Finite Automata Theory and Formal Languages

TMV027/DIT321– LP4 2015

Lecture 5
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Overview of today's lecture:

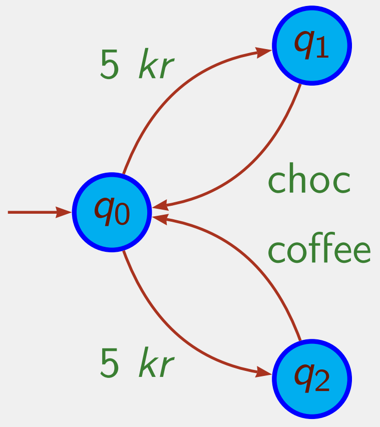
- NFA: Non-deterministic finite automata;
- Equivalence between DFA and NFA.

Recap: Deterministic Finite Automata

- Defined by a 5-tuple $(Q, \Sigma, \delta, q_0, F)$;
  Better represented by transition diagrams or tables;
- Why "finite"?;
- Why "deterministic"?;
- total $\delta : Q \times \Sigma \to Q$;
- Useful to model simple problems;
- Only accesible part is of interest;
- Accept set of words x such that $\hat{\delta}(q_0, x) \in F$;
- Accept the so called regular language;
- We can defined the products $\times$ and $\oplus$, and the complement...
- ... accepting the intersection, union and complement of the languages;
- Hence, regular languages are closed under intersection, union and complement.

Non-deterministic Finite Automata

Given a state and the next symbol, a non-deterministic finite automaton (NFA) can "move" to many states.



Intuitively, the vending machine can *choose* between different states.

Note: We will not need a *dead* state here.

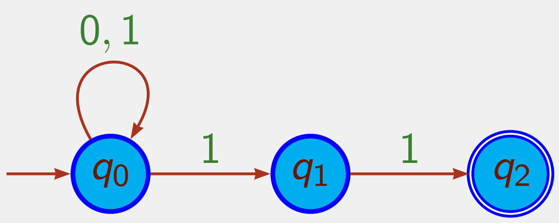
When Does a NFA Accepts a Word?

Intuitively: the automaton can *guess* a successful computation if there is one.

Formally: iff there is *at least one* path from the start state to an accepting state while reading the word.

Example:

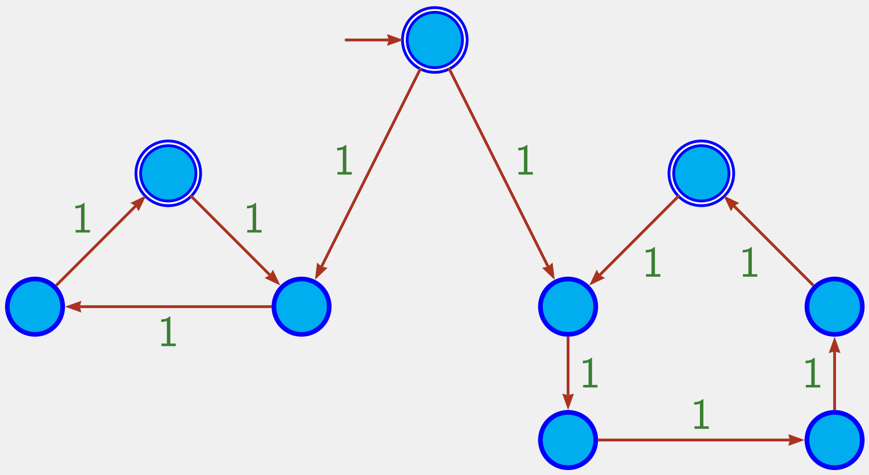
NFA accepting words that end in 11



What are all possible computations for the string 1011?
Will 1011 be accepted by the NFA?
And 110?

NFA Accepting Words of Length Divisible by 3 or by 5

Let $\Sigma = \{1\}$.



What would be the equivalent DFA?

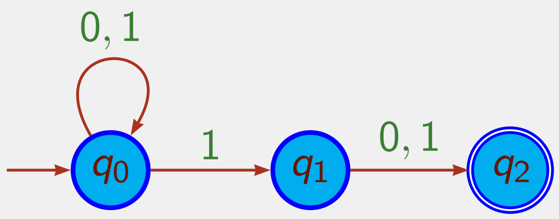
Non-deterministic Finite Automata

Definition: A *non-deterministic finite automaton* (NFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ consisting of:

1. A finite set Q of *states*;
2. A finite set Σ of *symbols* (alphabet);
3. A "partial" *transition function* $\delta : Q \times \Sigma \to \mathcal{P}ow(Q)$;
4. A *start state* $q_0 \in Q$;
5. A set $F \subseteq Q$ of *final* or *accepting* states.

Example: NFA

Let us define an automaton accepting only the words such that the second last symbol from the right is 1.



δ	0	1
$\rightarrow q_0$	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_2\}$	$\{q_2\}$
$*q_2$	$\emptyset$	$\emptyset$

Extending the Transition Function to Strings

As before, we define $\hat{\delta}(q, x)$ by recursion on x.

Definition:

$$\hat{\delta} : Q \times \Sigma^* \rightarrow \mathcal{P}ow(Q)$$
$$\hat{\delta}(q, \epsilon) = \{q\}$$
$$\hat{\delta}(q, ax) = \bigcup_{p \in \delta(q,a)} \hat{\delta}(p, x)$$

That is, if $\delta(q, a) = \{p_1, \ldots, p_n\}$ then

$$\hat{\delta}(q, ax) = \hat{\delta}(p_1, x) \cup \ldots \cup \hat{\delta}(p_n, x)$$

Language Accepted by a NFA

Definition: The *language* accepted by the NFA $N = (Q, \Sigma, \delta, q_0, F)$ is the set $\mathcal{L}(N) = \{x \in \Sigma^* \mid \hat{\delta}(q_0, x) \cap F \neq \emptyset\}$.

That is, a word x is accepted if $\hat{\delta}(q_0, x)$ contains at least one accepting state.

Note: Again, we could write a program that simulates a NFA and let it tell us whether a certain string is accepted or not.

Exercise: Do it!

Transforming a NFA into a DFA

For same examples it is much simpler to define a NFA than a DFA.

Example: The language with words of length divisible by 3 or by 5.

However, any language accepted by a NFA is also accepted by a DFA.

In general, the number of states of the DFA is about the number of states in the NFA although it often has many more transitions.

In the worst case, if the NFA has n states, a DFA accepting the same language might have 2^n states.

The *algorithm* transforming a NFA into an equivalent DFA is called the *subset construction*.

The Subset Construction

Definition: Given a NFA $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$ we construct a DFA $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$ such that $\mathcal{L}(D) = \mathcal{L}(N)$ as follows:

- $Q_D = \mathcal{P}ow(Q_N)$;
- $\delta_D : Q_D \times \Sigma \to Q_D$ (that is, $\delta_D : \mathcal{P}ow(Q_N) \times \Sigma \to \mathcal{P}ow(Q_N)$)

  $\delta_D(X, a) = \bigcup_{q \in X} \delta_N(q, a)$;
- $F_D = \{S \subseteq Q_N \mid S \cap F_N \neq \emptyset\}$.

Exercise: Implement the subset construction!

Remarks of the Subset Construction

- If $|Q_N| = n$ then $|Q_D| = 2^n$.

  Non accessible states in Q_D can be safely removed (we will see how to do this later on in the course).
- If $X = \{q_1, \ldots, q_n\}$ then $\delta_D(X, a) = \delta_N(q_1, a) \cup \ldots \cup \delta_N(q_n, a)$.

  In addition,

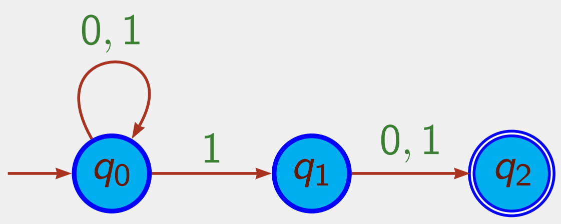
  $$\delta_D(\emptyset, a) = \emptyset \qquad \delta_D(\{q\}, a) = \delta_N(q, a) \qquad \delta_D(X, a) = \bigcup_{q \in X} \delta_D(\{q\}, a)$$

  and

  $$\delta_D(X_1 \cup X_2, a) = \delta_D(X_1, a) \cup \delta_D(X_2, a)$$
- Each accepting state (set) S in F_D contains at least one accepting state of N.

Example: Subset Construction

Let us convert this NFA into a DFA



The DFA starts from $\{q_0\}$. Only accessible states matter.

From $\{q_0\}$, with 0, we go to q_0 so $\delta_D(\{q_0\}, 0) = \{q_0\}$.

From $\{q_0\}$, with 1, we go to q_0 or to q_1. Then, $\delta_D(\{q_0\}, 1) = \{q_0, q_1\}$.

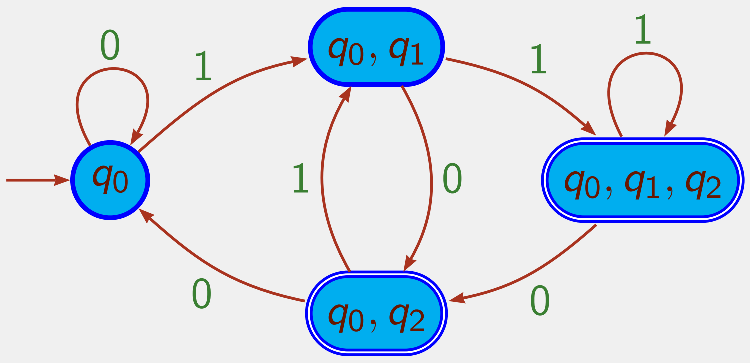
From $\{q_0, q_1\}$, with 0, we go to q_0 or to q_2. Then, $\delta_D(\{q_0, q_1\}, 0) = \{q_0, q_2\}$.

From $\{q_0, q_1\}$, with 1, we go to q_0 or q_1 or q_2. Then, $\delta_D(\{q_0, q_1\}, 1) = \{q_0, q_1, q_2\}$.

etc...

Example: Subset Construction (cont.)

The complete (and simplified) DFA from the previous NFA is:



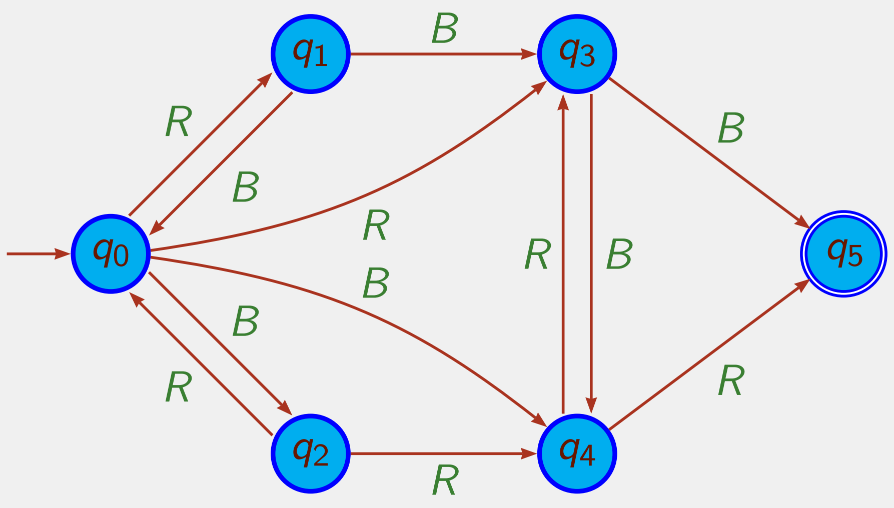
The DFA *remembers* the last two bits seen and accepts a word if the next-to-last bit is 1.

By only computing the *accessible* states (from the start state) we are able to keep the total number of states to 4 (and not 8).

Example: NFA Representation of Gilbreath's Principle

Let us shuffle 2 non-empty alternating decks of cards, one starting with a red card and one starting with a black one.
How does the resulting deck look like?

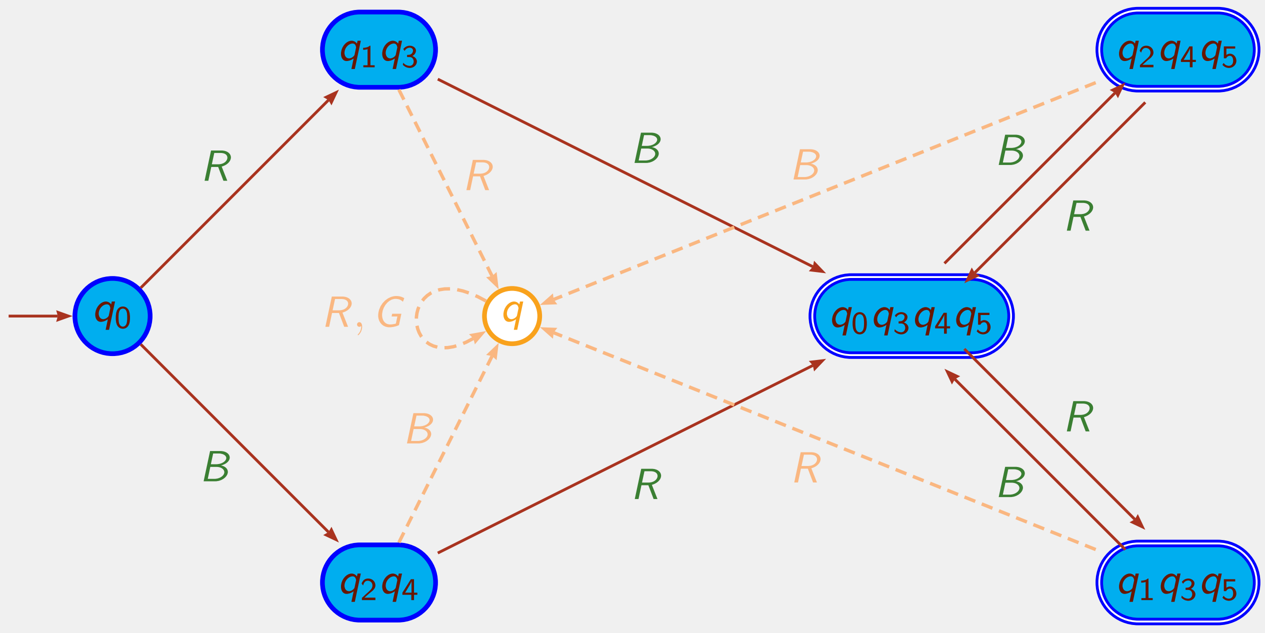
Let $\Sigma = \{B, R\}$ represent a black or red card respectively.



q_0 starts with B and R
q_1 both start with B
q_2 both start with R
q_3 starts with B and ϵ
q_4 starts with R and ϵ
q_5 both ϵ

How does the resulting deck look like? Build the corresponding DFA!

Example: DFA Representation of Gilbreath's Principle



How does the resulting deck look like?

Application of NFA: Text Search

Suppose we want to find occurrences of certain *keywords* in a text.

We could design a NFA that enters in an accepting state when it has recognised one of these keywords.

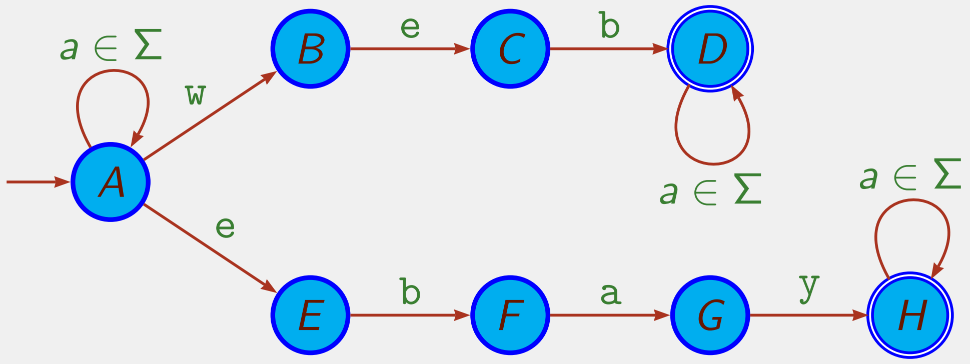
Then we could either implement the NFA or transform it to a DFA and get a "deterministic" (efficient) program.

Once we prove the subset construction correct, then we know the DFA will be correct (if the NFA is!).

This is a good example of a derivation of a *program* (the DFA) from a *specification* (the NFA).

Application of NFA: Text Search

The following (easy to write) NFA searches for the keywords `web` and `ebay`:



If one applies the subset construction one obtains the (complicated) DFA of page 71 in the book.

Observe that the obtained DFA has the same number of states as the NFA, but it is much more difficult to define directly!

Towards the Correction of the Subset Construction

Proposition: $\forall x . \forall q .\ \hat{\delta}_N(q, x) = \hat{\delta}_D(\{q\}, x)$.

Proof: By induction on x we prove $P(x) : \forall q .\ \hat{\delta}_N(q, x) = \hat{\delta}_D(\{q\}, x)$.

Base case: trivial.

Inductive step: Assuming $P(x)$ we prove $P(ax)$.

$$\begin{array}{rcll}
\hat{\delta}_N(q, ax) & = & \bigcup_{p \in \delta_N(q,a)} \hat{\delta}_N(p, x) & \text{by definition of } \hat{\delta}_N \\
& = & \bigcup_{p \in \delta_N(q,a)} \hat{\delta}_D(\{p\}, x) & \text{by IH with state } p \\
& = & \hat{\delta}_D(\delta_N(q, a), x) & \text{see lemma below} \\
& = & \hat{\delta}_D(\delta_D(\{q\}, a), x) & \text{remark on slide 11} \\
& = & \hat{\delta}_D(\{q\}, ax) & \text{by definition of } \hat{\delta}_D
\end{array}$$

Lemma: *For all words x and sets of states S,* $\hat{\delta}_D(S, x) = \bigcup_{p \in S} \hat{\delta}_D(\{p\}, x)$.

Correction of the Subset Construction

Theorem: *Given a NFA N, if D is the DFA constructed from N by the subset construction then $\mathcal{L}(N) = \mathcal{L}(D)$.*

Proof: $x \in \mathcal{L}(N)$ iff $\hat{\delta}_N(q_0, x) \cap F_N \neq \emptyset$ iff $\hat{\delta}_N(q_0, x) \in F_D$.

By the previous proposition, then $\hat{\delta}_D(\{q_0\}, x) \in F_D$.

Since $\{q_0\}$ is the starting state in D, then $x \in \mathcal{L}(D)$.

Equivalence between DFA and NFA

Theorem: *A language $\mathcal{L}$ is accepted by some DFA iff $\mathcal{L}$ is accepted by some NFA.*

Proof: The "if" part is the result of the previous theorem (correctness of subset construction).

For the "only if" part we need to transform the DFA into a NFA.

Intuitively: each DFA can be seen as a NFA where there exists only one choice at each stage.

Formally: given $D = (Q, \Sigma, \delta_D, q_0, F)$ we define $N = (Q, \Sigma, \delta_N, q_0, F)$ such that $\delta_N(q, a) = \{\delta_D(q, a)\}$.

It only remains to show (by induction on x) that if $\hat{\delta}_D(q_0, x) = p$ then $\hat{\delta}_N(q_0, x) = \{p\}$.

Regular Languages

Recall: A language $\mathcal{L} \subseteq \Sigma^*$ is *regular* iff there exists a DFA D on the alphabet Σ such that $\mathcal{L} = \mathcal{L}(D)$.

Proposition: *A* language $\mathcal{L} \subseteq \Sigma^*$ is *regular* iff there exists a NFA N such that $\mathcal{L} = \mathcal{L}(N)$.

Proof: If $\mathcal{L}$ is regular then $\mathcal{L} = \mathcal{L}(D)$ for some DFA D.
To any DFA D we can associate a NFA N_D such that $\mathcal{L}(D) = \mathcal{L}(N_D)$ as in previous theorem.

In the other direction, if $\mathcal{L} = \mathcal{L}(N)$ for some NFA N then, the subset construction gives a DFA D such that $\mathcal{L}(N) = \mathcal{L}(D)$ so $\mathcal{L}$ is regular.

Overview of Next Lecture (in HC2)

Sections 2.3.6, 2.5–2.5.5:

- More on NFA;
- NFA with ϵ-transitions;
- Equivalence between DFA and ϵ-NFA.