Brief sample solutions for the examination of Models of Computation (DIT310/TDA183/TDA184) from 2018-04-05

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1. (a) $A = \emptyset$, $B = \mathbb{N}$.

   (b) It is countable.

   Note first that, for any set A, any two functions f, $g \in A \to \{ 0 \}$ are equal, because we have $\forall x \in A. f\, x = 0 = g\, x$. Thus there is exactly one element in $A \to \{ 0 \}$, the function that maps every input to 0.

   We get that $\mathbb{N} \to \{ 0 \}$ is in bijective correspondence with $\{ 0 \}$, and that $(\mathbb{N} \to \{ 0 \}) \to (\mathbb{N} \to \{ 0 \})$ is in bijective correspondence with $(\mathbb{N} \to \{ 0 \}) \to \{ 0 \}$. The argument used above implies that there is exactly one element in $(\mathbb{N} \to \{ 0 \}) \to \{ 0 \}$ as well, so this set is countable.

   We can conclude that $(\mathbb{N} \to \{ 0 \}) \to (\mathbb{N} \to \{ 0 \})$ is countable, because if a countable set A is in bijective correspondence with a set B, then B is also countable.

2. **rec** $x = \mathsf{True}(x)$.

3. Yes. Given an implementation of *terminates-in* it is easy to implement:

   - Given the pair $\ulcorner (e, 0) \urcorner$, use *terminates-in* to check if e terminates in (at most) 0 steps.
   - Given the pair $\ulcorner (e, n) \urcorner$, with $n > 0$, use *terminates-in* to check if e terminates in *exactly* n steps by checking if it terminates in at most n steps but not in at most $n - 1$ steps.
   - In either case, if the answer is affirmative, return $\ulcorner * \urcorner$, and otherwise go into an infinite loop.

4. No. We can prove this by reducing the halting problem (which is not χ-decidable) to g.

   If g is χ-computable, then there is a closed χ expression $\underline{g}$ witnessing the computability of g. We can use this expression to construct a closed χ

expression $\underline{halts}$ (written using a mixture of concrete syntax and meta-level notation):

$$
\begin{aligned}
&\underline{halts} = \lambda\, e.\, \textbf{case}\ \underline{g}\ e\ \textbf{of} \\
&\quad \{\ \mathsf{Star}() \ \to \mathsf{False}() \\
&\quad ;\ \mathsf{Zero}() \ \to \mathsf{True}() \\
&\quad ;\ \mathsf{Suc}(n) \to \mathsf{True}() \\
&\quad \}.
\end{aligned}
$$

Note that if a closed expression $e \in \mathit{Exp}$ terminates, then it terminates in n steps for some $n \in \mathbb{N}$. Thus $\underline{halts}$ witnesses the decidability of the halting problem.

5. (a) If the machine is run with 1 as the input string, then it will move to the right forever (reading 1 once and then ␣) and never halt.

   (b) Yes, the machine implements the successor function: For every input of the form 1^n0 the machine will move to the right past all the ones, replace the zero with a one, add a new zero at the end, and halt with the head above the final one. The final string is $1^{1+n}0$.

6. This follows from the following lemma (where PRF_n^- is the variant of PRF_n obtained by removing suc and rec, and $\mathit{Fin}\ n = \{\ i \in \mathbb{N} \mid 0 \leq i < n\ \}$):

   Lemma. *For any $n \in \mathbb{N}$, $f \in PRF_n^-$, and $\rho \in \mathbb{N}^n$, we either have that $[\![f]\!]\ \rho = 0$, or there is some index $i \in \mathit{Fin}\ n$ such that $[\![f]\!]\ \rho \leq \mathit{index}\ \rho\ i$.*

   Similarly, for any m, $n \in \mathbb{N}$, $fs \in (PRF_m^-)^n$, $\rho \in \mathbb{N}^m$, and $i \in \mathit{Fin}\ n$, we either have that $\mathit{index}\ ([\![fs]\!]{\star}\ \rho)\ i = 0$, or there is some index $j \in \mathit{Fin}\ m$ such that $\mathit{index}\ ([\![fs]\!]{\star}\ \rho)\ i \leq \mathit{index}\ \rho\ j$.

   Proof. The two statements can be proved simultaneously, using induction on the structure of f and fs. I only include one case here, that in which f is $\mathsf{comp}\ g\ fs$ (where $g \in PRF_m^-$ and $fs \in (PRF_n^-)^m$ for some $m \in \mathbb{N}$). The inductive hypothesis for g and $[\![fs]\!]{\star}\ \rho$ leads to one of the following cases:

   - $[\![g]\!]\ ([\![fs]\!]{\star}\ \rho) = 0$. We get that

     $$
     \begin{aligned}
     &[\![\mathsf{comp}\ g\ fs]\!]\ \rho = \\
     &[\![g]\!]\ ([\![fs]\!]{\star}\ \rho)\ = \\
     &0.
     \end{aligned}
     $$

   - There is some $i \in \mathit{Fin}\ m$ such that $[\![g]\!]\ ([\![fs]\!]{\star}\ \rho) \leq \mathit{index}\ ([\![fs]\!]{\star}\ \rho)\ i$. Now we can use the inductive hypothesis for fs, ρ and i. Again we have two cases:

     – $\mathit{index}\ ([\![fs]\!]{\star}\ \rho)\ i = 0$. This implies that

     $$
     \begin{aligned}
     &[\![\mathsf{comp}\ g\ fs]\!]\ \rho && = \\
     &[\![g]\!]\ ([\![fs]\!]{\star}\ \rho) && \leq
     \end{aligned}
     $$

$$index\ ([\![fs]\!]\star\ \rho)\ i = 0.$$

– There is some $j \in Fin\ n$ such that $index\ ([\![fs]\!]\star\ \rho)\ i \leq index\ \rho\ j$. We get that

$$\begin{aligned} &[\![\mathsf{comp}\ g\ fs]\!]\ \rho &= \\ &[\![g]\!]\ ([\![fs]\!]\star\ \rho) &\leq \\ &index\ ([\![fs]\!]\star\ \rho)\ i &\leq \\ &index\ \rho\ j. \end{aligned}$$

□

As an aside the lemma above holds also for the variant of PRF obtained by removing only suc.