Finite Automata Theory and Formal Languages
TMV027/DIT321 – LP4 2015

Formal Proofs, Alphabets and Words

Week 2

In these exercises, book sections and pages refer to those in the third edition of the course book.

Let $\mathbb{N}$ be the set of all non-negative integers $\{0, 1, 2, \ldots\}$ (see page 22 of the text book: "Integers as recursively defined concepts").

Proofs by Counterexample and Contradiction

1. If $\Sigma = \{0, 1\}$, find a counterexample to the following alleged theorem: $\forall x, y \in \Sigma^*$ we have (cf. section 1.3.4)
$$x^2y = xyx$$

2. Suppose we put infinitely many pigeons into two pigeonholes. Show that one of the pigeonholes contains infinitely many pigeons. *Hint:* Prove by contradiction!

Mathematical and Course-of-values Induction

1. Prove that $\sum_{0 \leqslant k}^{n} k = n(n + 1)/2$.

2. Prove that $\sum_{1 \leqslant k}^{n} (2k - 1) = n^2$.

3. Prove that $\sum_{1 \leqslant k}^{n} k^2 = n(n + 1)(2n + 1)/6$.

4. Prove that $\forall n \geqslant 4. n^2 \leqslant 2^n$.

5. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by recursion as
$$f(0) = 0 \qquad f(n + 1) = f(n) + n$$
What are the values of $f(2)$ and $f(3)$?

   Use mathematical induction to show that for all $n \in \mathbb{N}$ we have
$$2f(n) = n^2 - n$$

6. Suppose that we have stamps of 4 kr and 3 kr. Show that any amount of postage over 5 kr can be made with some combinations of these stamps.

7. Let us define by recursion the following function:

$$0! = 1 \qquad (n+1)! = (n+1) \times n!$$

Show that $n! \geqslant 2^n$ for $n \geqslant 4$ by analogy with the proof of example 1.17, page 21 of the text book.

8. Consider the following definitions for $f, g : \mathbb{N} \to \mathbb{N}$:

$$\begin{array}{ll} f(0) = 0 & g(0) = 0 \\ f(1) = 1 & g(n+1) = 1 - g(n) \\ f(n+2) = f(n) & \end{array}$$

Compute $f(2), f(3), g(1), g(2)$ and $g(3)$.

Prove that $\forall n \in \mathbb{N}.\ f(n) = g(n)$.

9. Let us define the Fibonacci function:

$$f(0) = 0 \qquad f(1) = 1 \qquad f(n+2) = f(n+1) + f(n)$$

We then define $s(0) = 0,\ s(n+1) = s(n) + f(n+1)$.

Prove by induction that we have

$$\forall n. s(n) = f(n+2) - 1.$$

Now we define

$$l(0) = 2,\ l(1) = 1,\ l(n+2) = l(n+1) + l(n)$$

Prove by induction that we have $l(n+1) = f(n) + f(n+2)$.

Mutual Induction

1. Let us define by recursion the following two functions $f, g : \mathbb{N} \to \mathbb{N}$

$$\begin{array}{ll} f(0) = 0 & g(0) = 1 \\ f(n+1) = g(n) & g(n+1) = f(n) \end{array}$$

What are the values of $g(2)$ and $f(4)$? Show by mathematical induction that for all $n \in \mathbb{N}$ we have

$$f(n) + g(n) = 1 \qquad f(n)g(n) = 0$$

Show by *mutual* induction that $f(n) = 0$ iff $g(n) = 1$ iff n is even, and that $f(n) = 1$ iff $g(n) = 0$ iff n is odd, in analogy to the proof in pages 26–28 in the text book.

2. Consider the following definitions for $f, g, h : \mathbb{N} \to \mathbb{N}$:

$$\begin{array}{lll} f(0) = 0 & g(0) = 0 & h(0) = 1 \\ f(n+1) = 2 + f(n) & g(n+1) = 2h(n) & h(n+1) = g(n) + 2 - n \end{array}$$

Compute $f(1), g(1), h(1), f(2), g(2), h(2), f(3), g(3)$ and $h(3)$.

Prove that $\forall n \in \mathbb{N}.\ f(n) = g(n)$.

Hint: Try to prove by mutual induction that $f(n) = g(n) = 2n$ and $h(n) = n + 1$.

Alphabets, Words and Induction

1. If $\Sigma = \{a, b, c\}$, what are Σ^0, Σ^1 and Σ^2?

2. Let $\Sigma = \{0, 1\}$. We define $\phi : \Sigma^* \to \Sigma^*$ by recursion as follows

   $$\phi(\epsilon) = \epsilon \qquad \phi(0w) = 1\phi(w) \qquad \phi(1w) = 0\phi(w)$$

   What are $\phi(1011)$ and $\phi(1101)$?

   Show by induction on w that
   $$|\phi(w)| = |w|.$$

3. Let $\Sigma = \{0, 1\}$. We define the reverse function on Σ^* by the equations

   $$rev(\epsilon) = \epsilon \qquad rev(ax) = rev(x)a$$

   What are $rev(010)$ and $rev(10)$?

   Show by induction on y that we have

   $$rev(yx) = rev(x)rev(y).$$

   Show by induction on $n \in \mathbb{N}$ that we have

   $$rev(x^n) = (rev(x))^n.$$

4. Let $\Sigma = \{0, 1\}$. We define $\varphi : \Sigma^* \to \Sigma^*$ by recursion as follows

   $$\varphi(\epsilon) = \epsilon \qquad \varphi(0w) = \varphi(w)1 \qquad \varphi(1w) = \varphi(w)0$$

   What are $\varphi(1011)$ and $\varphi(1101)$?

   Show by induction on w that

   $$\varphi(w) = rev(\phi(w))$$

   where ϕ and rev are those from exercises 2) and 3), respectively, of this section.

5. Given a finite alphabet Σ, when can we have $x^2 = y^3$ with $x, y \in \Sigma^*$?

Inductive Sets and Structural Induction

1. A binary tree with information on the nodes is either a leaf with no information or a node containing some information and exactly 2 subtrees. You may assume the information in the nodes being of any suitable type (string, Natural numbers, ...).

   (a) Give the inductive definition of the set `Tree` of trees .

   (b) Define the functions that count the number of leaves nl, the number of nodes that are not leaves nn, and the number of subtrees nt of a tree.

   (c) Prove by structural induction on the trees that $\forall t \in \texttt{Tree}.\, \mathsf{nl}(t) = \mathsf{nn}(t) + 1$.

(d) Prove by structural induction on the trees that $\forall t \in \texttt{Tree}.\, 2\mathsf{nn}(t) = \mathsf{nt}(t)$.

2. Consider expressions that consist of either an identifier, the addition of two expressions, or the multiplication of two expressions. You may assume the identifiers being of any suitable type (string, Natural numbers, ...).

   (a) Give the inductive definition of the set `Exp` of expressions;

   (b) Define the function `op` that counts the number of additions and multiplications in an expression, and the function `id` that counts the number of identifiers in an expression;

   (c) Prove by structural induction on the expressions that $\forall e \in \texttt{Exp}.\, \mathsf{op}(e) + 1 = \mathsf{id}(e)$.