Finite Automata Theory and Formal Languages

TMV027/DIT321– LP4 2015

Lecture 14
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Overview of today's lecture:

- Turing machines.
- Guest lecture by Prof. *Aarne Ranta* on Automata and Grammars in Programming Language Technology

Recap: Context-free Languages

- Closure properties for CFL:
  - Union, concatenation, closure, reversal, prefix and homomorphism;
  - Intersection and difference with a RL;
  - No closure under complement;
- Decision properties for CFL:
  - Is the language empty?
  - Does a word belong to the language of a certain grammar?
- The following problems are undecidable:
  - Is the CFG G ambiguous?
  - Is the CFL $\mathcal{L}$ inherently ambiguous?
  - If $\mathcal{L}_1$ and $\mathcal{L}_2$ are CFL, is $\mathcal{L}_1 \cap \mathcal{L}_2 = \emptyset$?
  - If $\mathcal{L}_1$ and $\mathcal{L}_2$ are CFL, is $\mathcal{L}_1 = \mathcal{L}_2$? is $\mathcal{L}_1 \subseteq \mathcal{L}_2$?
  - If $\mathcal{L}$ is a CFL and $\mathcal{P}$ a RL, is $\mathcal{P} = \mathcal{L}$? is $\mathcal{P} \subseteq \mathcal{L}$?
  - If $\mathcal{L}$ is a CFL over Σ, is $\mathcal{L} = \Sigma^*$?
- Push-down automata.

Undecidable Problems

Definition: An *undecidable problem* is a decision problem for which it is impossible to construct a single algorithm that always leads to a yes-or-no answer.

Example: Halting problem: does this program terminate?

To prove that a certain problem P is undecidable one usually *reduces* an already known undecidable problem U to the problem P: instances of U become instances of P.

(Can be seen like one "transforms" U so it "becomes" P).

That is, $w \in U$ iff $w' \in P$ for certain w and w'.
Then, a solution to P would serve as a solution to U.

However, we know there are no solutions to U since U is known to be undecidable.
Then we have a contradiction.

Example of Undecidable Problem: Post's Correspondence

It is an undecidable decision problem introduced by Emil Post in 1946.

> *Given words $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ in $\{0, 1\}^*$, is it possible to find $i_1, \ldots, i_k$ such that $u_{i_1} \ldots u_{i_k} = v_{i_1} \ldots v_{i_k}$?*

Example: Given $u_1 = 1, u_2 = 10, u_3 = 001, v_1 = 011, v_2 = 11, v_3 = 00$ we have that $u_3 u_2 u_3 u_1 = v_3 v_2 v_3 v_1 = 001100011$.

We can use grammars to show that the Post's correspondence problem is undecidable by showing that a grammar is ambiguous iff the PCP has a solution.

(See Section 9.4 in the book.)

Undecidable and Intractable Problems

The theory of undecidable problems provides a guidance about what we may or may not be able to perform with a computer.

One should though distinguish between undecidable problems and *intractable problems*, that is, problems that are decidable but require a large amount of time to solve them.

(In daily life, intractable problems are more common than undecidable ones.)

To reason about both kind of problems we need to have a basic notion of *computation*.

Entscheidungsproblem (Decision Problem)

The *Entscheidungsproblem* (David Hilbert 1928) asks for an *algorithm* to decide whether a given statement is provable from the axioms using the rules of first-order logic.

To answer the question, the notion of *algorithm* had to be formally defined.

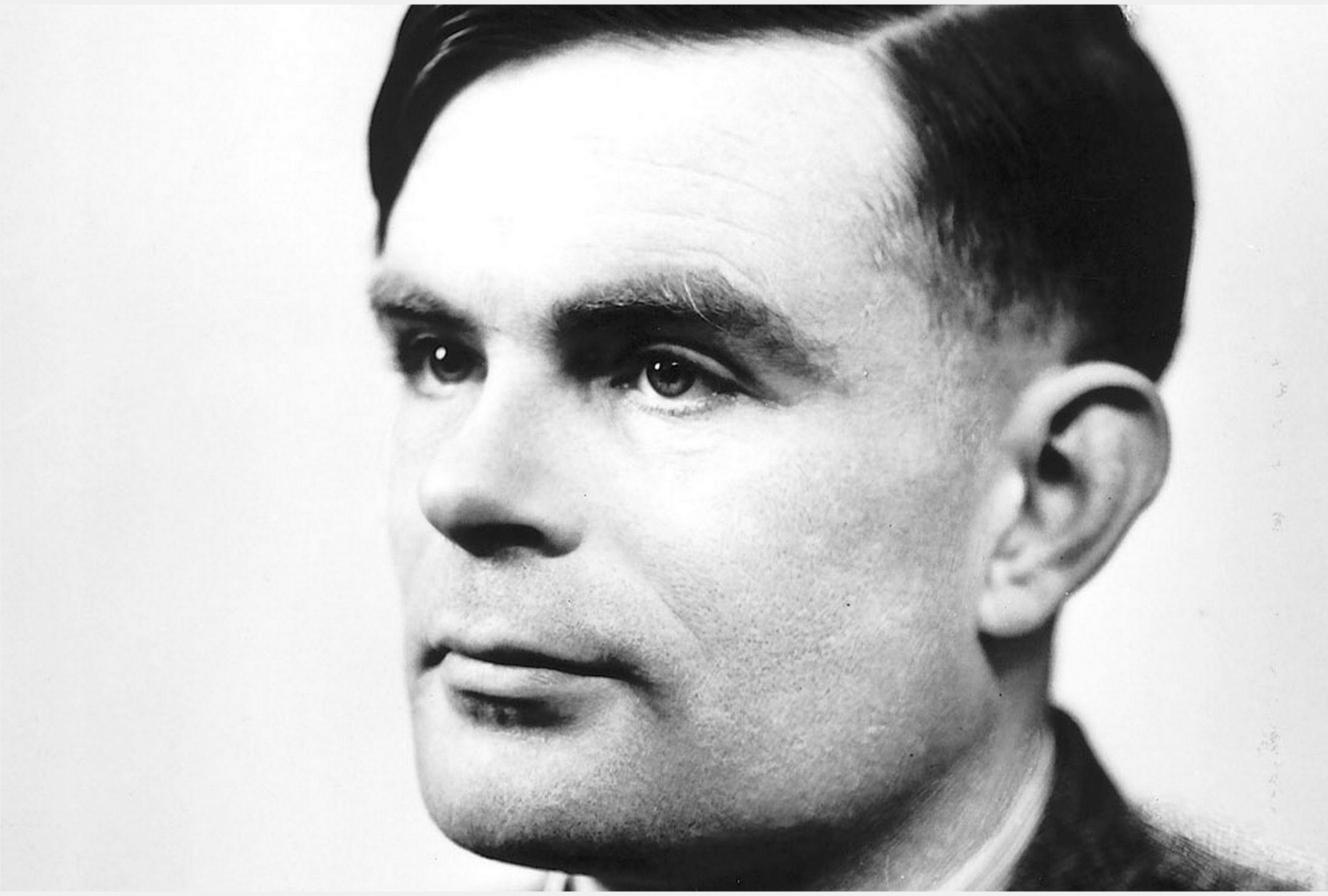
In 1936, Alonzo Church defined the concept of *effective calculable* based on his λ-calculus.

Also in 1936, Alan Turing presented the *Turing machines*.

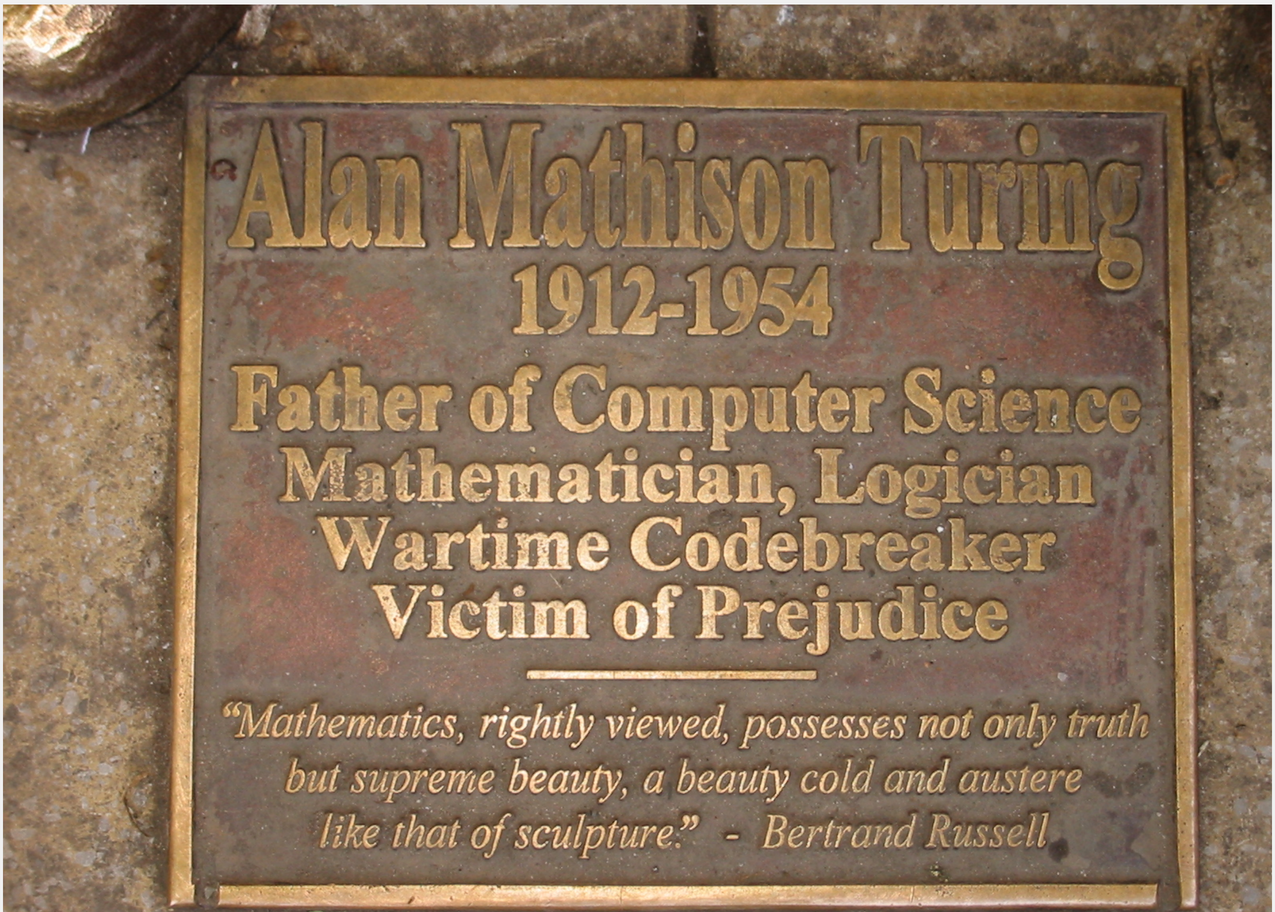
(It was then proved that λ-calculus and Turing machines are equivalent *models of computation*.)

In 1936, both published independent papers showing that a general solution to the Entscheidungsproblem is impossible.

Alan Mathison Turing (23 June 1912 – 7 June 1954)



Alan Mathison Turing



Alan Mathison Turing

- British computer scientist, mathematician, logician and cryptanalyst;
- Considered the father of theoretical computer science and artificial intelligence;
- Philosopher, mathematical biologist;
- Marathon and ultra distance runner;
- In the 50' he also became interested in chemistry.

Alan Mathison Turing

- He took his Ph.D. in 1938 at Princeton with Alonzo Church;
- He invented the concept of a computer, called *Turing Machine* (TM);

  Turing showed that TM could perform any kind of *computation*;

  He also showed that his notion of *computable* was equivalent to Church's notion of *effective calculable*;
- During the WWII he helped Britain to break the German Enigma machines which shortened the war by 2-4 years and saved many lives!
- Since 1966, ACM annually gives the *Turing Award* for contributions to the computing community.

Turing Machines (1936)

- Theoretically, a TM is just as *powerful* as any other computer!
  Powerful here refers only to which computations a TM is capable of doing, not to how *fast* or *efficiently* it does its job.
- Conceptually, a TM has a finite set of states, a finite alphabet (containing a blank symbol), and a finite set of instructions;
- Physically, it has a *head* that can read, write, and move along an *infinitely long tape* (on both sides) that is divided into *cells*.
- Each cell contains a symbol of the alphabet (possibly the blank symbol):

$\cdots$		a_1	a_2	a_3	a_4	a_5		$\cdots$
			$\uparrow$					

Turing Machines: More Concretely

- Let $\square$ represents the *blank* symbol and let Σ be a non-empty alphabet of symbols such that $\{\square, \mathsf{L}, \mathsf{R}\} \cap \Sigma = \emptyset$.
  Now, we define $\Sigma' = \Sigma \cup \{\square\}$;
- The read/write head of the TM is always placed over one of the cells. We said that that particular cell is being *read*, *examined* or *scanned*;
- At every moment, the TM is in a certain state $q \in Q$, where Q is a non-empty and finite set of states;
- In some cases, we consider a set F of final states.

Turing Machines: Transition Functions

In one *move*, the TM will:

1. Change to a (possibly) new state;
2. Replace the symbol below the head by a (possibly) new symbol;
3. Move the head to the left (denoted L) or to the right (denoted R).

The behaviour of a TM is given by a possibly partial *transition function*

$$\delta \in Q \times \Sigma' \to Q \times \Sigma' \times \{\mathsf{L}, \mathsf{R}\}$$

δ is such that for every $q \in Q$, $a \in \Sigma'$ there is *at most* one instruction.

Note: We have a *deterministic* TM.

How to Compute with a TM?

Before the execution starts, the tape of a TM looks as follows:

$\cdots$		a_1	a_2	$\cdots$	a_{n-1}	a_n		b_1	$\cdots$	b_m		$\cdots$
		$\uparrow$										

- The input data is placed on the tape, if necessary separated with blanks;
- There are infinitely many blank to the left and to the right of the input;
- The head is placed on the first symbol of the input;
- The TM is in a special *initial state* $q_0 \in Q$;
- The machine then proceeds according to the transition function δ.

Turing Machine: Formal Definition

Definition: A *TM* is a 6-tuple $(Q, \Sigma, \delta, q_0, \Box, F)$ where:

- Q is a non-empty, finite set of states;
- Σ is a non-empty alphabet such that $\{\Box, \mathsf{L}, \mathsf{R}\} \cap \Sigma = \emptyset$;
- $\delta \in Q \times \Sigma' \to Q \times \Sigma' \times \{\mathsf{L}, \mathsf{R}\}$ is a transition function, where $\Sigma' = \Sigma \cup \{\Box\}$;
- $q_0 \in Q$ is the initial state;
- $\Box$ is the blank symbol, $\Box \notin \Sigma$;
- F is a non-empty, finite set of final or accepting states, $F \subseteq Q$.

Note: In some cases, the set F is not relevant (compare with FA).

Result of a Turing Machine

Definition: Let $M = (Q, \Sigma, \delta, q_0, \Box, F)$ be a TM.
We say that M *halts* if for certain $q \in Q$ and $a \in \Sigma$, $\delta(q, a)$ is undefined.

Whatever is written in the tape when the TM *halts* can be considered as the *result* of the computation performed by the TM.

If we are only interested in the result of a computation, we can omit F from the formal definition of the TM.

Examples

Example: Let $\Sigma = \{0, 1\}$, $Q = \{q_0\}$ and let δ be as follows:

$$\delta(q_0, 0) = (q_0, 1, \text{R})$$
$$\delta(q_0, 1) = (q_0, 0, \text{R})$$

What does this TM do?

Example: The execution of a TM might loop.

Consider the following set of instructions for Σ and Q as above.

$$\delta(q_0, a) = (q_0, a, \text{R}) \qquad \text{with } a \in \Sigma \cup \{\square\}$$

Recursive and Recursively Enumerable Languages

Definition: Let $M = (Q, \Sigma, \delta, q_0, \square, F)$ be a TM.
The TM M accepts a word $w \in \Sigma^*$ if when we run M with w as input data, the TM is in a final state when it halts.

Definition: The *language* accepted by a TM is the set of words that are accepted by the TM.

Definition: A languages is called *recursively enumerable* if there is a TM accepting the words in that language.

Definition: A *Turing decider* is a TM that never loops, that is, the TM halts.

Definition: A language is *recursive* or *decidable* if there is a Turing decider accepting the words in the language.

Example of a Turing Decider

How to define a TM that accepts the language $\mathcal{L} = \{ww^r \mid w \in \{0,1\}^*\}$?
(One can prove using the Pumping lemma that this language is not context-free.)

Let $\Sigma = \{0, 1, X, Y\}$, $Q = \{q_0, \ldots, q_7\}$ and $F = \{q_7\}$,

Let $a \in \{0, 1\}$, $b \in \{X, Y, \Box\}$, and $c \in \{X, Y\}$.

$$
\begin{array}{lll}
\delta(q_0, 0) = (q_1, X, \mathrm{R}) & \delta(q_0, 1) = (q_3, Y, \mathrm{R}) & \delta(q_0, \Box) = (q_7, \Box, \mathrm{R}) \\
\delta(q_1, a) = (q_1, a, \mathrm{R}) & \delta(q_3, a) = (q_3, a, \mathrm{R}) & \\
\delta(q_1, b) = (q_2, b, \mathrm{L}) & \delta(q_3, b) = (q_4, b, \mathrm{L}) & \\
\delta(q_2, 0) = (q_5, X, \mathrm{L}) & \delta(q_4, 1) = (q_5, Y, \mathrm{L}) & \\
\delta(q_5, a) = (q_6, a, \mathrm{L}) & & \delta(q_5, c) = (q_7, c, \mathrm{R}) \\
\delta(q_6, a) = (q_6, a, \mathrm{L}) & \delta(q_6, c) = (q_0, c, \mathrm{R}) &
\end{array}
$$

What happens with the input 0110?
And with the input 010?

Overview of Next Lecture (in HC2)

- More on Turing machines;
- Summary of the course.