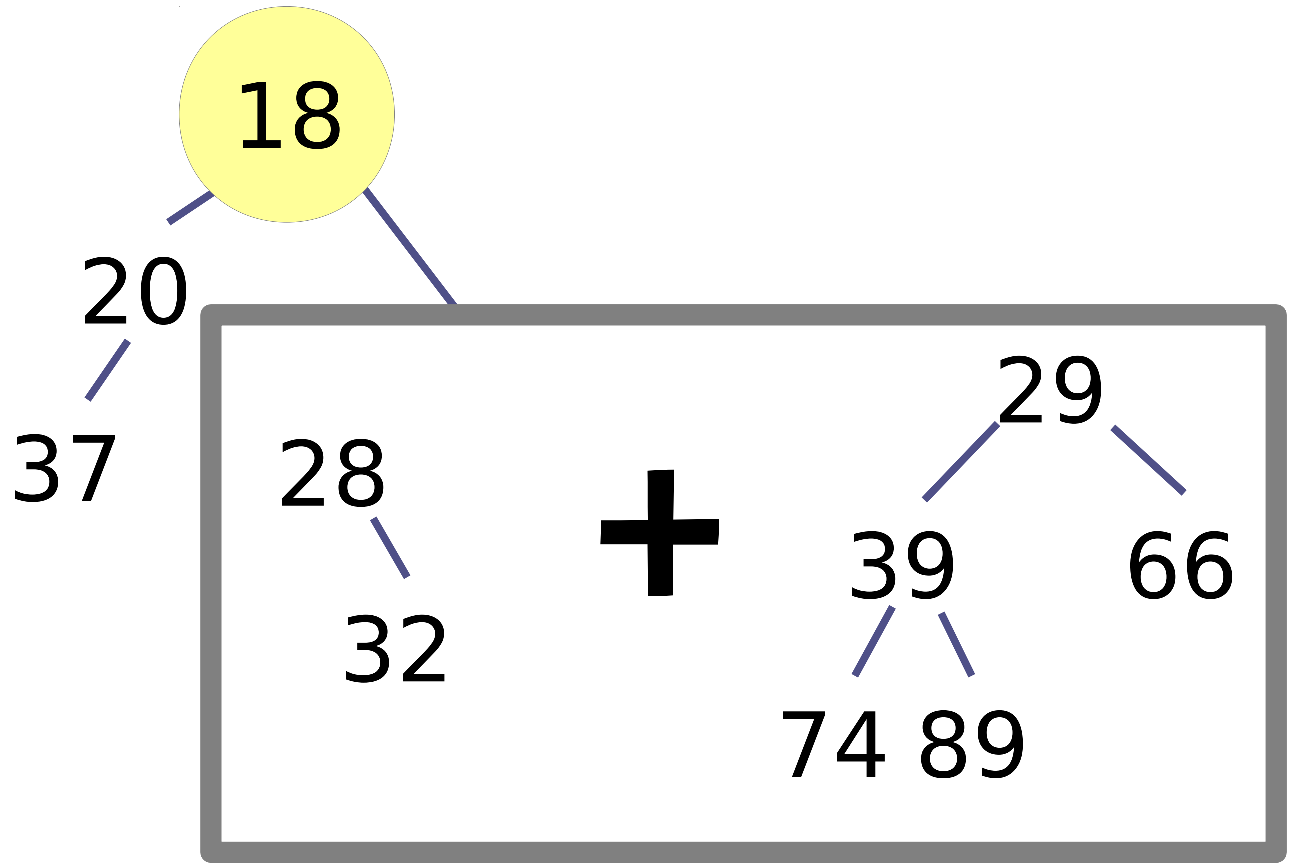
1. Look at the roots of the two trees



We are going to pick the smaller one as the root of the new tree

Naive merging

2. *Recursively merge* the right branch and the second tree



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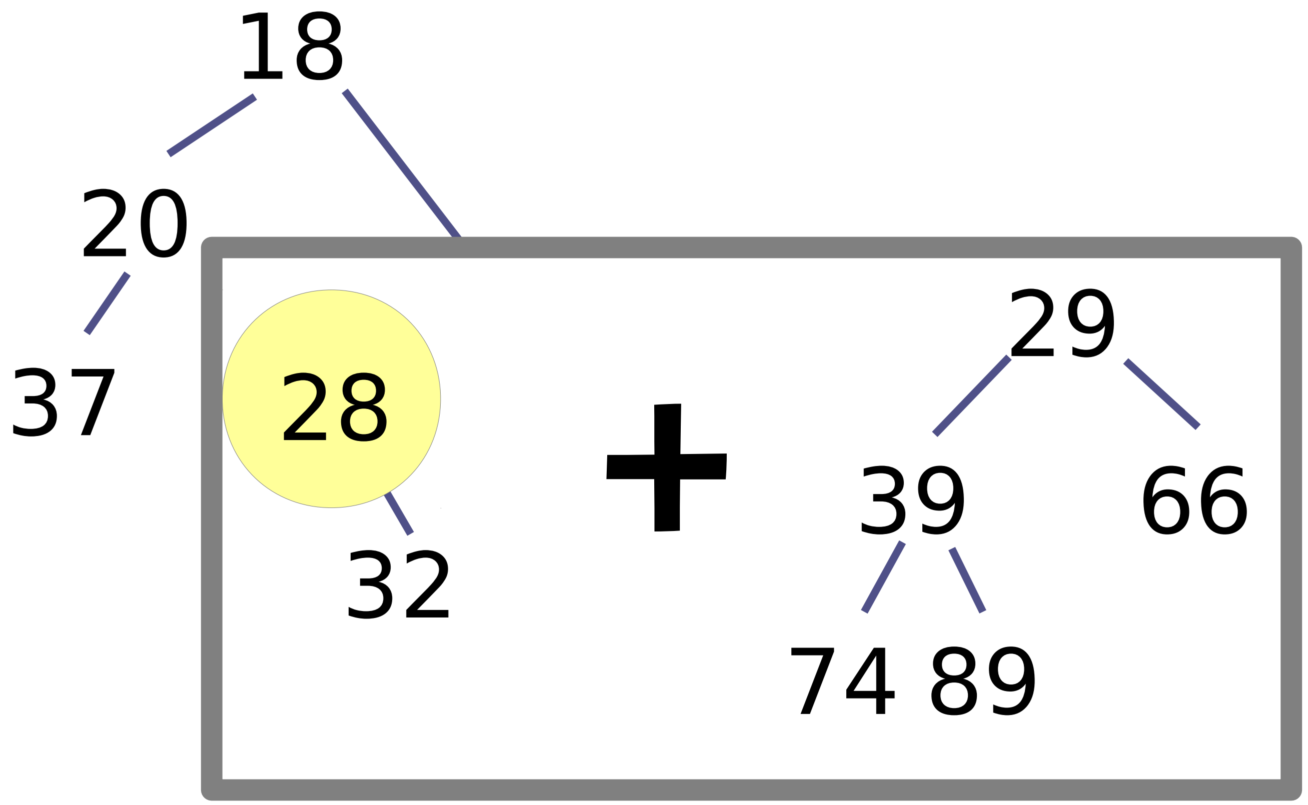

Naive merging

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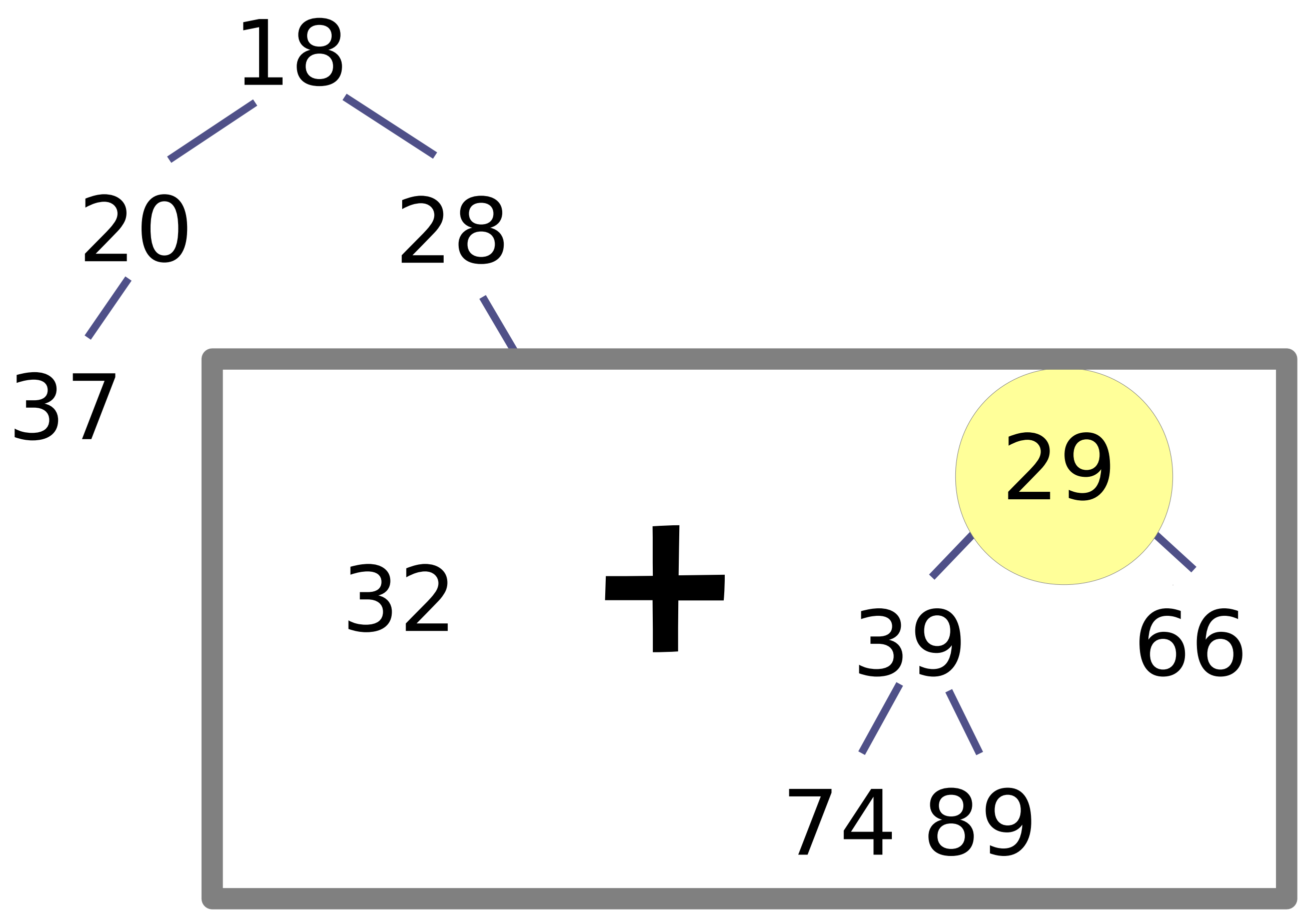

Naive merging

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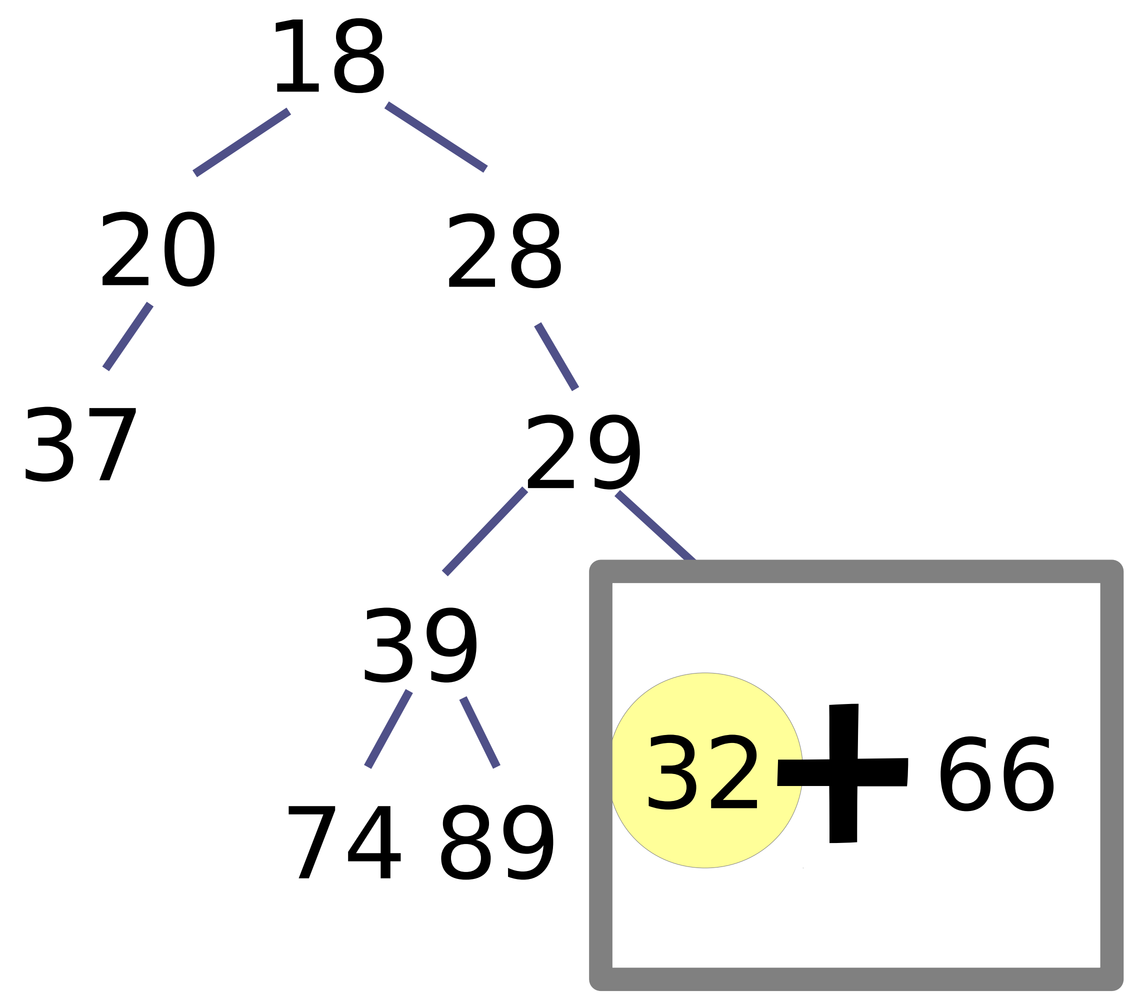

Naive merging

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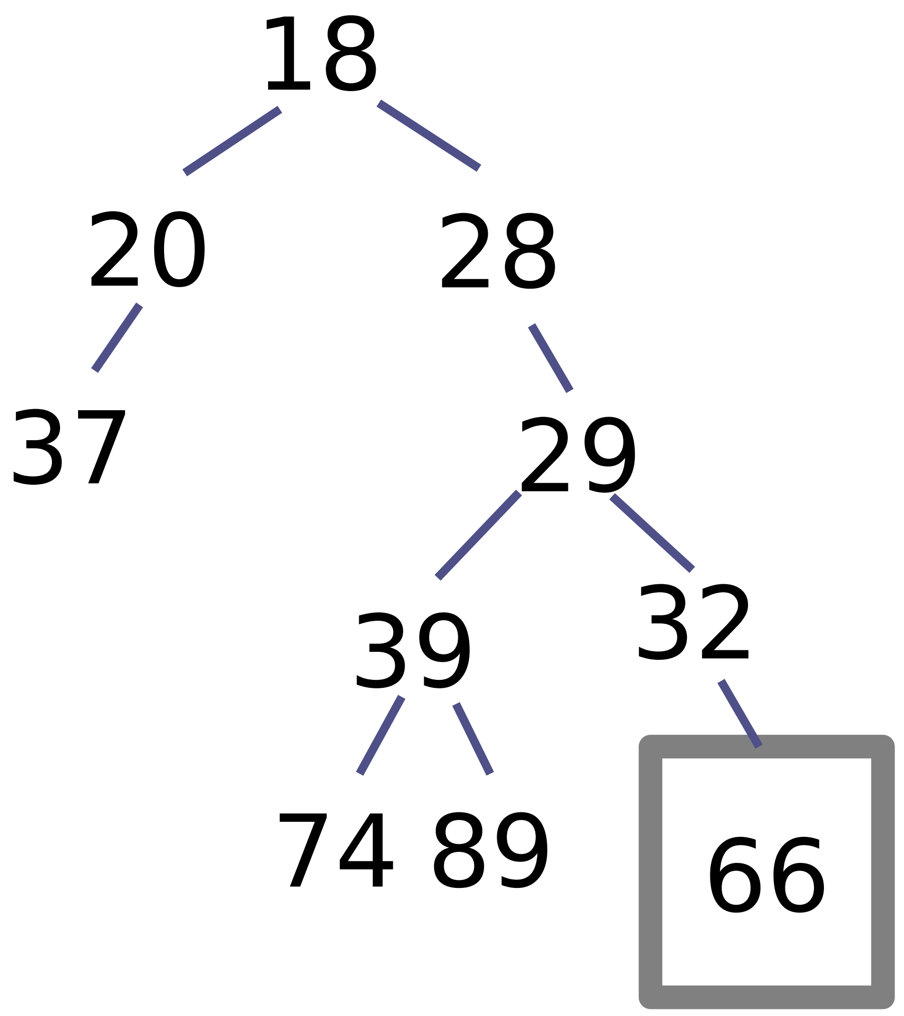

Naive merging

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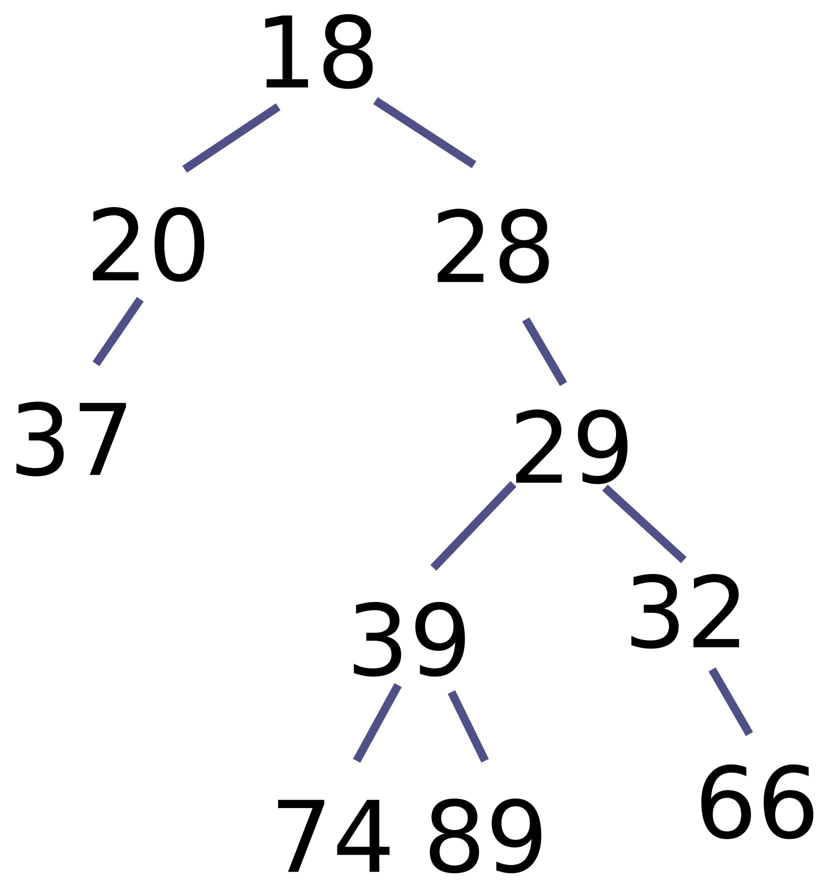
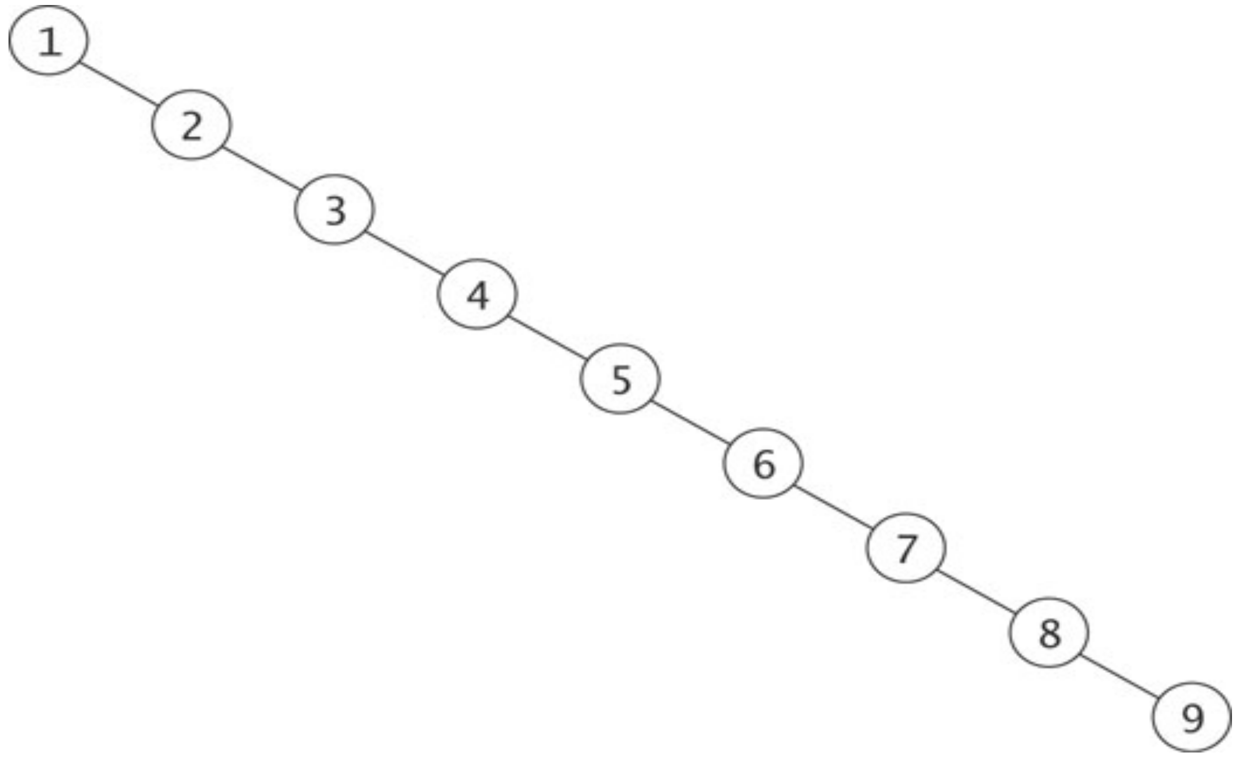


Performance of naive merging

- The merge algorithm descends down the *right branch* of both trees
- So the runtime depends on *how many times you can follow the right branch before you get to the end of the tree*
  - Let's call this the *right null path length*
- Complexity: $O(m+n)$
  - where m and n are the right null path lengths of the two trees
- Logarithmic complexity for balanced trees, but can become linear if the trees are heavily “right- biased”

Worst case for naive merging

- A heavily right-biased tree:



- Unfortunately, you get this just by doing insertions! So insert takes O(n) time...

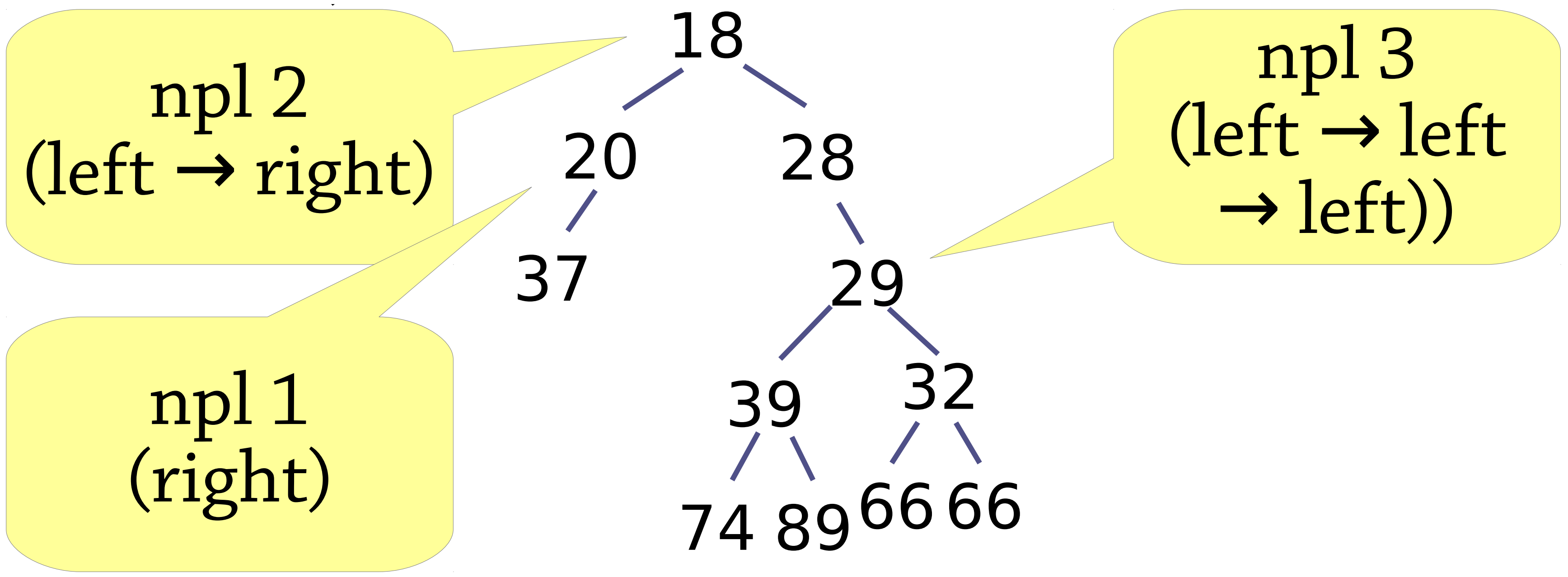
- How can we stop the tree from becoming right-heavy?

Leftist heaps – observation

- Naive merging is:
  - bad (linear complexity) for right-biased trees
  - good (logarithmic or better) for left-biased trees
- Idea of leftist heaps:
  - Add an invariant that stops the tree becoming right-biased
  - In other words, by repeatedly following the right branch, you quickly reach the end of the tree

Null path length

- We define the null path length (npl) of a node to be the shortest path that leads to the end of the tree (a null in Java)

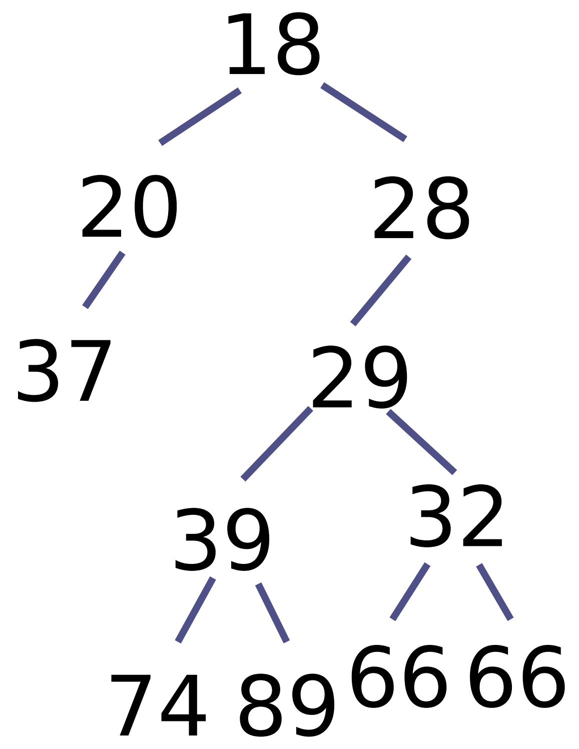


- The null path length of null itself is 0

- Similar concept to height, but with height we measure the longest path in the tree

Leftist heaps

- Leftist invariant: the npl of the left child ≥ the npl of the right child



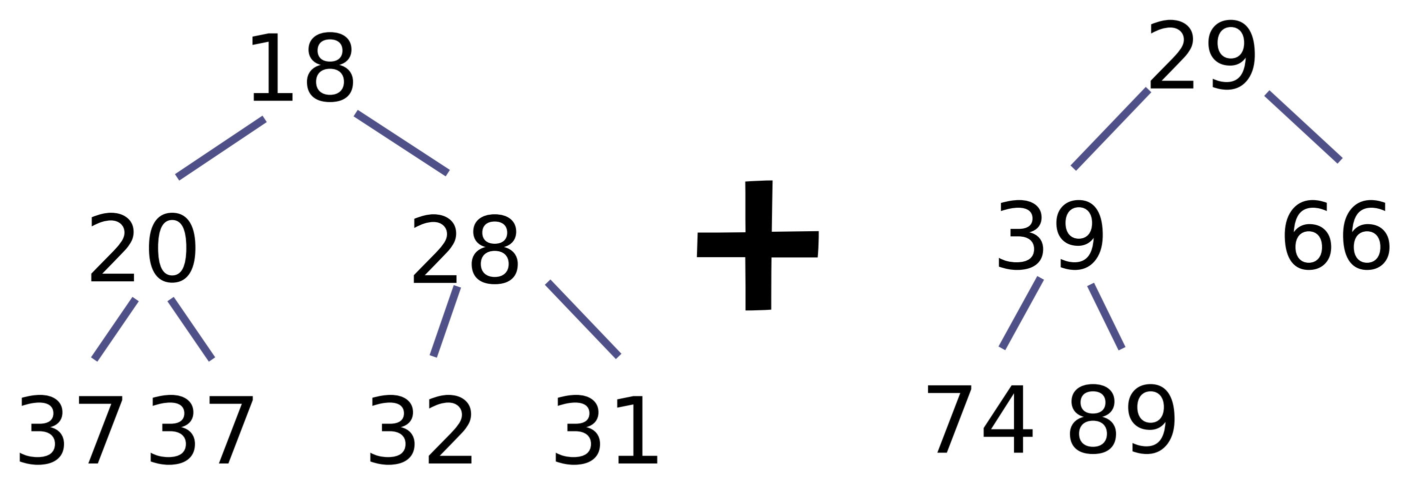
- This means: the quickest way to reach a *null* is to follow the right branch

Leftist merging

- We start with the naive merging algorithm from earlier:
  - The leftist invariant means that naive merging stops after $O(\log n)$ steps
- But the merge might break the leftist invariant!
  - When we descend into the right child, its npl might increase, and become greater than the left child
- Fix it by:
  - Going *upwards* in the tree from where the merge finished, and wherever we encounter a node where left child's npl < right child's npl, *swap the two children*!

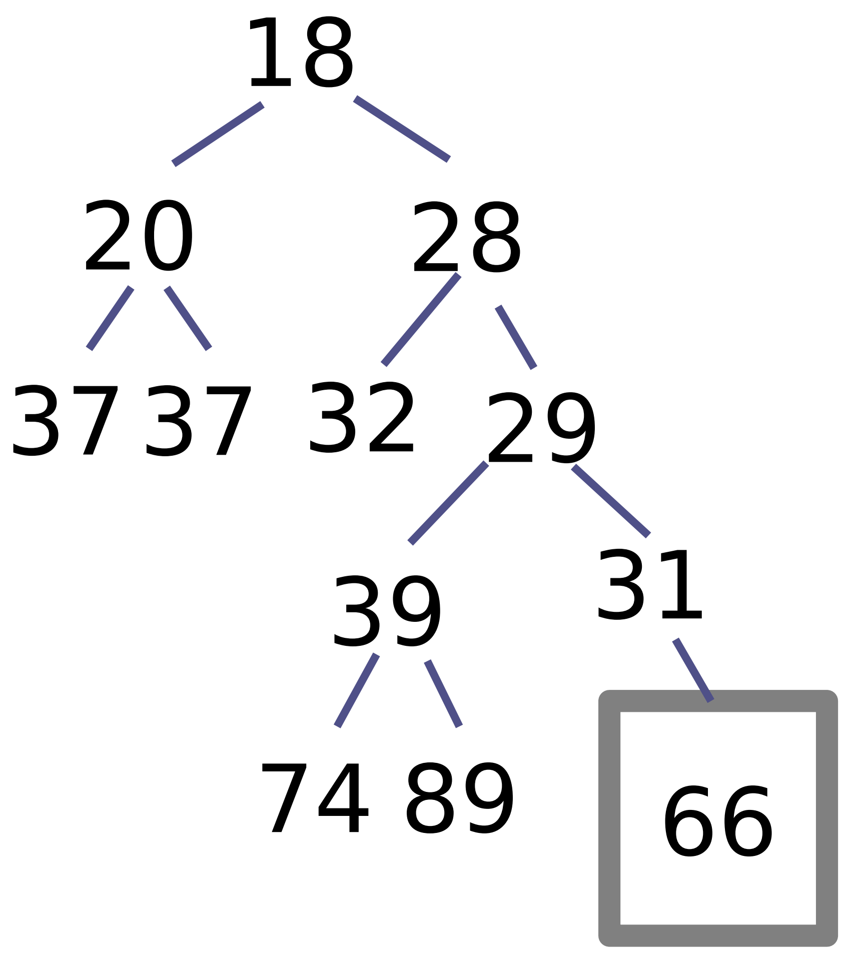
Leftist merging

1. Start with naive merging from earlier



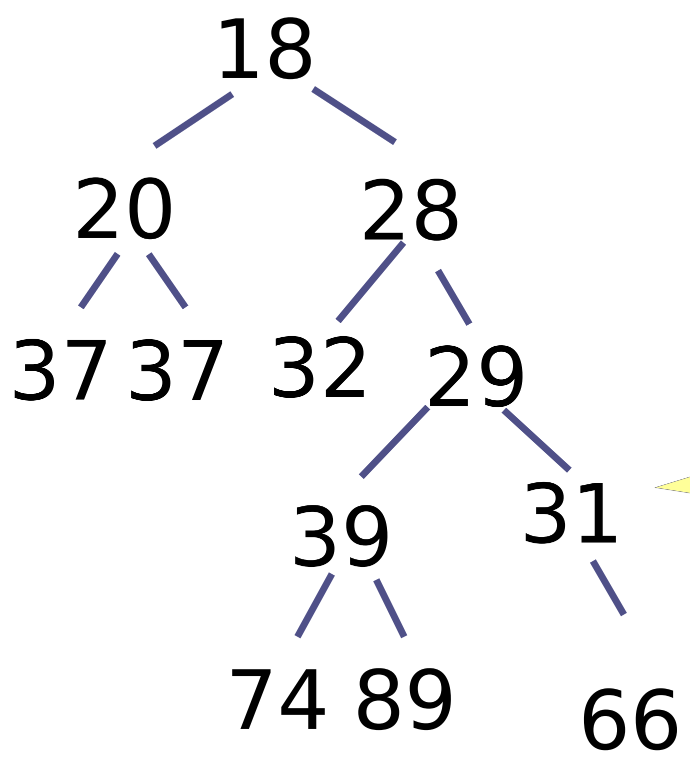
Leftist merging

2. The recursion “bottomed out” at 66 here



Leftist merging

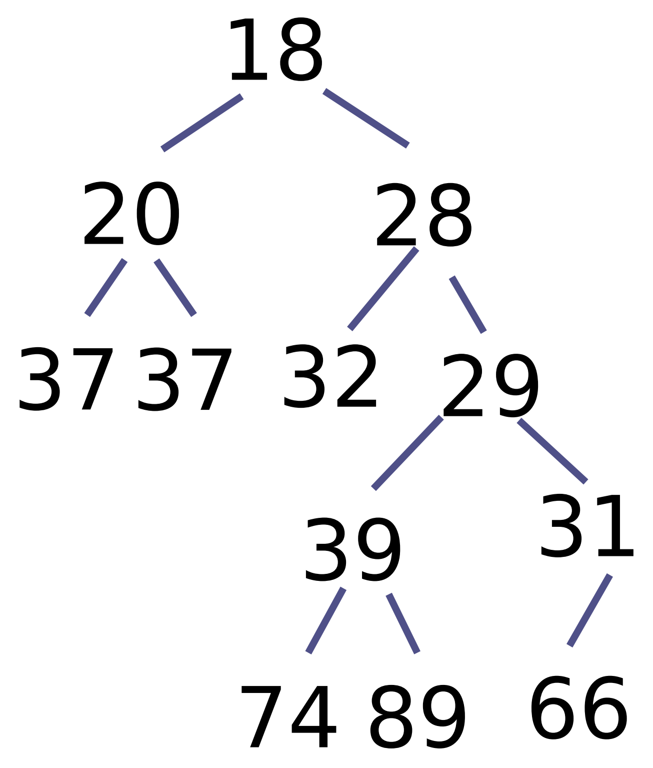
3. Go up to the parent, compare left and right child's npl



left npl: 0
right npl: 1
Invariant broken!

Leftist merging

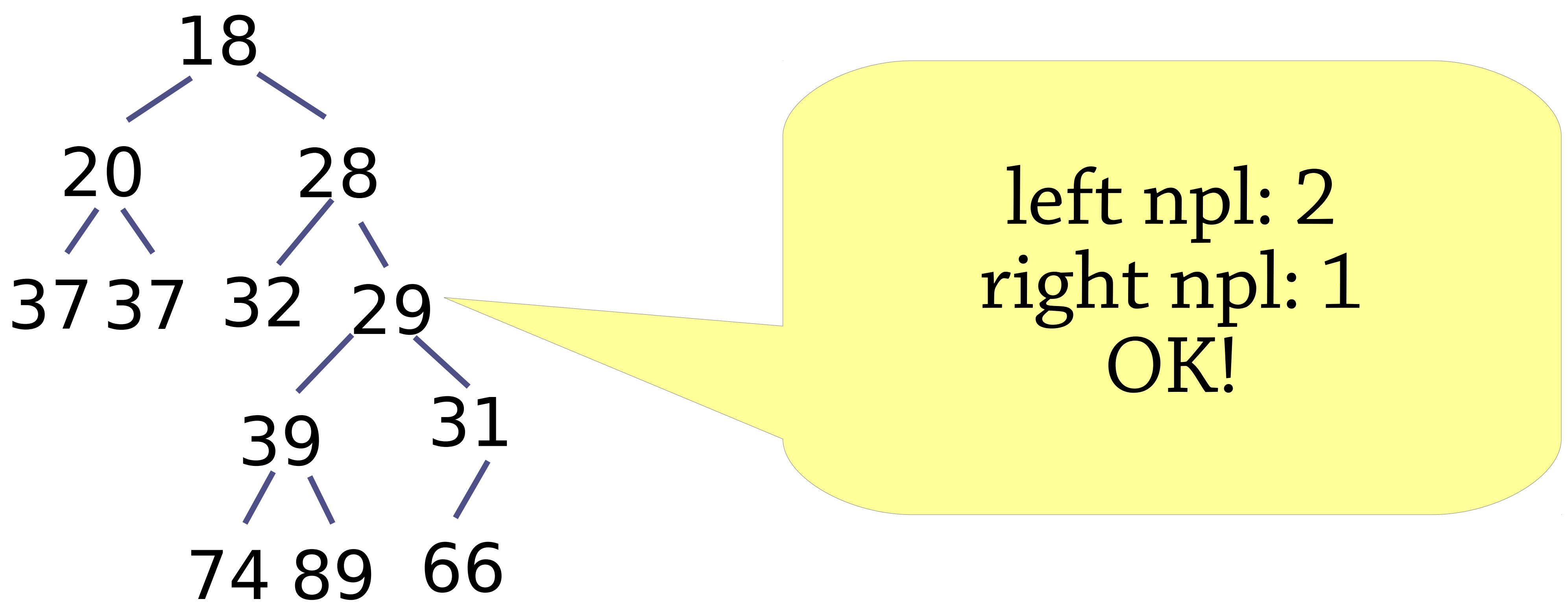
4. If the leftist invariant is broken, swap the left and right children



66 becomes the left child instead

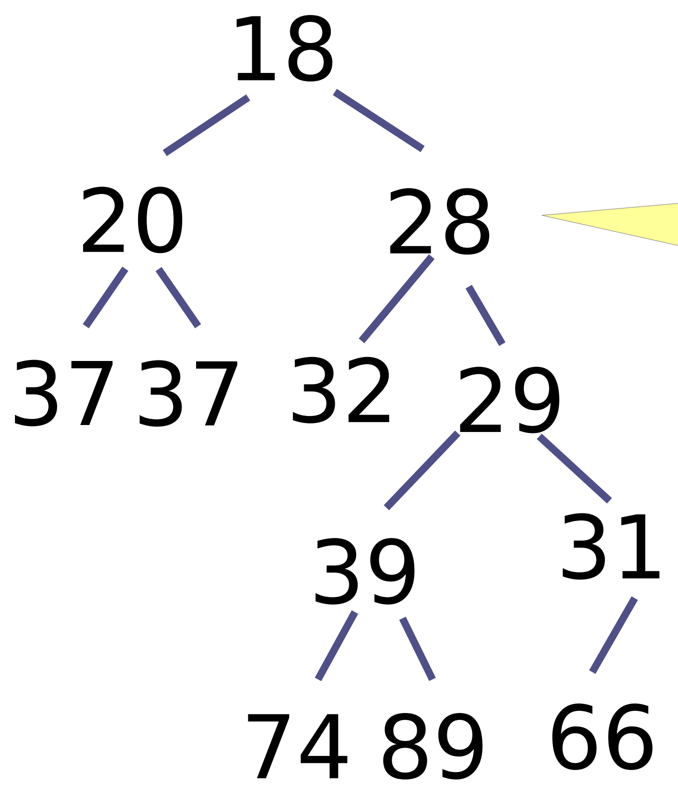
Leftist merging

5. Go up again and repeat!



Leftist merging

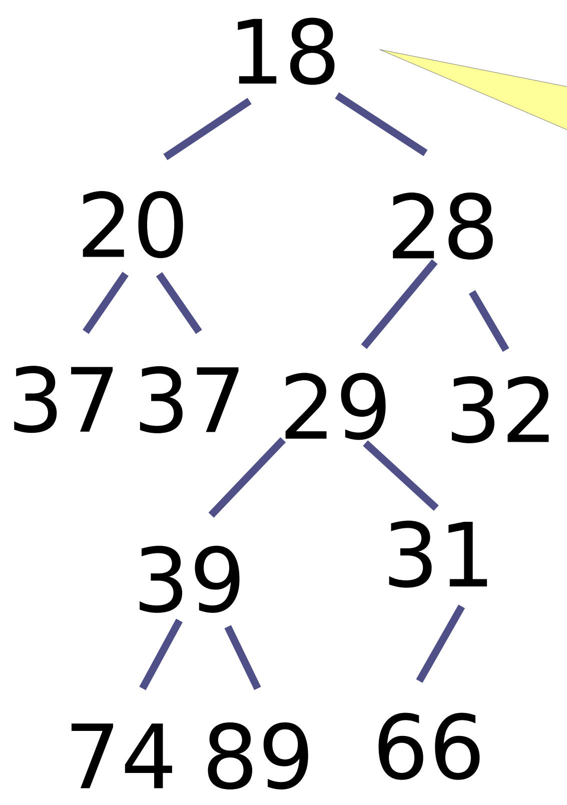
5. Go up again and repeat!



left npl: 1
right npl: 2
Invariant broken!
Swap left and right.

Leftist merging

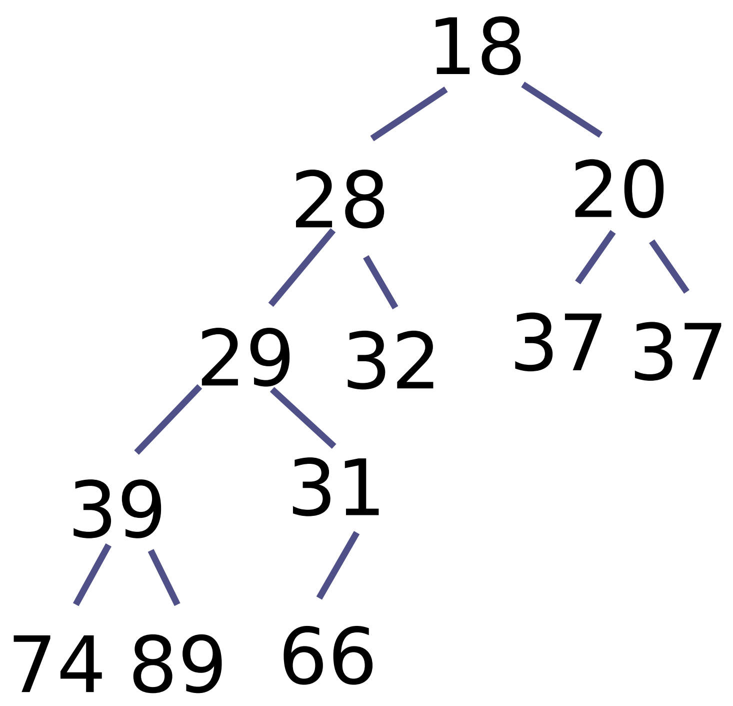
5. Go up again and repeat!



left npl: 1
right npl: 2
Invariant broken!
Swap left and right.

Leftist merging

6. When we've reached the root, we've finished!



Notice how the final heap “leans to the left”.

Implementation

- Implementation:
  - Need to be able to compute npl efficiently
  - Add a field for the npl to each node, and update it whenever we modify the node
  - Update by computing: npl = 1 + right child's npl

Complexity of leftist merging

- I claim: the npl of a tree of size n is $O(\log n)$
  - Check it for yourself :)
  - For balanced trees, the npl is $O(\log n)$, much like height
  - By unbalancing a tree, we make some paths longer, and some shorter. This increases the height, but decreases the npl!
- Hence, in a leftist heap, by following the right branch $O(\log n)$ times, you reach a null
- So merge takes $O(\log n)$ time!
  - $\log n$ steps down the tree to do the naive merge
  - then $\log n$ steps upwards while repairing the leftist invariant

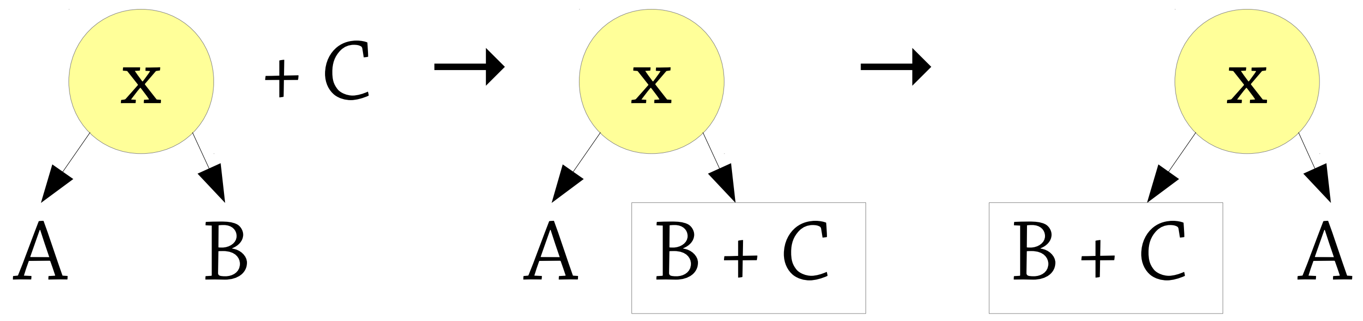
Leftist heaps

- Implementation of priority queues:
  - binary trees with heap property
  - leftist invariant for $O(\log n)$ merging
  - other operations are based on merge
- A good fit for functional languages:
  - based on trees rather than arrays

Skew heap

Skew merging

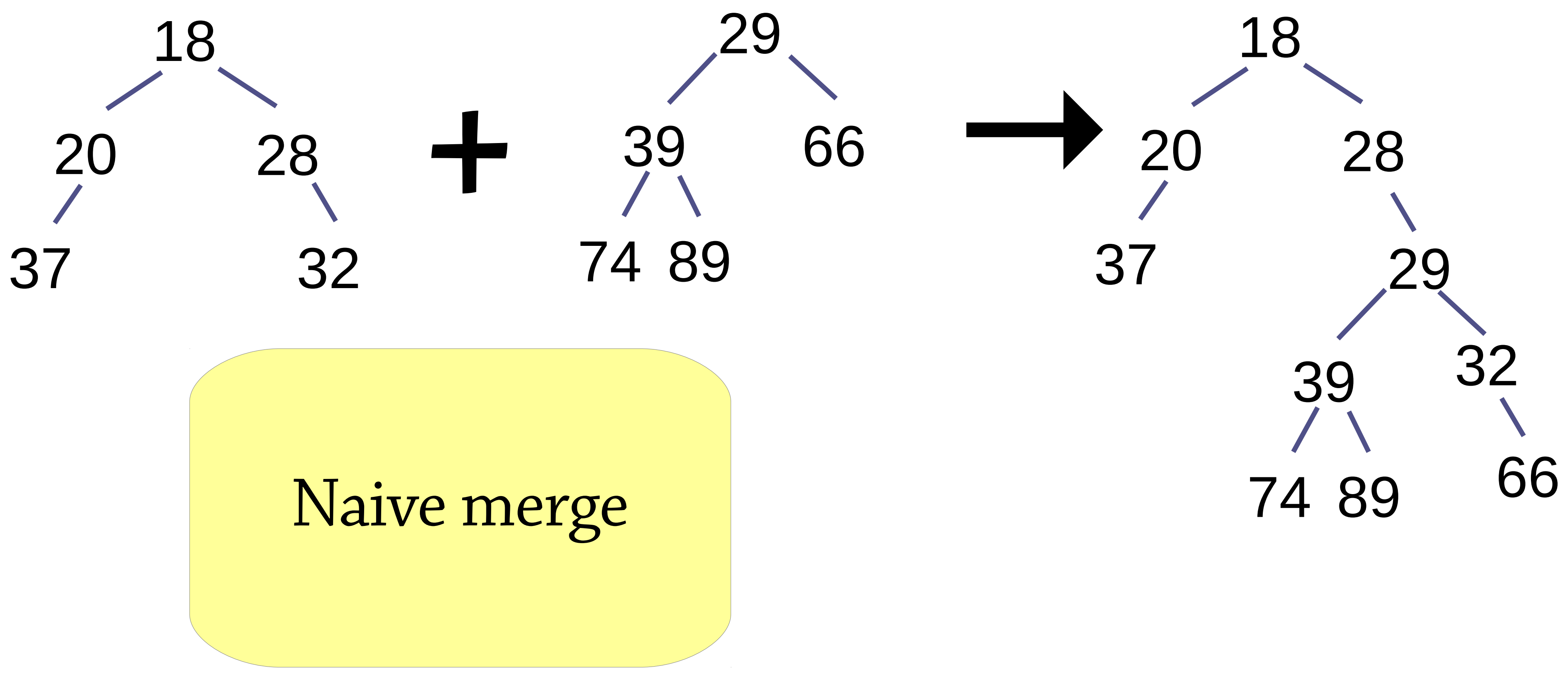
- In a skew heap, after making a recursive call to merge, we *swap the two children*:



- Amazingly, this small change completely fixes the performance of merge!
- We almost never end up with right-heavy trees
- We get $O(\log n)$ amortised complexity

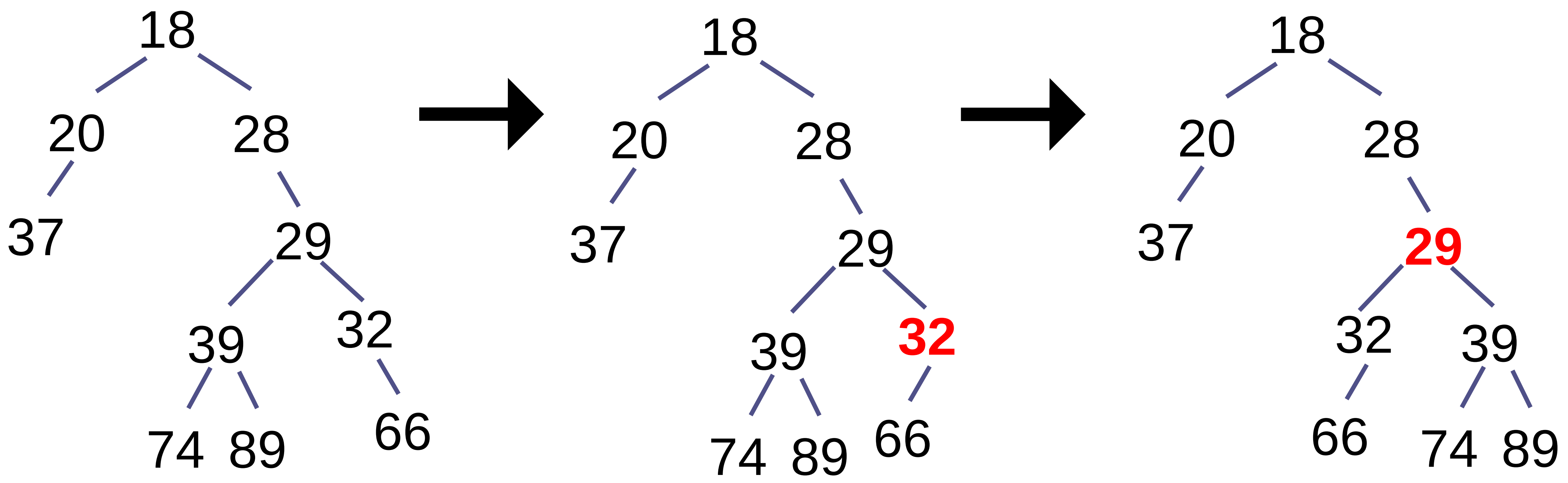
Example

- One way to do skew merge is to first do naive merge, then go up the tree swapping left and right children...



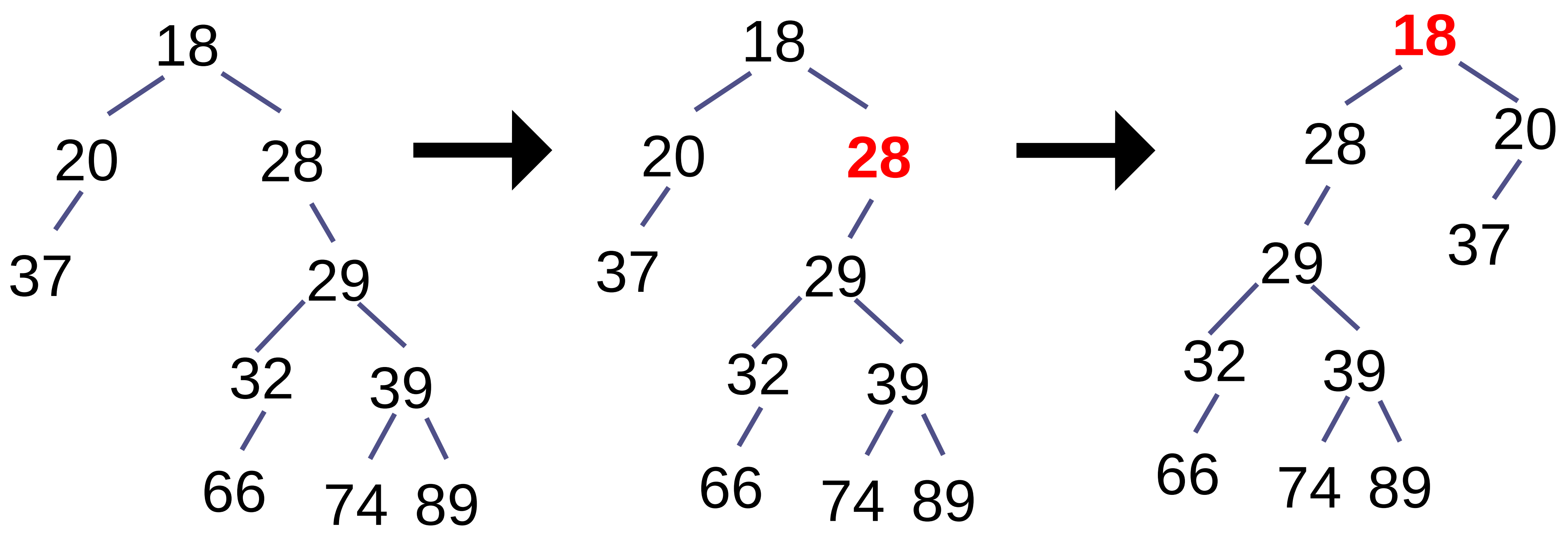
Example

- ... like this:



Example

- ... like this:



Skew heaps

- Implementation of priority queues:
  - binary trees with heap property
  - skew merging avoids right-heavy trees, gives $O(\log n)$ amortised complexity
  - other operations are based on merge
- A good fit for functional languages:
  - based on trees rather than arrays, tiny implementation!
- Based on same idea as leftist heaps: naive merging + avoiding right heavy trees
- See webpage for link to visualisation site!