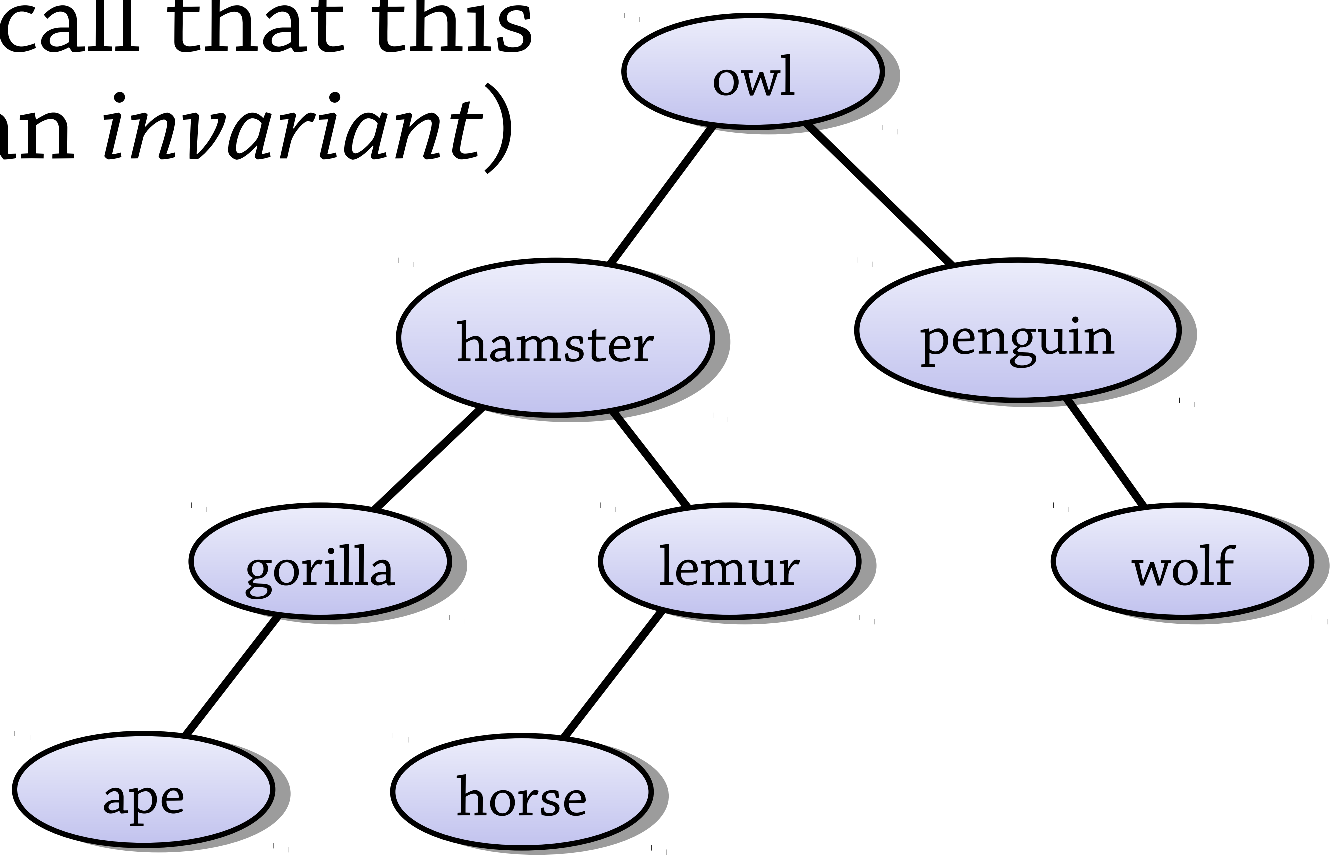
Binary search trees
(chapters 18.1 – 18.3)

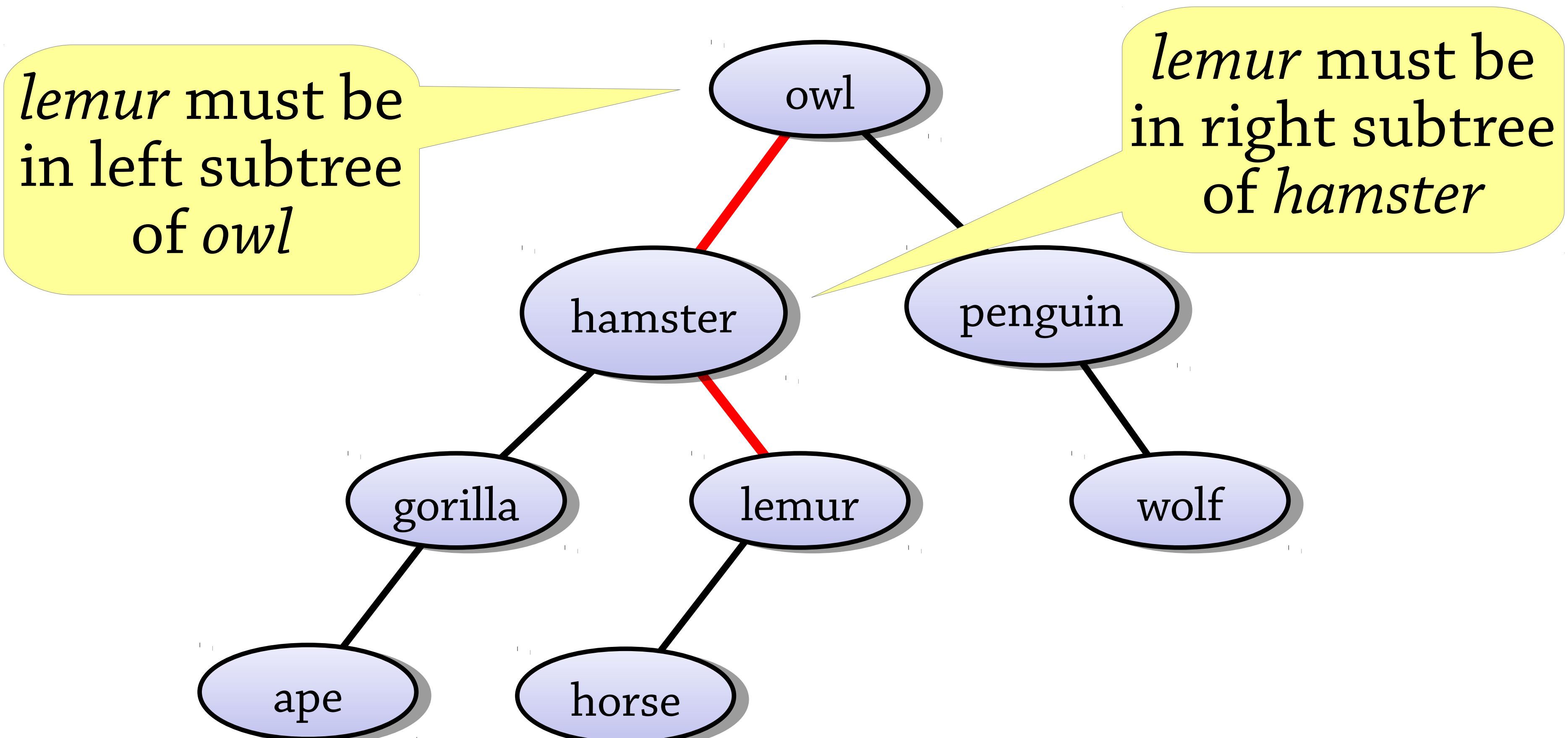
Binary search trees

In a *binary search tree* (BST), every node is greater than all its left descendants, and less than all its right descendants (recall that this is an *invariant*)



Searching in a BST

Finding an element in a BST is easy, because by looking at the root you can tell which subtree the element is in



Searching in a binary search tree

To search for *target* in a BST:

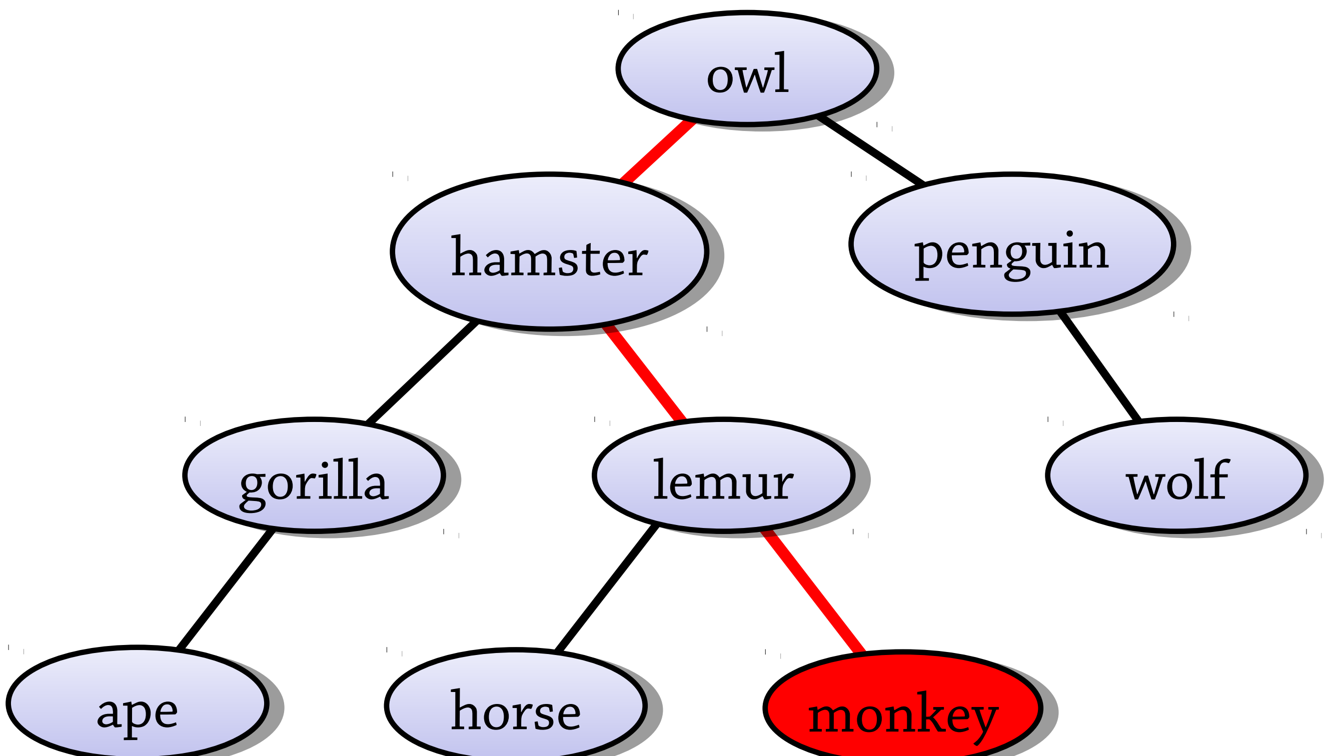
- If the target matches the root node's data, we've found it
- If the target is *less* than the root node's data, recursively search the left subtree
- If the target is *greater* than the root node's data, recursively search the right subtree
- If the tree is empty, fail

A BST can be used to implement a set, or a map from keys to values

Inserting into a BST

To insert a value into a BST:

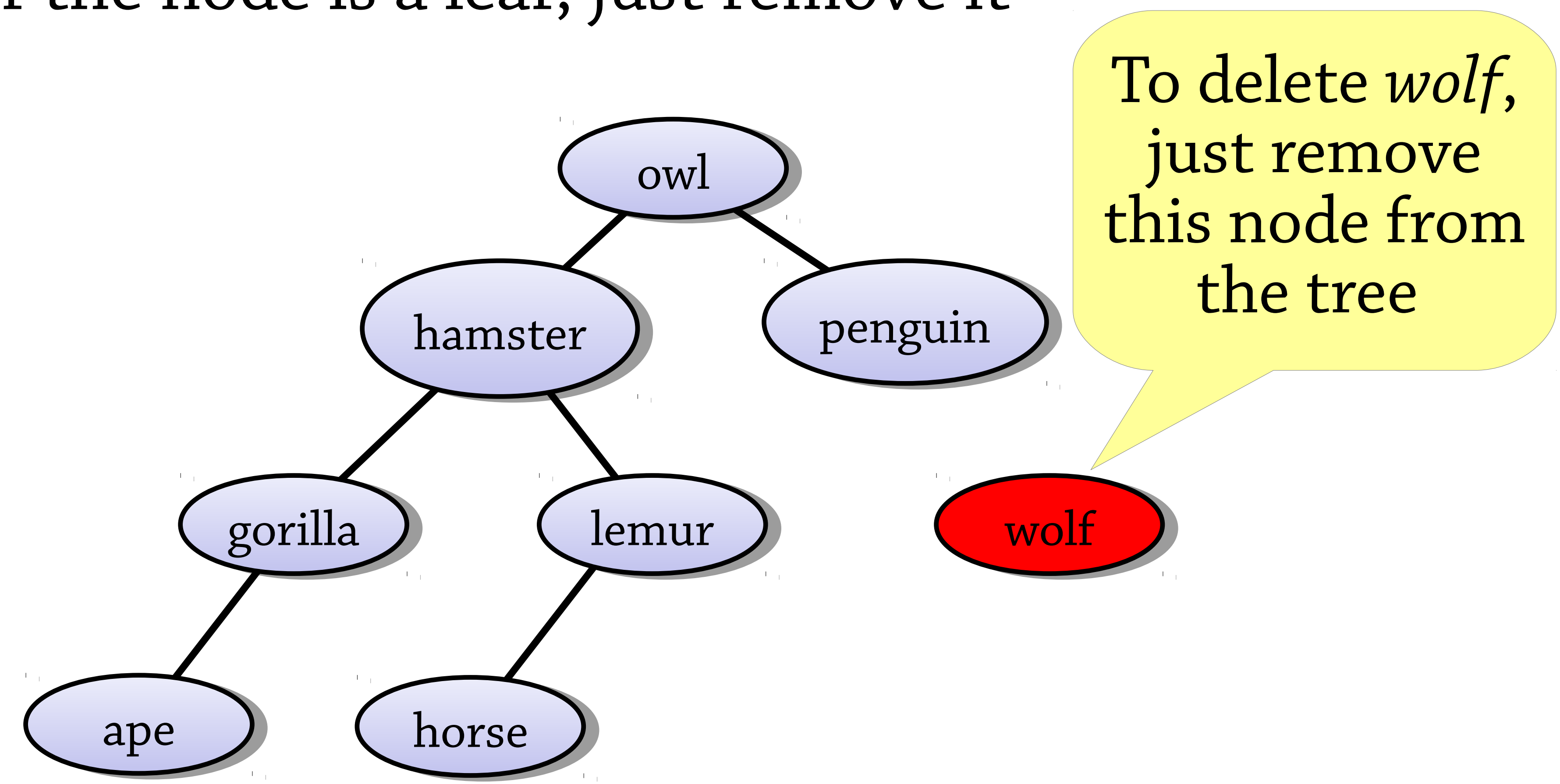
- Start by searching for the value
- But when you get to *null* (the empty tree), make a node for the value and place it there



Deleting from a BST

To delete a value into a BST:

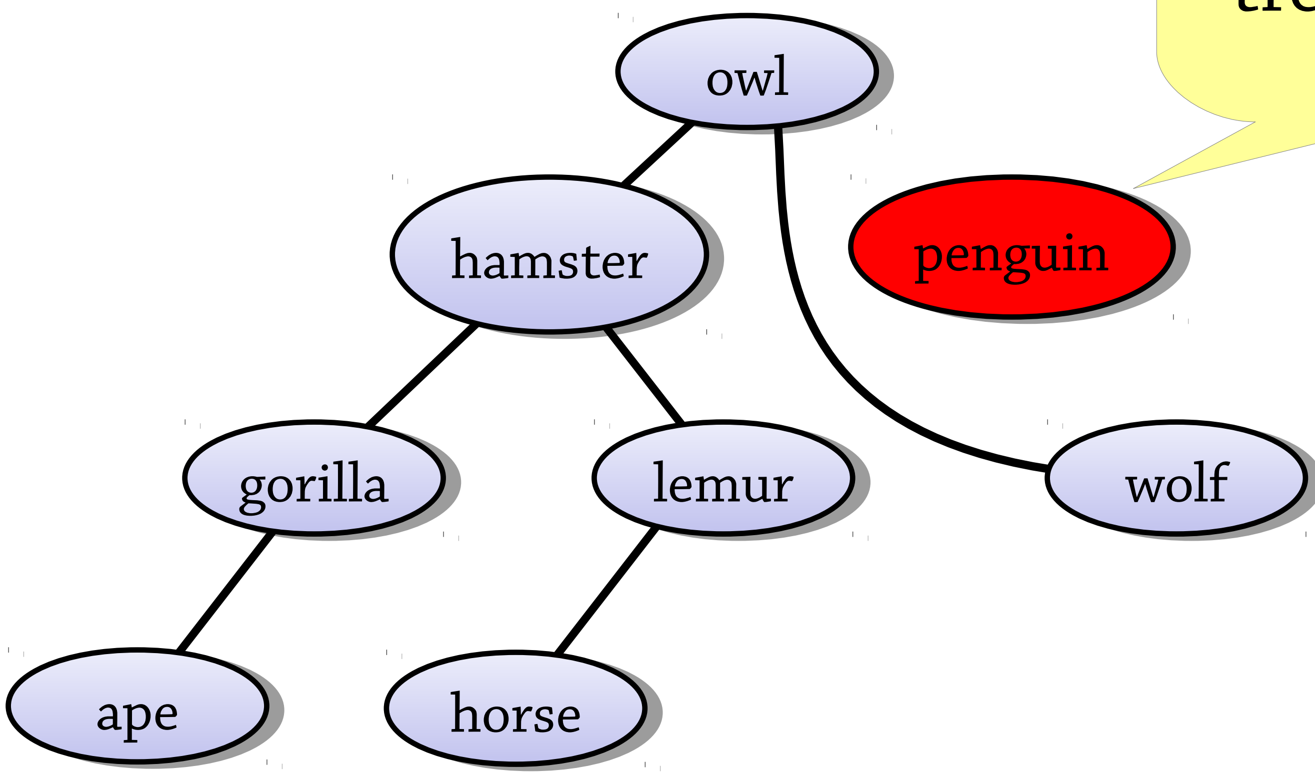
- Find the node containing the value
- If the node is a leaf, just remove it



Deleting from a BST, continued

If the node has *one* child, replace the node with its child

To delete *penguin*, replace it in the tree with *wolf*



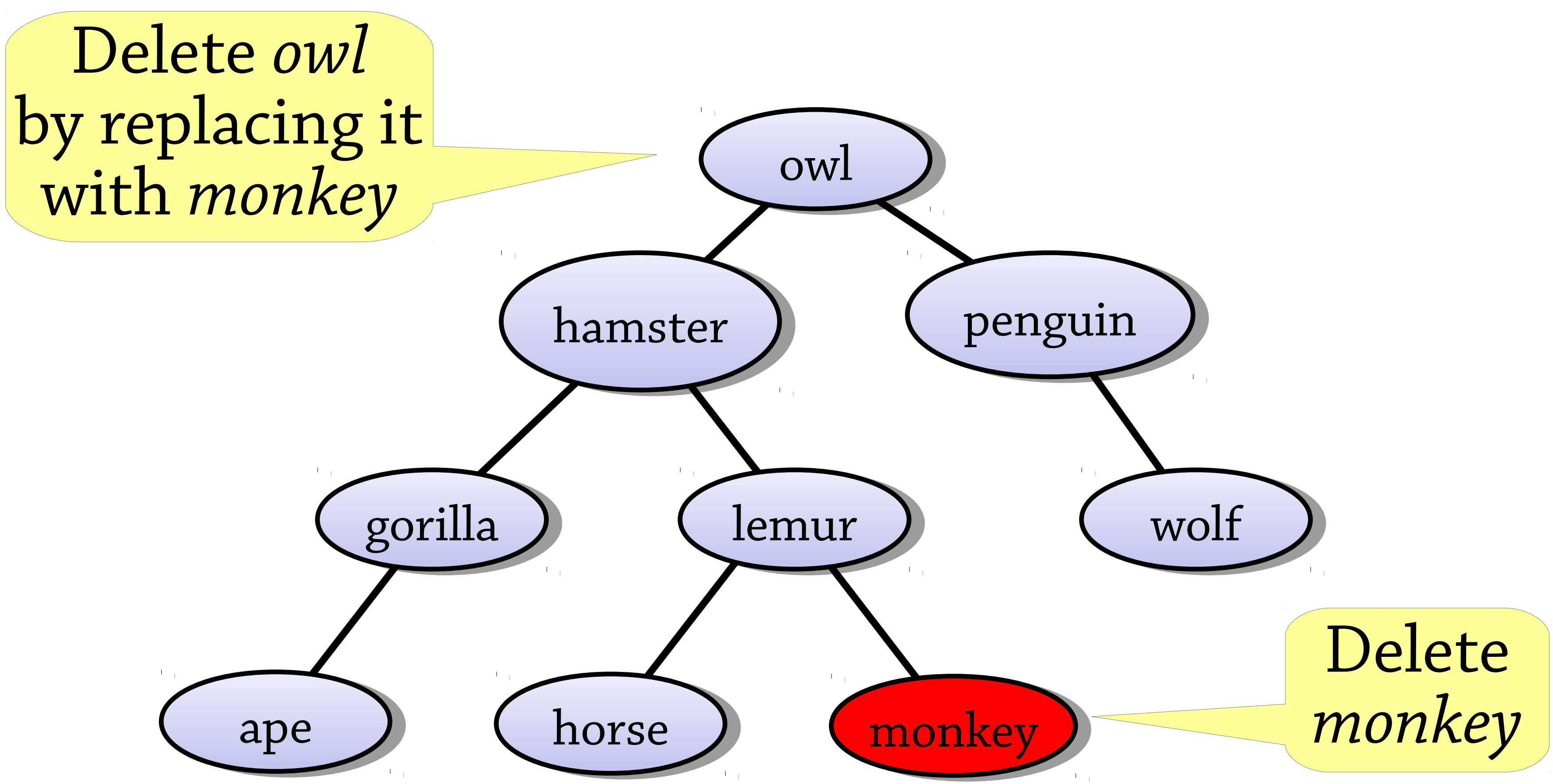
Deleting from a BST

To delete a value from a BST:

- Find the node
- If it has no children, just remove it from the tree
- If it has one child, replace the node with its child
- If it has two children…?
  Can't remove the node without removing its children too!

Deleting a node with two children

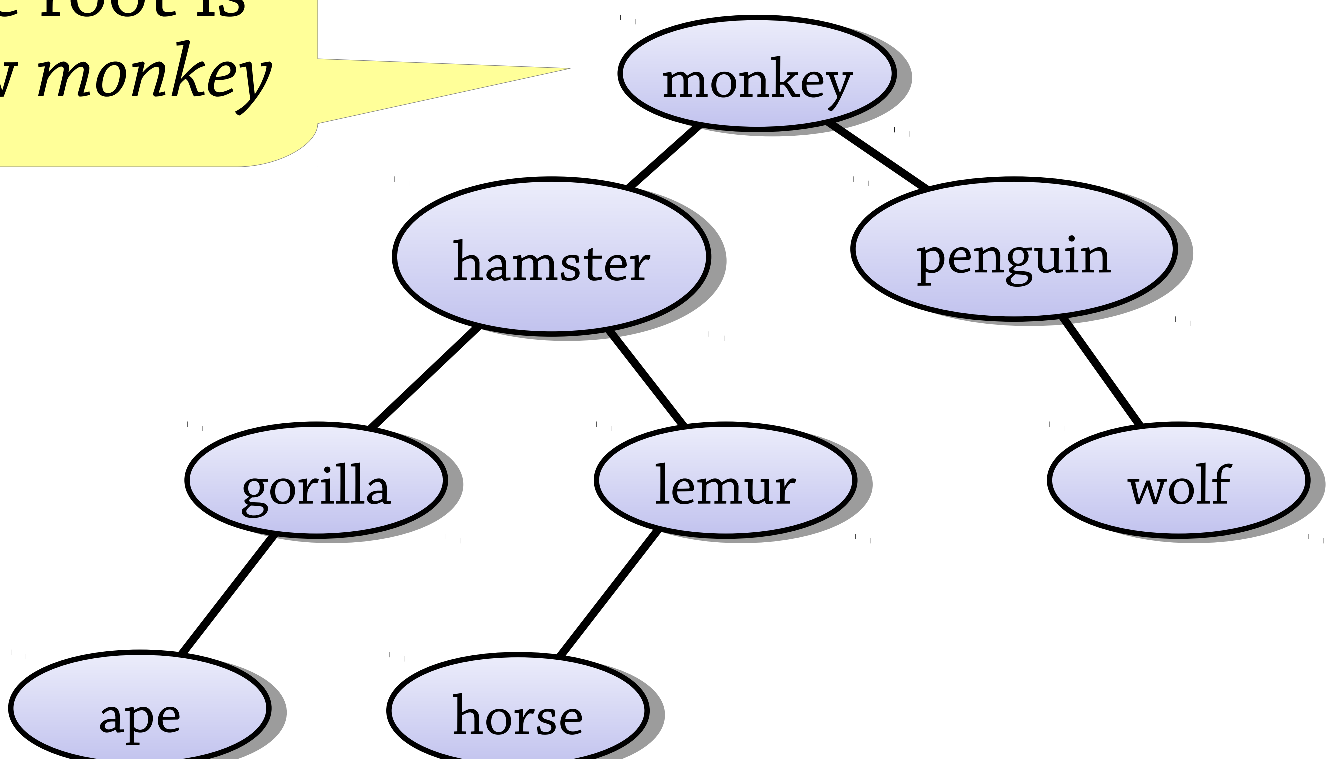
Delete the *biggest value from the node's left subtree* and put this value [why this one?] in place of the node we want to delete



Deleting a node with two children

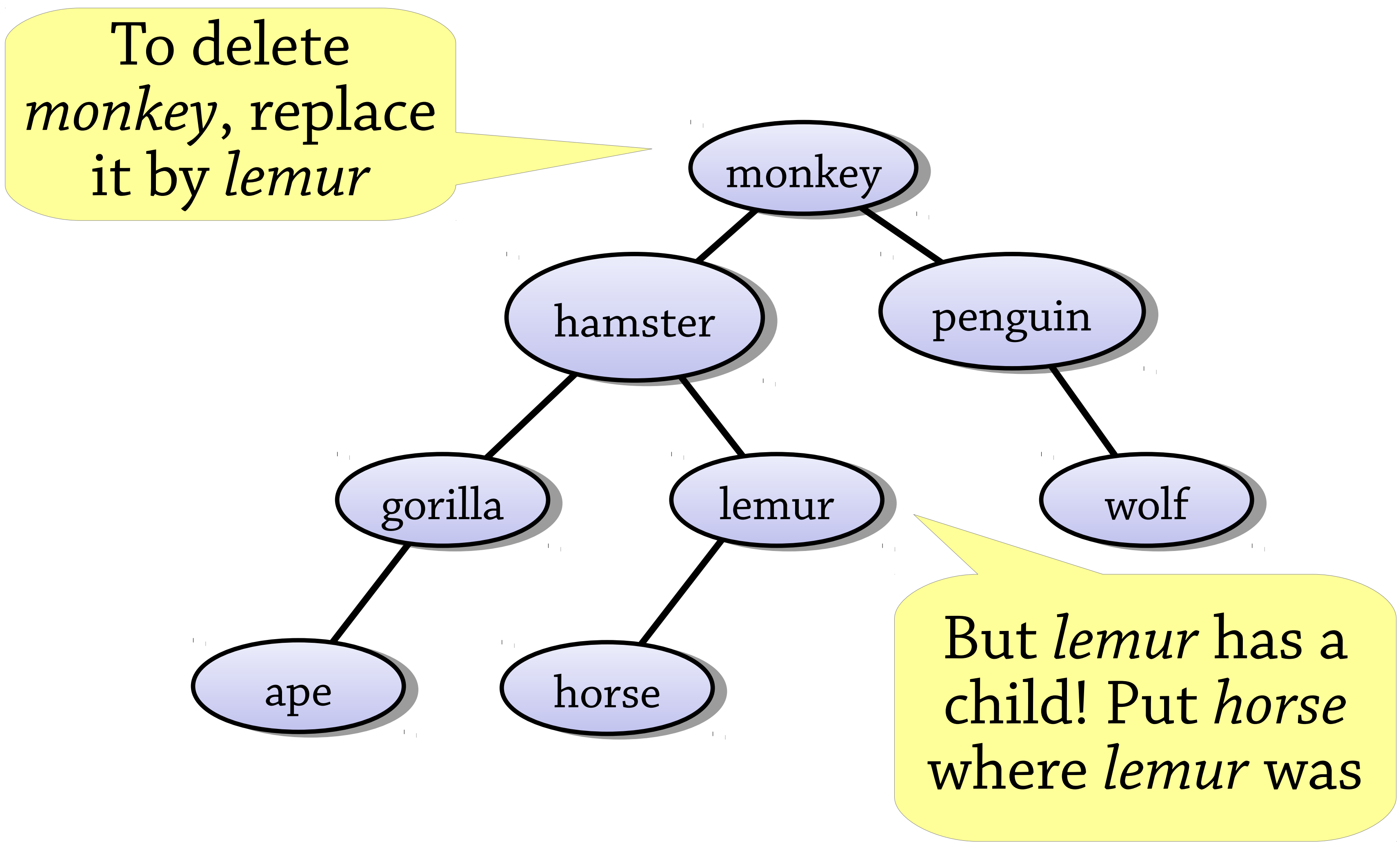
Delete the *biggest value from the node's left subtree* and put this value [why this one?] in place of the node we want to delete

The root is now *monkey*



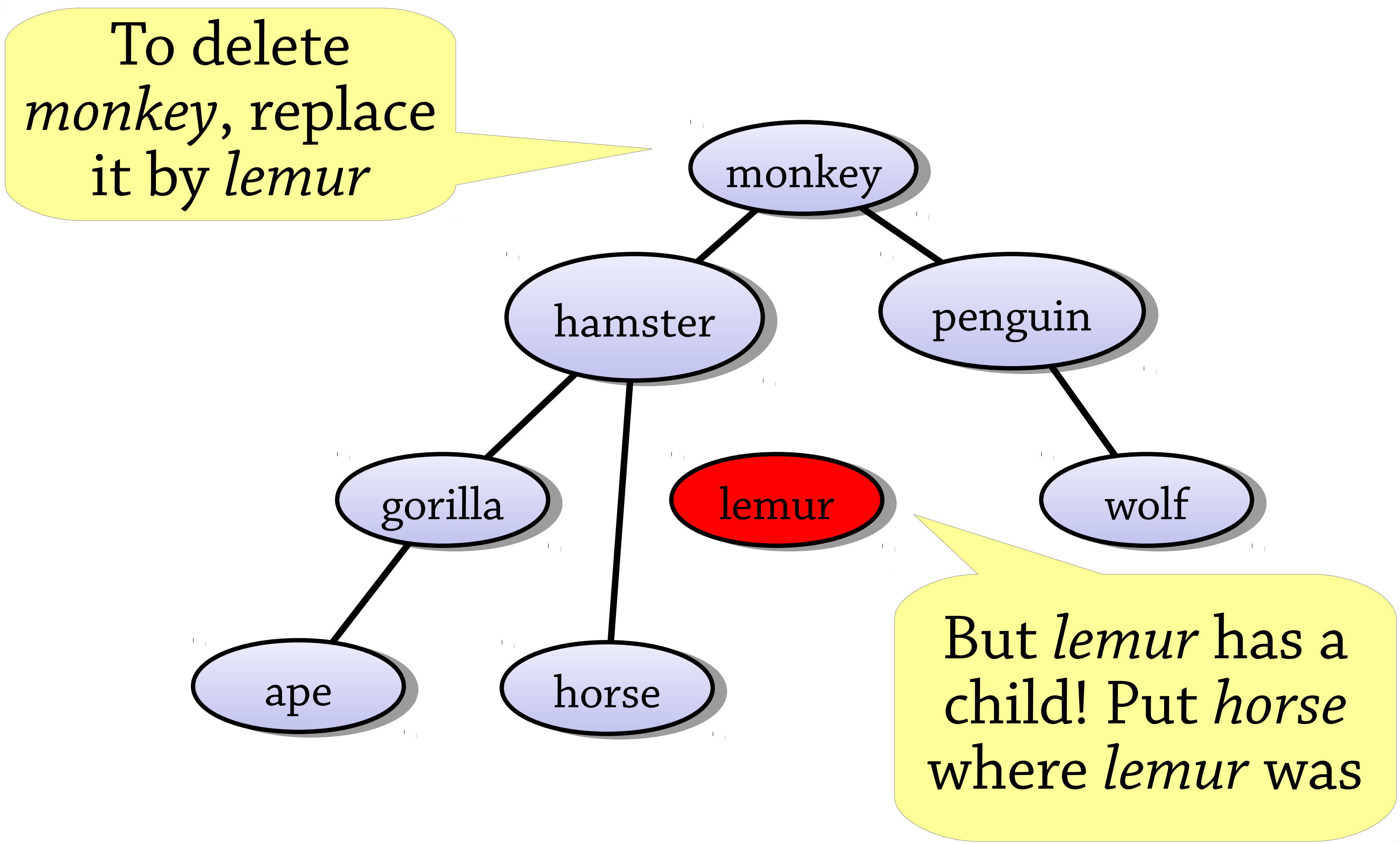
Deleting a node with two children

Here is the most complicated case:



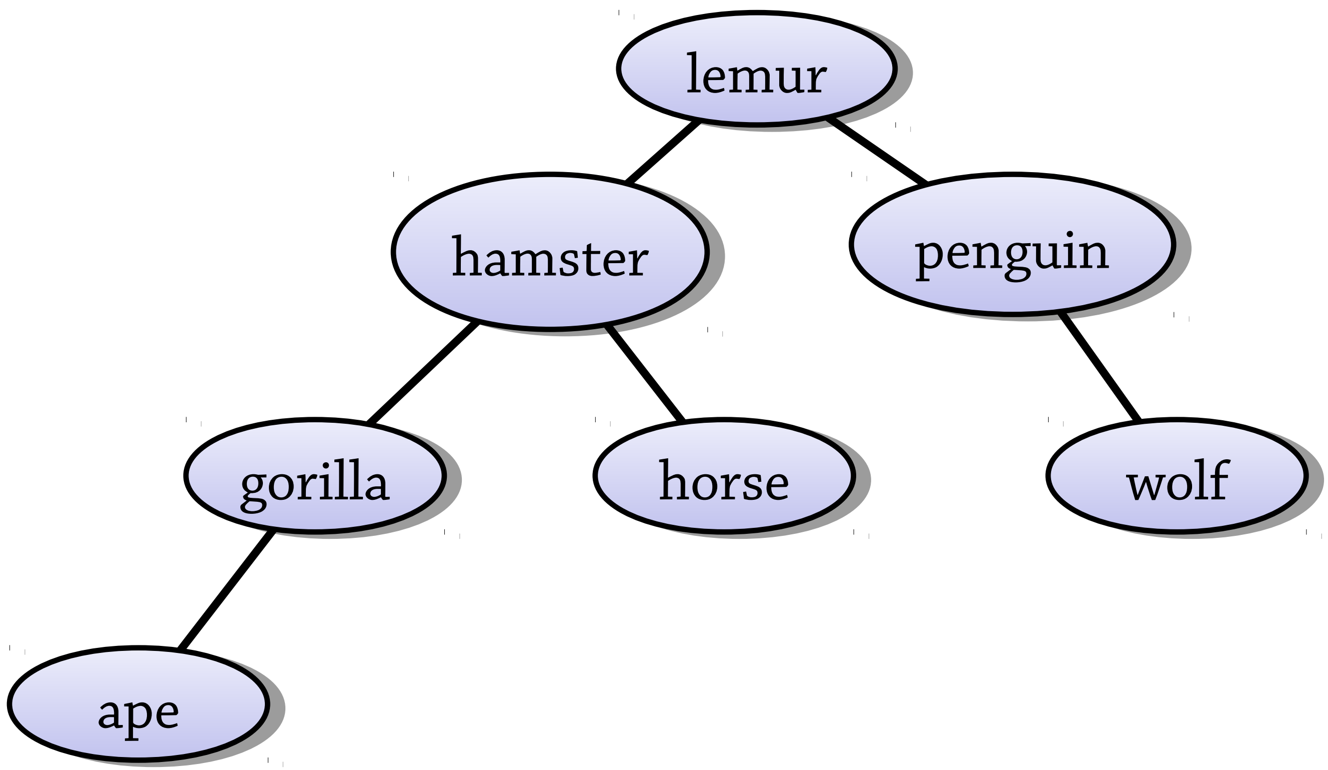
Deleting a node with two children

Here is the most complicated case:



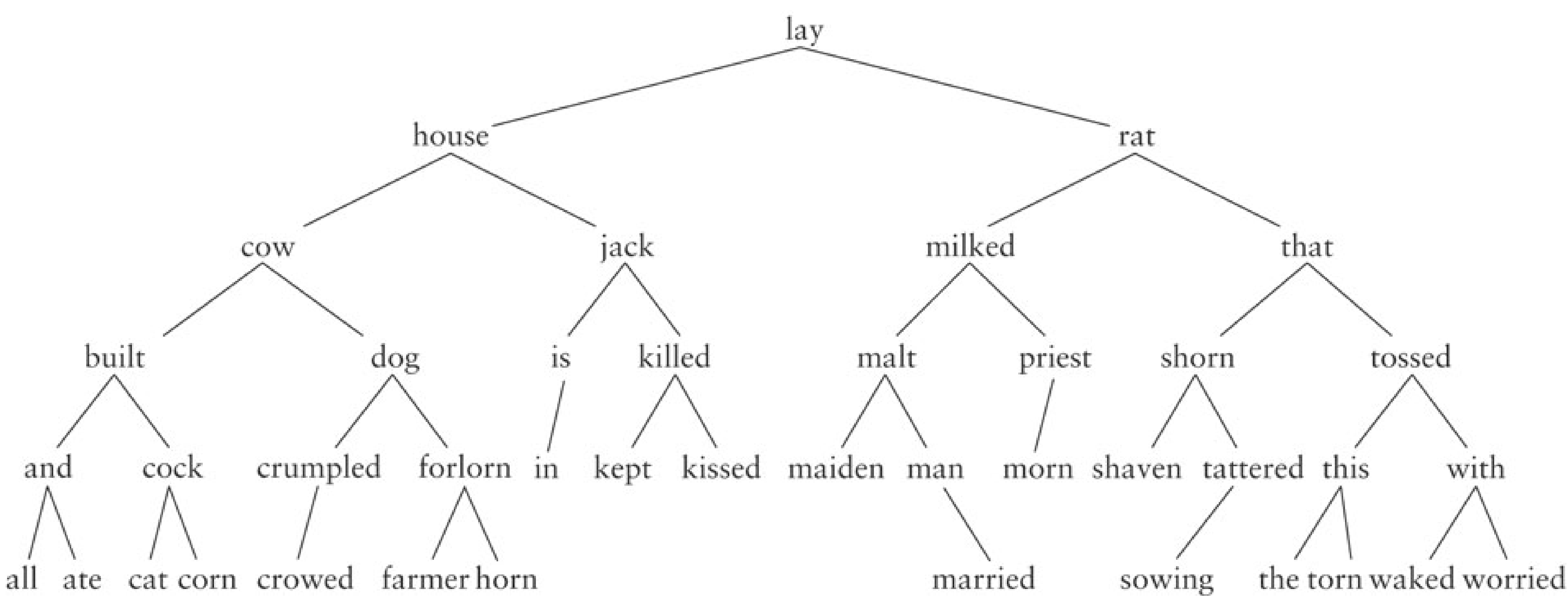
Deleting a node with two children

Here is the most complicated case:



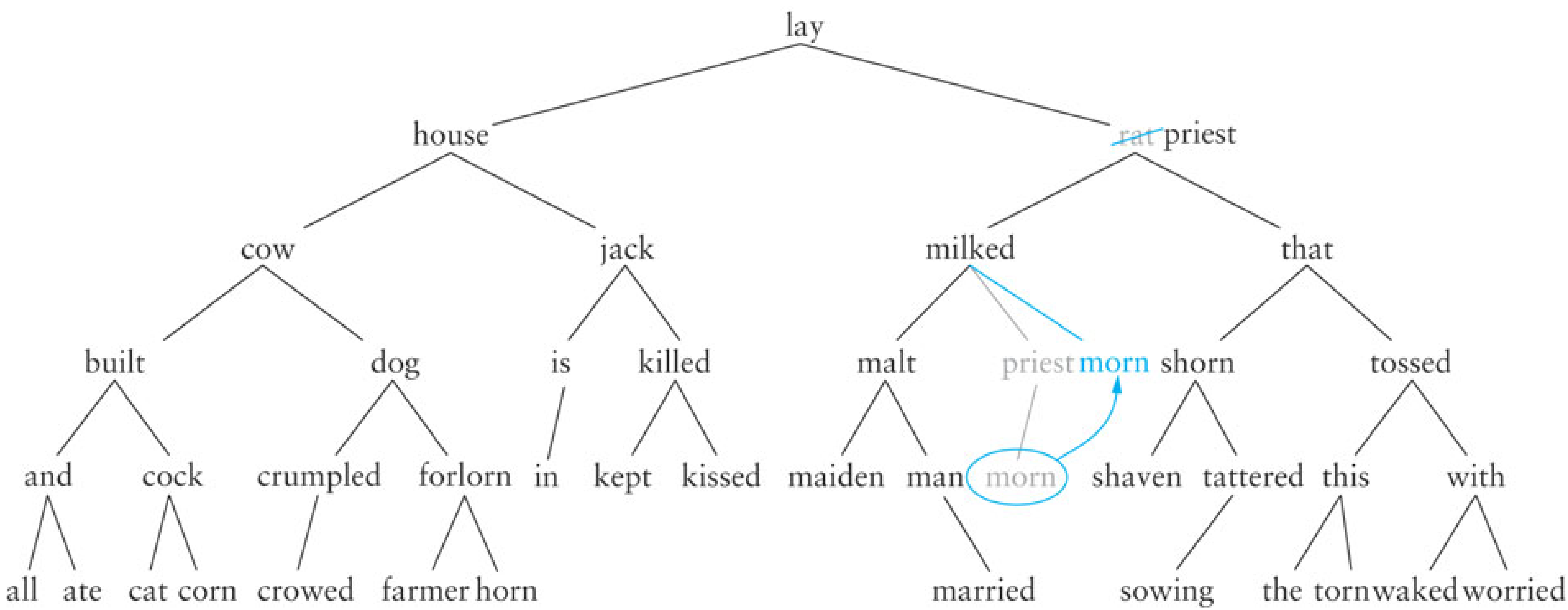
A bigger example

What happens if we delete farmer? is? cow? rat?



Deleting a node with two children

Deleting *rat*, we replace it with *priest*; now we have to delete *priest* which has a child, *morn*



Deleting a node with two children

Find and delete the *biggest value* in the *left subtree* and put that value in the deleted node

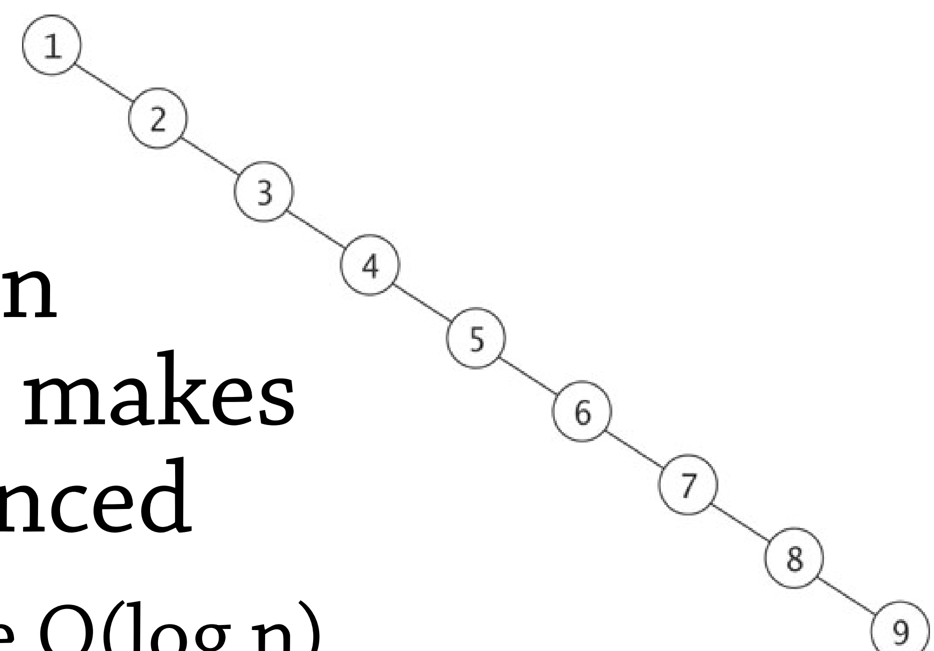
- Using the biggest value preserves the invariant (check you understand why)
- To find the biggest value: repeatedly descend into the right child until you find a node with no right child
- The biggest node can't have two children, so deleting it is easier

Complexity of BST operations

All our operations are O(height of tree)

This means O(log n) if the tree is balanced, but O(n) if it's unbalanced (like the tree on the right)

- how might we get this tree?



Balanced BSTs add an extra invariant that makes sure the tree is balanced

- then all operations are O(log n)

Summary of BSTs

Binary trees with *BST invariant*

Can be used to implement sets and maps

- lookup: can easily find a value in the tree
- insert: perform a lookup, then put the new value at the place where the lookup would terminate
- delete: find the value, then remove its node from the tree – several cases depending on how many children the node has

Complexity:

- all operations O(height of tree)
- that is, O(log n) if tree is balanced, O(n) if unbalanced
- inserting random data tends to give balanced trees, sequential data gives unbalanced ones

Tree traversal

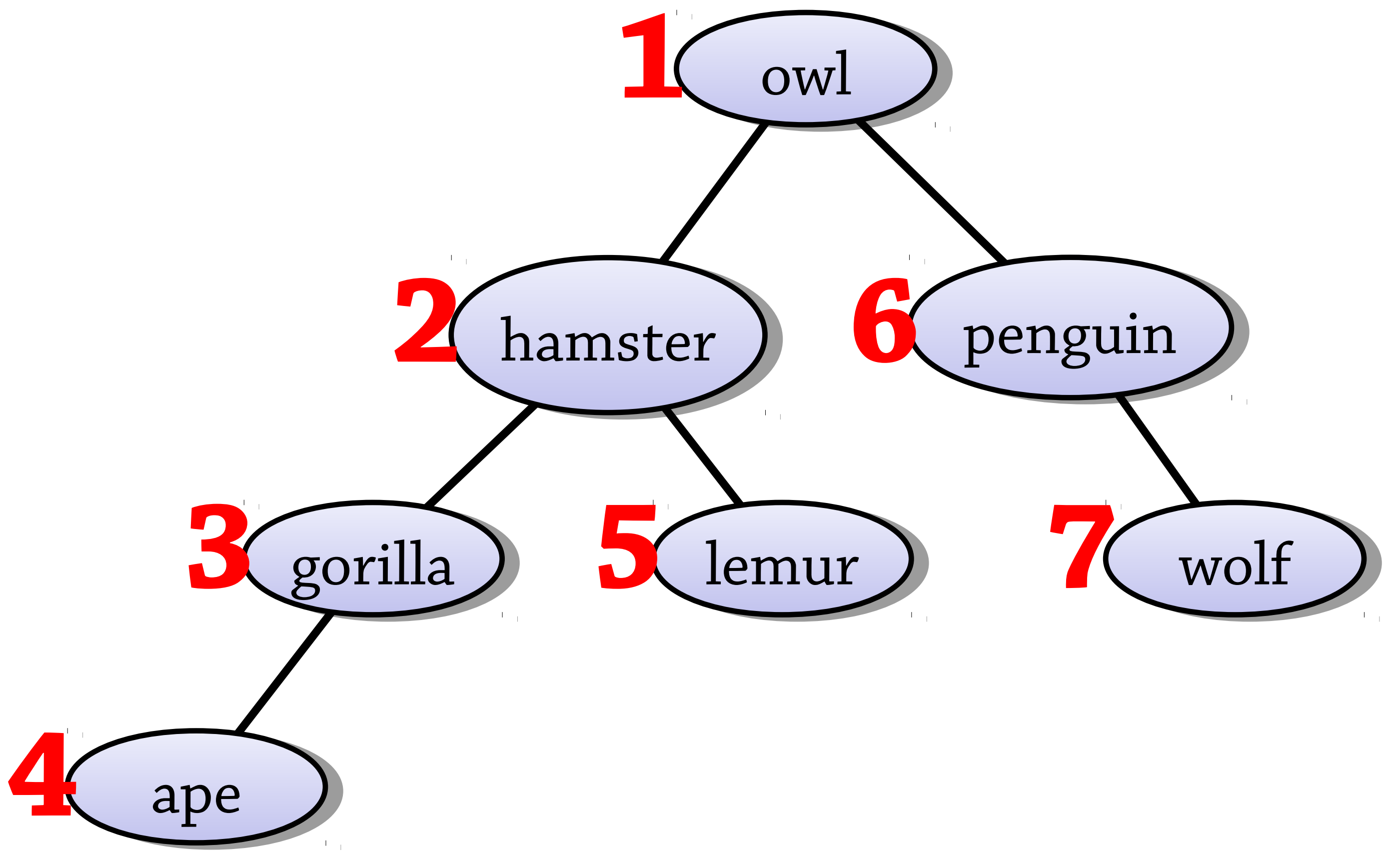
Traversing a tree means visiting all its nodes in some order

A *traversal* is a particular order that we visit the nodes in

Three common traversals: preorder, inorder, postorder

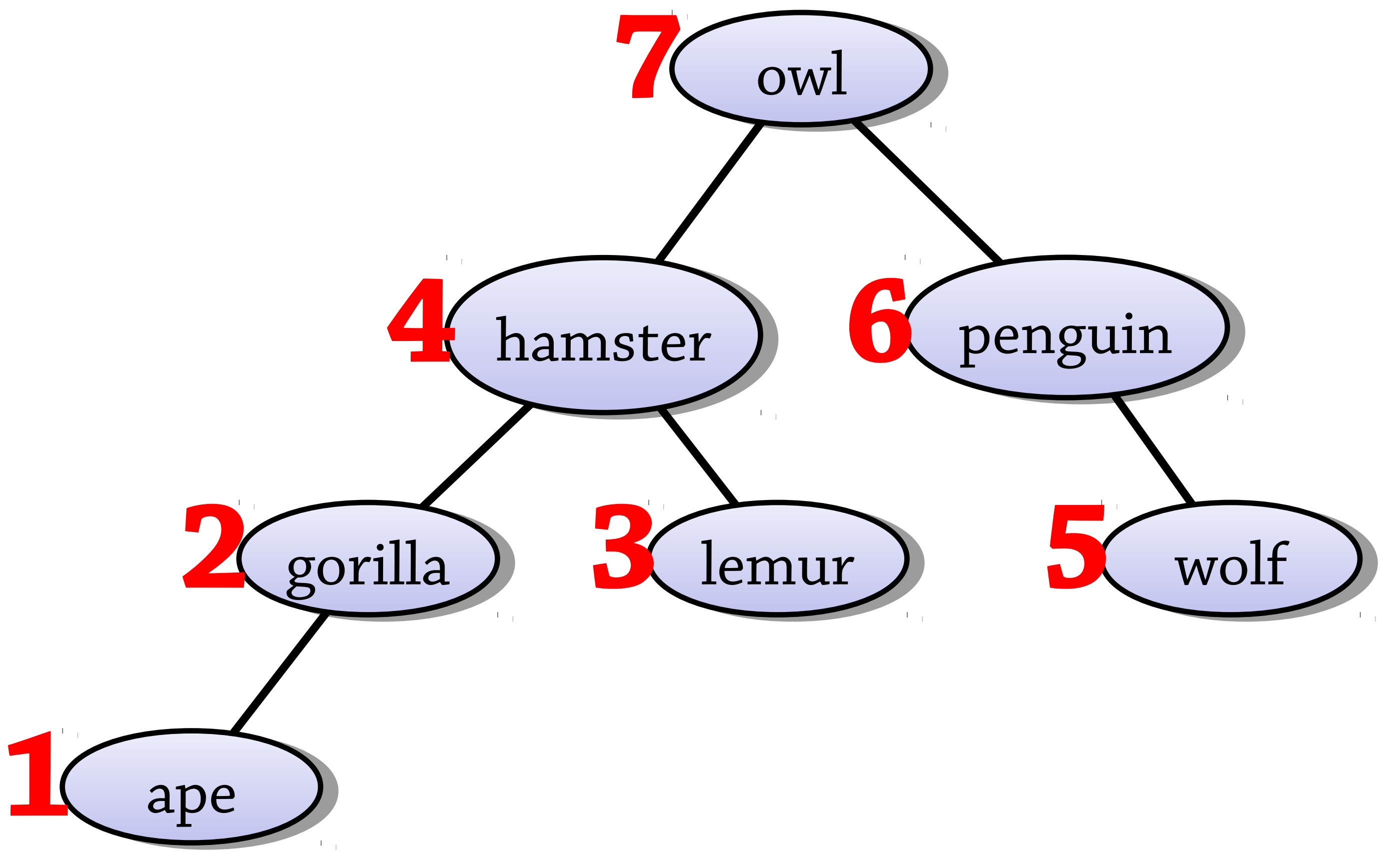
Preorder traversal

Visit root node, then left subtree, then right (root node first)



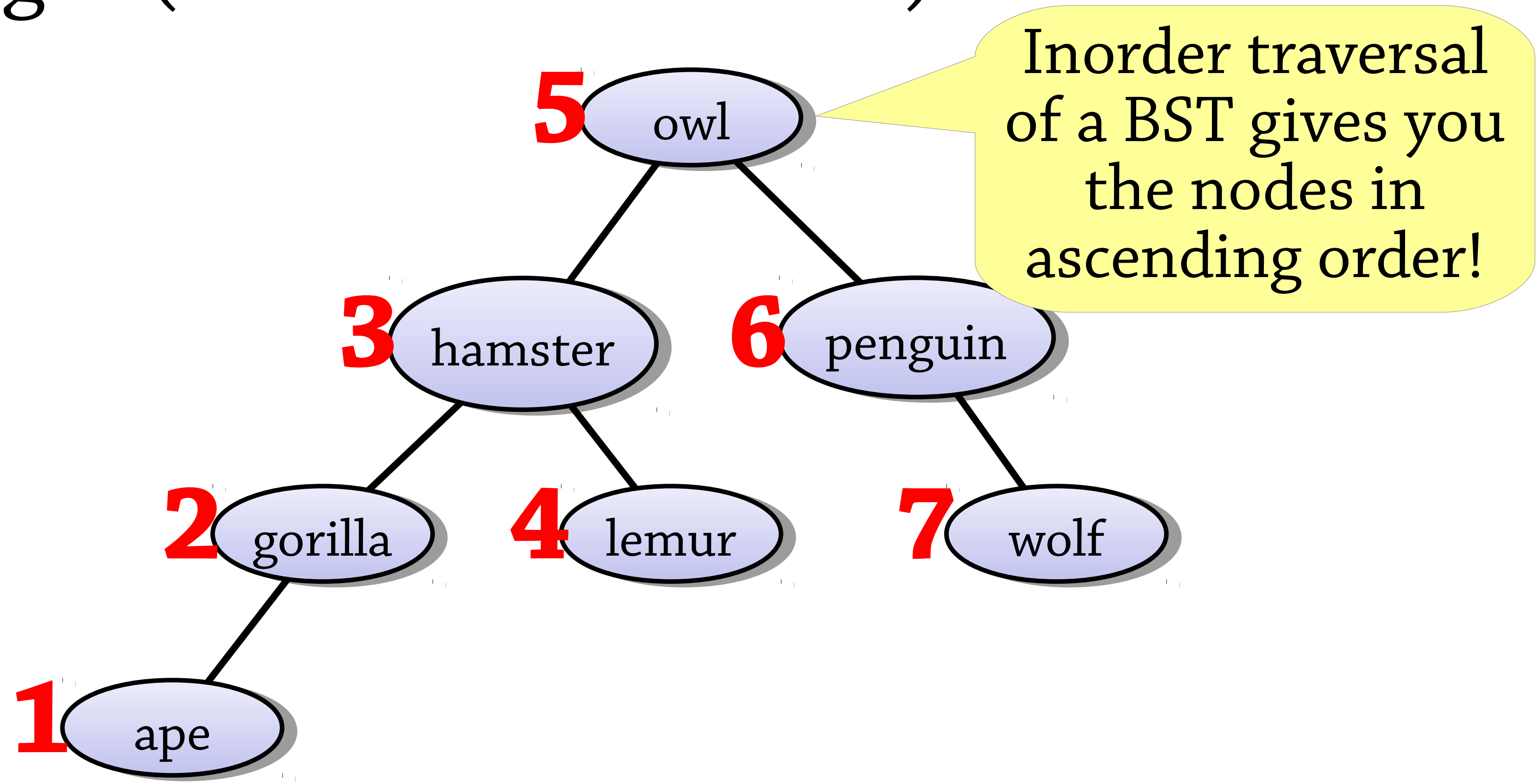
Postorder traversal

Visit left subtree, then right, then root node (root node last)



Inorder traversal

Visit left subtree, then root node, then right (root node in middle)



In-order traversal – printing

```
void inorder(Node<E> node) {
  if (node == null) return;
  inorder(node.left);
  System.out.println(node.value);
  inorder(node.right);
}
```

But nicer to define an iterator!

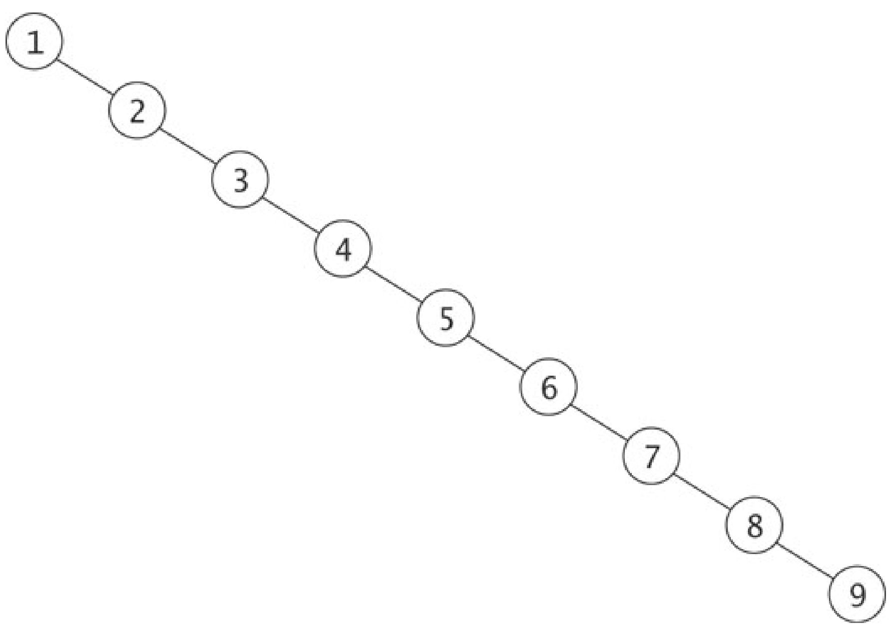
```
Iterator<Node<E>> inorder(Node<E>
node);
```

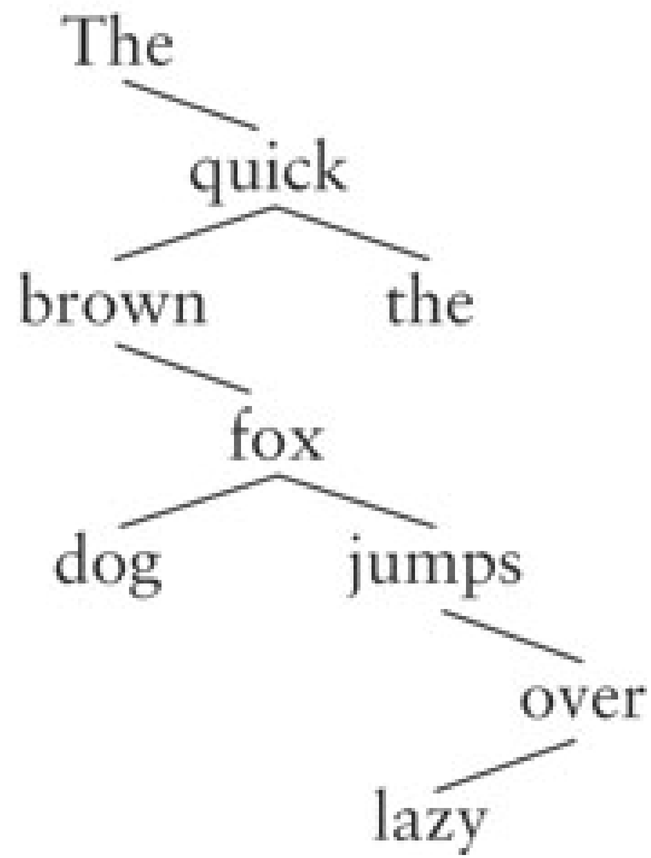
See 17.4

AVL trees
(chapter 18.4)

Balanced BSTs: the problem

The BST operations take O(height of tree), so for unbalanced trees can take O(n) time





Balanced BSTs: the solution

Take BSTs and add an extra invariant that makes sure that the tree is balanced

- Height of tree must be O(log n)
- Then all operations will take O(log n) time

One possible idea for an invariant:

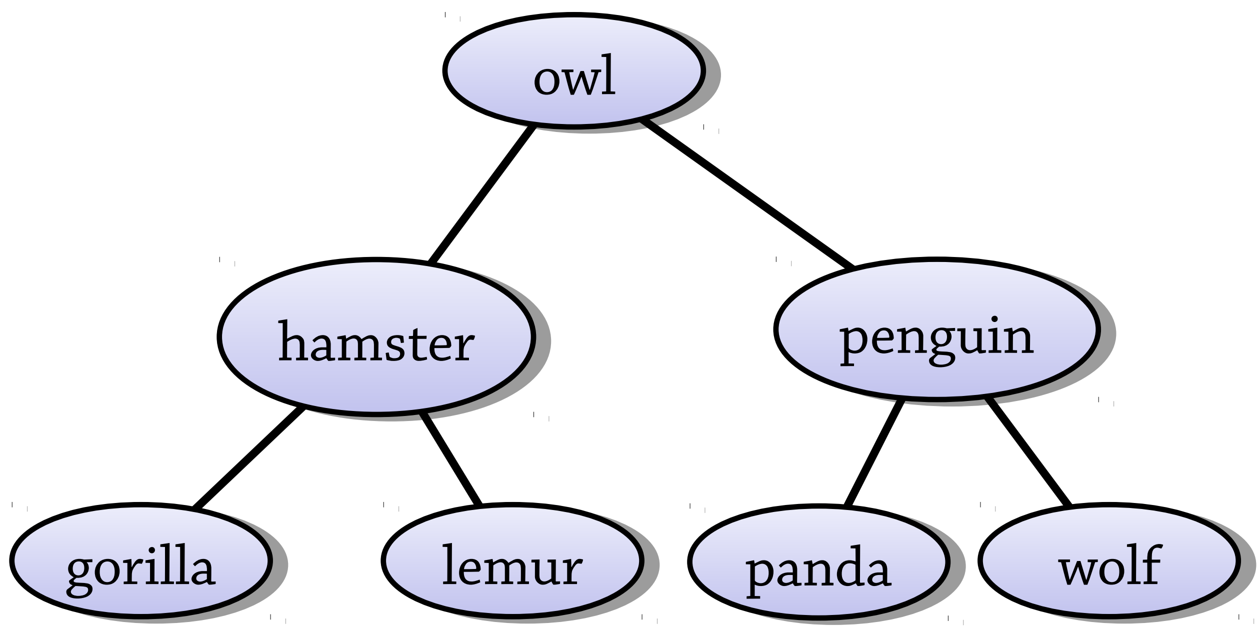
- Height of left child = height of right child (for all nodes in the tree)
- Tree would be sort of "perfectly balanced"

What's wrong with this idea?

A too restrictive invariant

Perfect balance is too restrictive!

Number of nodes can only be 1, 3, 7, 15, 31, …



AVL trees – a less restrictive invariant

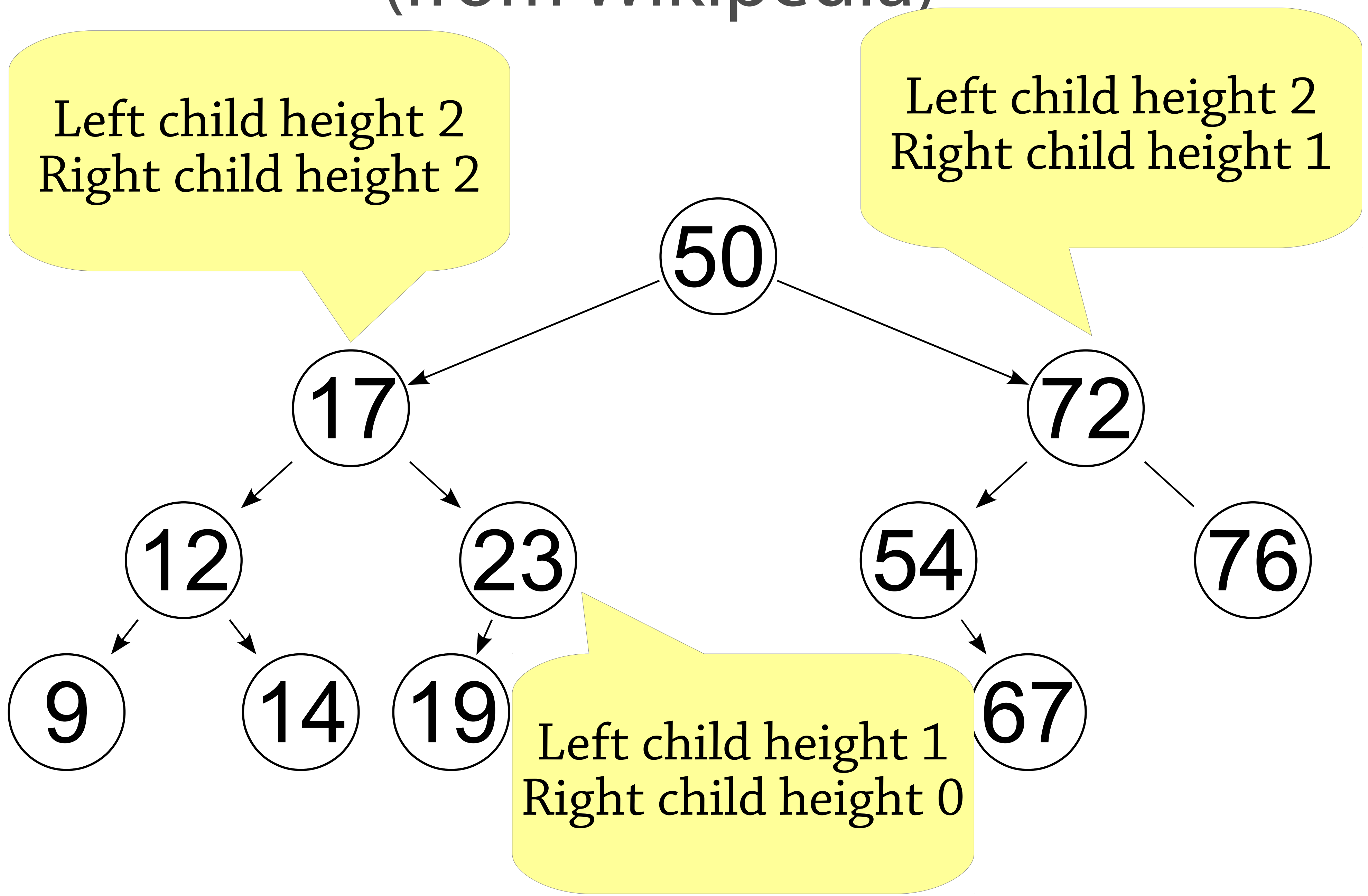
The AVL tree is the first balanced BST discovered (from 1962) – it's named after Adelson-Velsky and Landis

It's a BST with the following invariant:

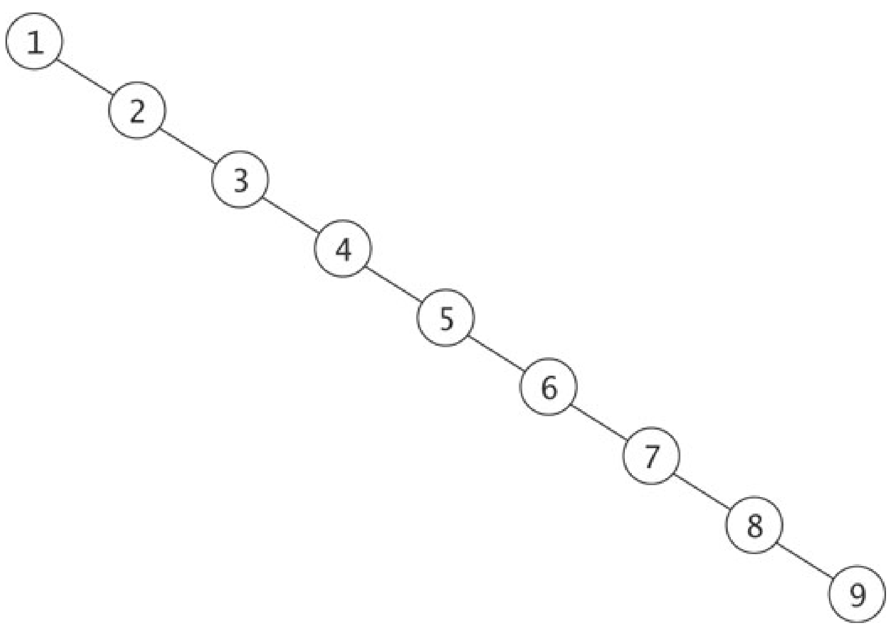
- The *difference in heights* between the left and right children of any node is at most 1

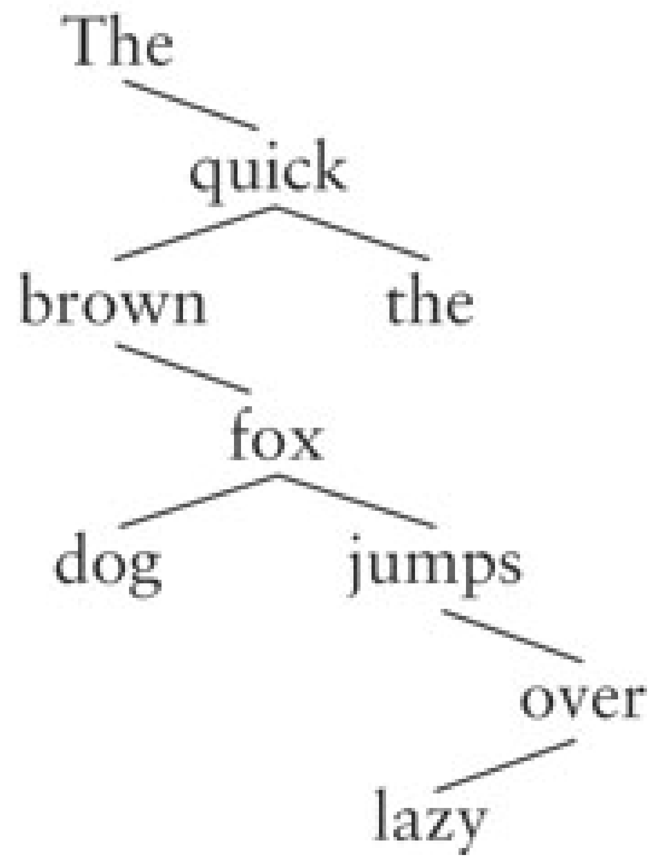
This makes the tree's height O(log n), so it's balanced

Example of an AVL tree (from Wikipedia)

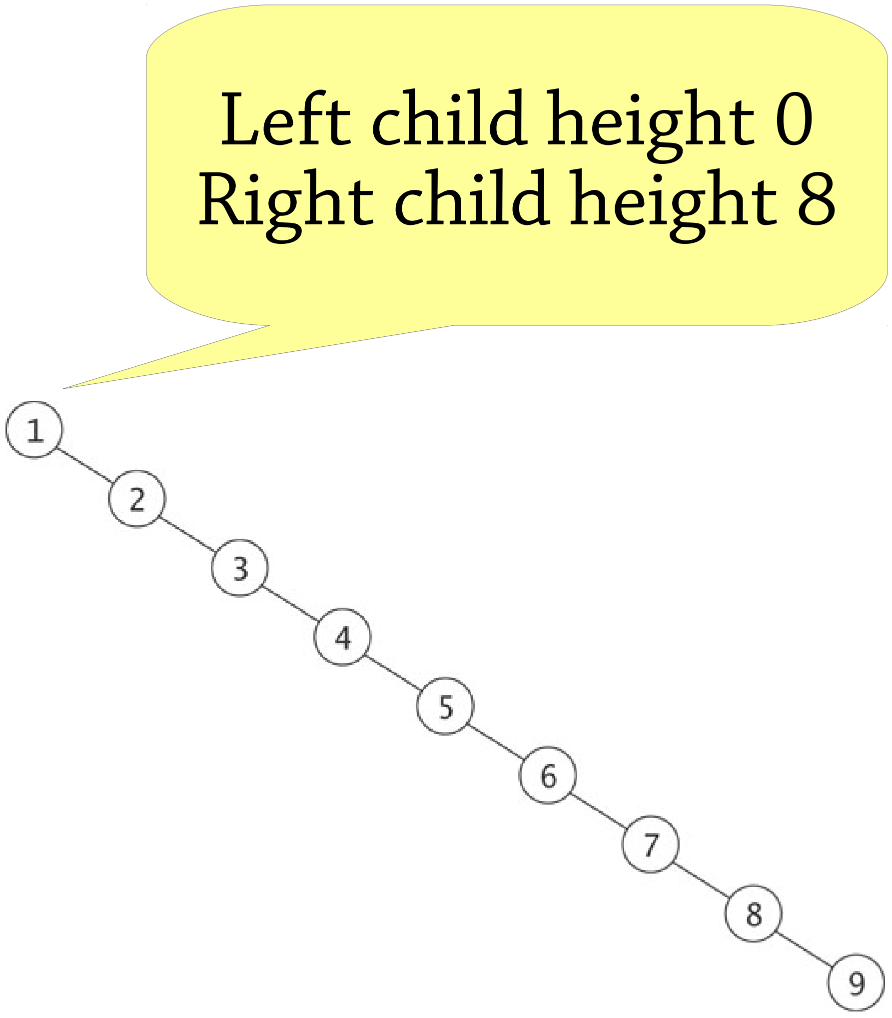


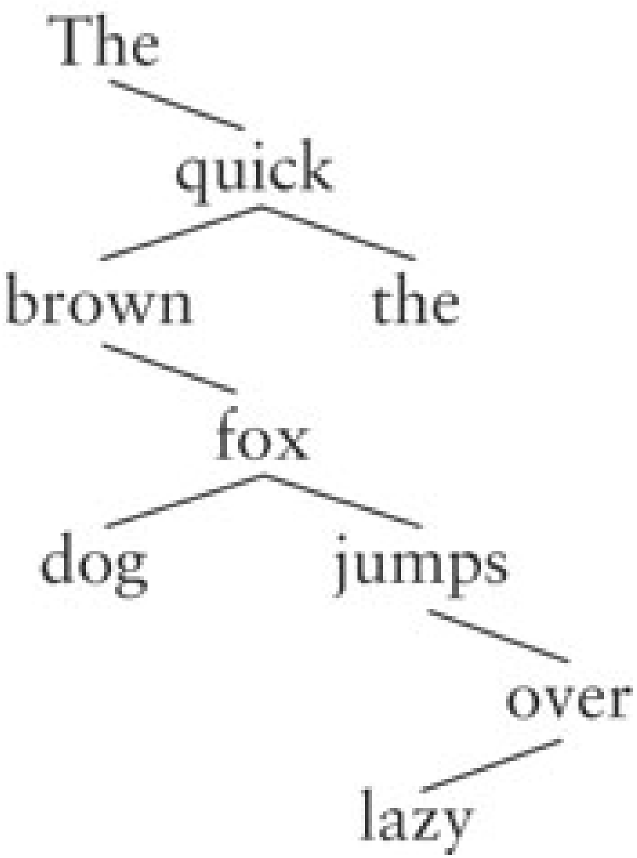
Why are these not AVL trees?



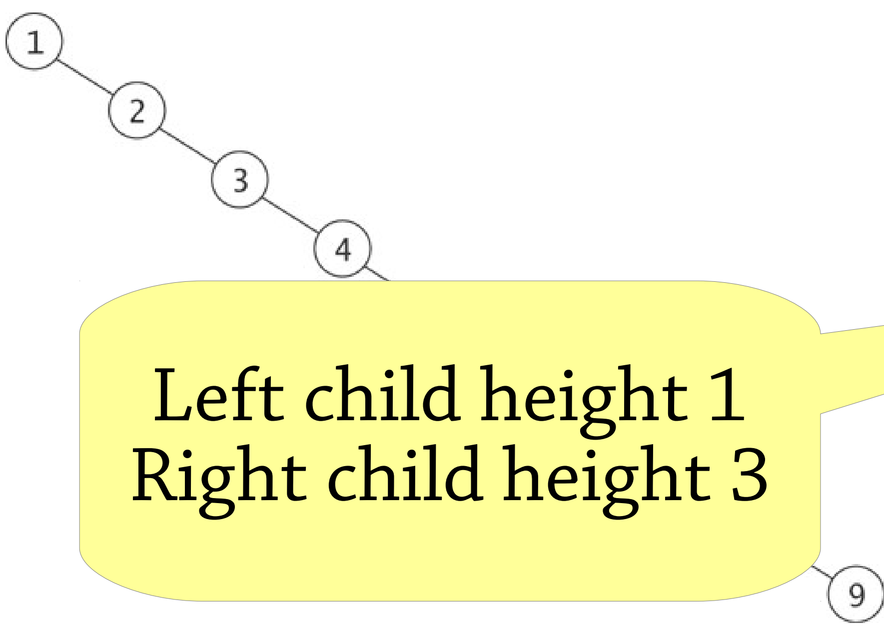


Why are these not AVL trees?

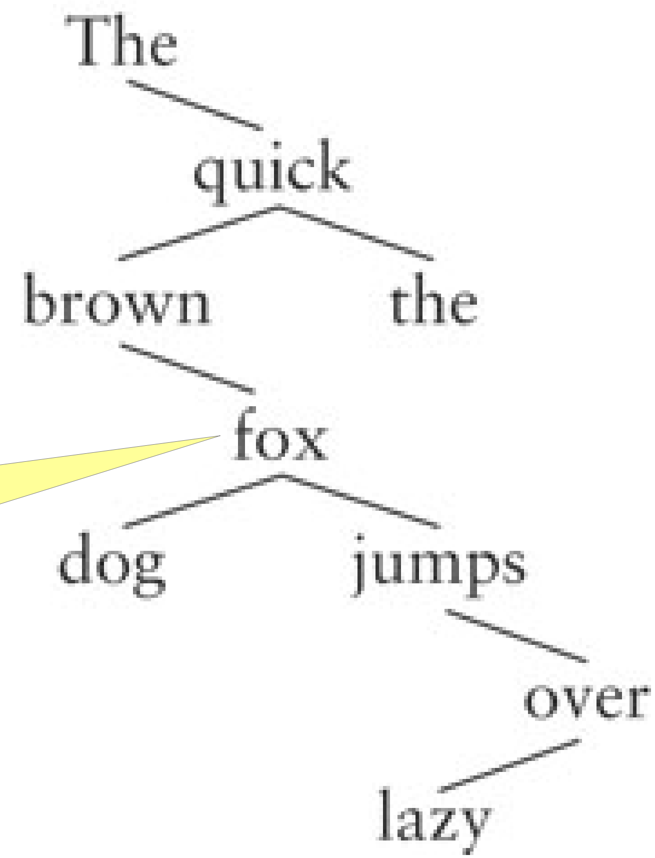




Why are these not AVL trees?

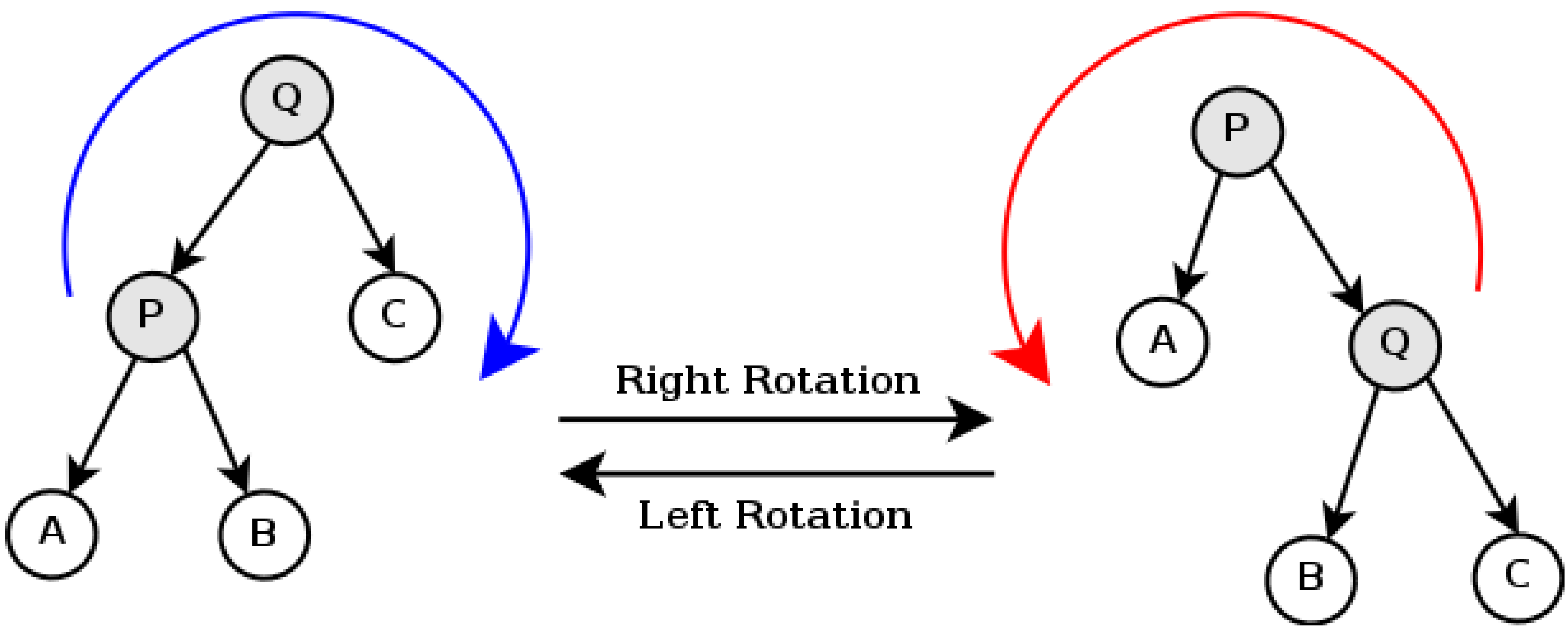


Left child height 1
Right child height 3



Rotation

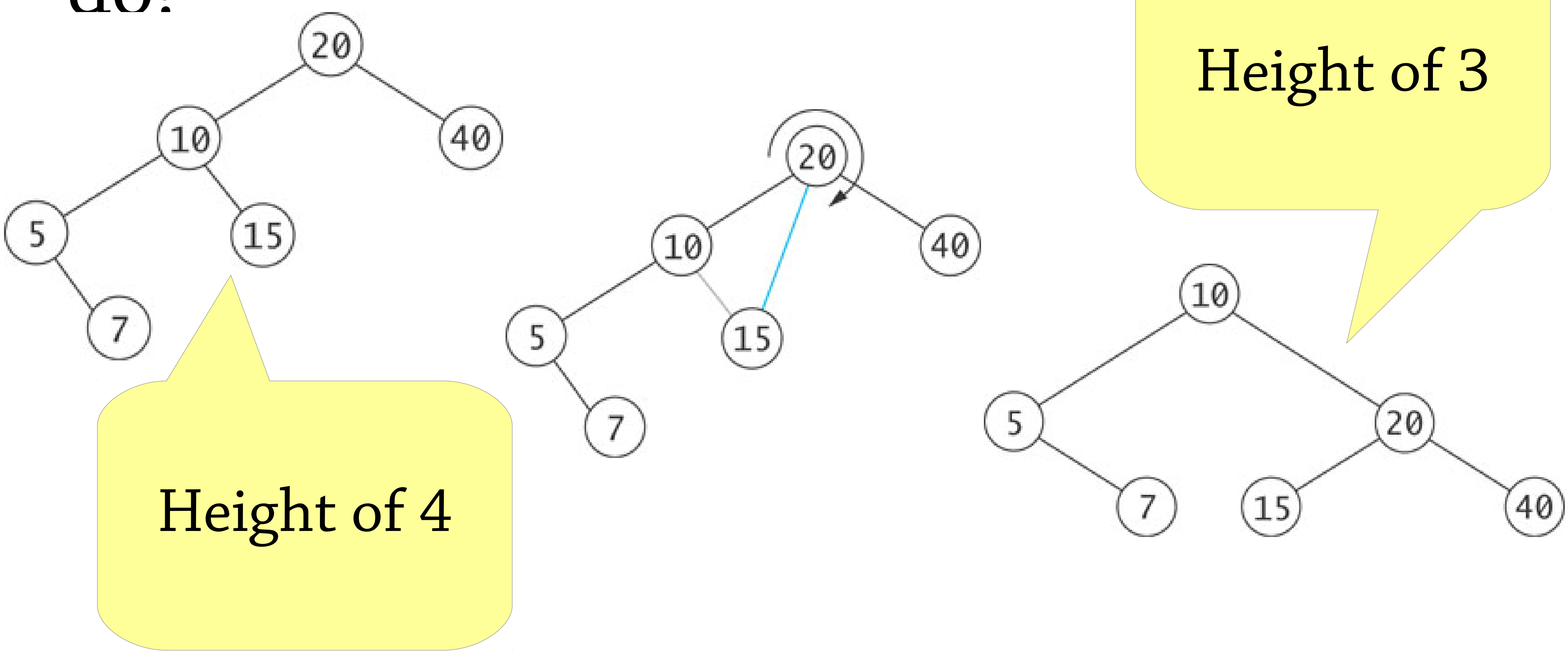
Rotation rearranges a BST by moving a different node to the root, without changing the BST's contents



(pic from Wikipedia)

Rotation

We can strategically use rotations to rebalance an unbalanced tree.
This is what most balanced BST variants do!



AVL insertion

Start by doing a BST insertion

- This might break the AVL (balance) invariant

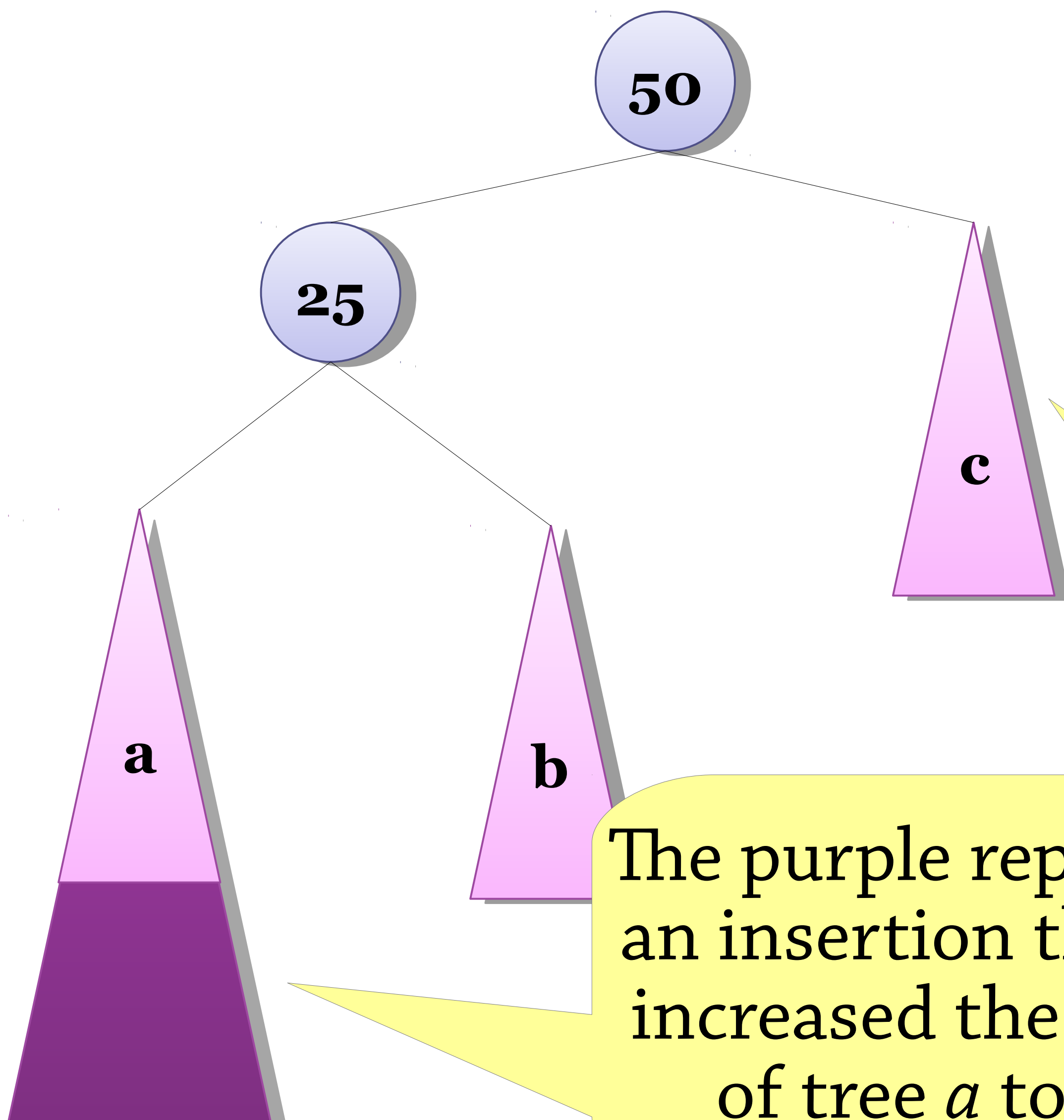
Then go upwards from the newly-inserted node, looking for nodes that break the invariant (unbalanced nodes)

Whenever you find one, rotate it

- Then continue upwards in the tree

There are several cases depending on *how* the node became unbalanced

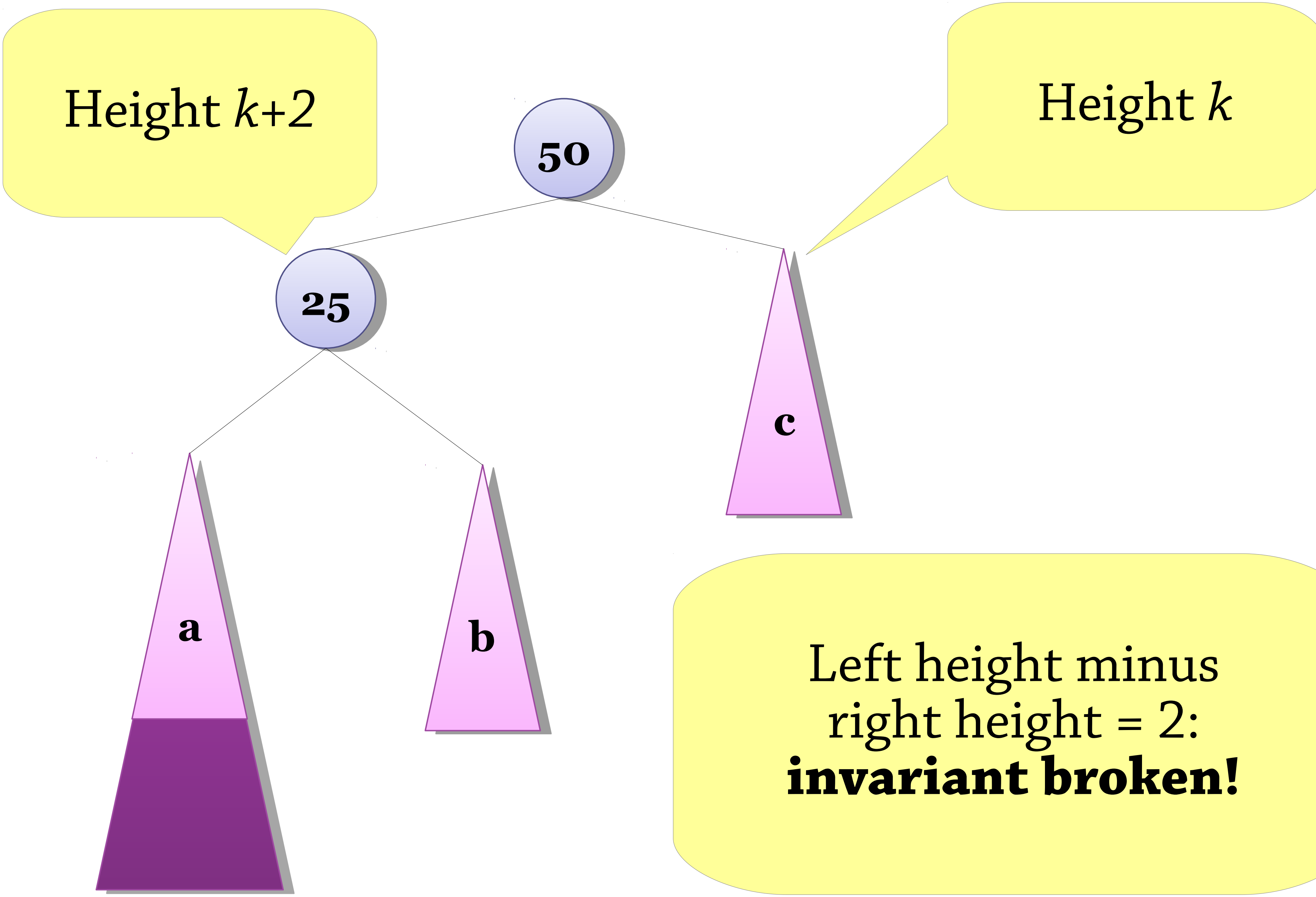
Case 1: a *left-left* tree



Each pink triangle represents an AVL tree with height k

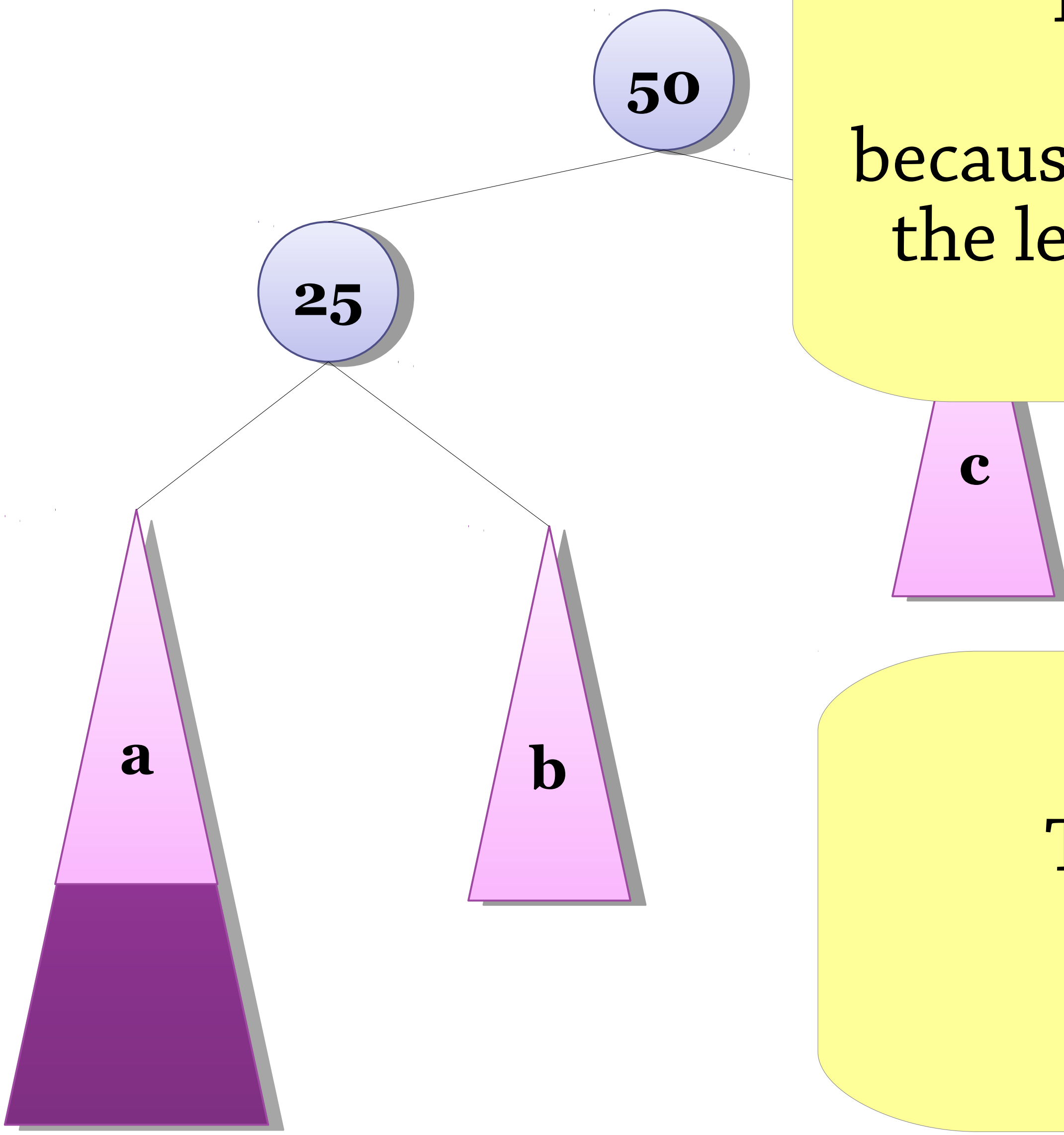
The purple represents an insertion that has increased the height of tree a to $k+1$

Case 1: a *left-left* tree



Left height minus right height = 2: **invariant broken!**

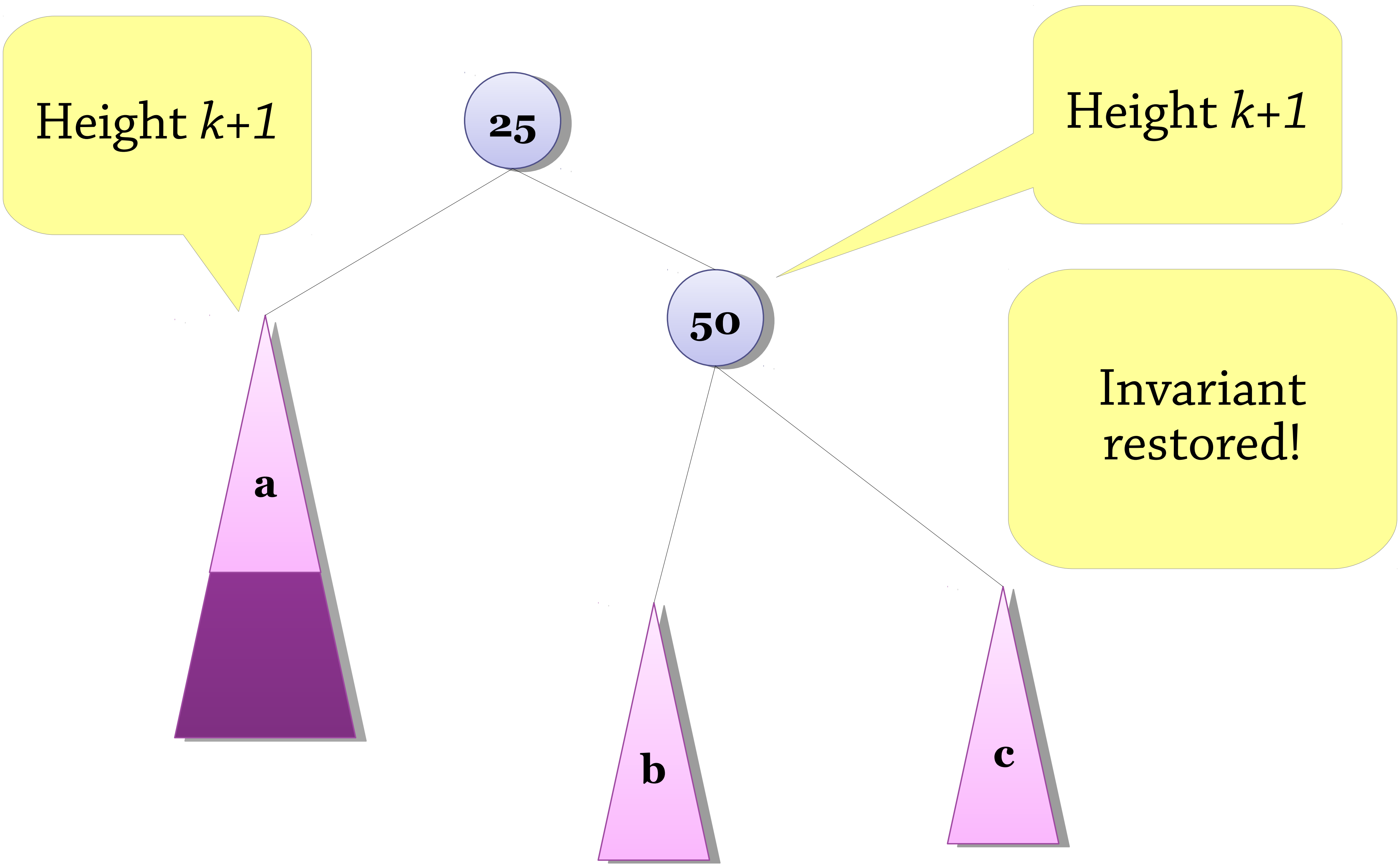
Case 1: a *left-left* tree



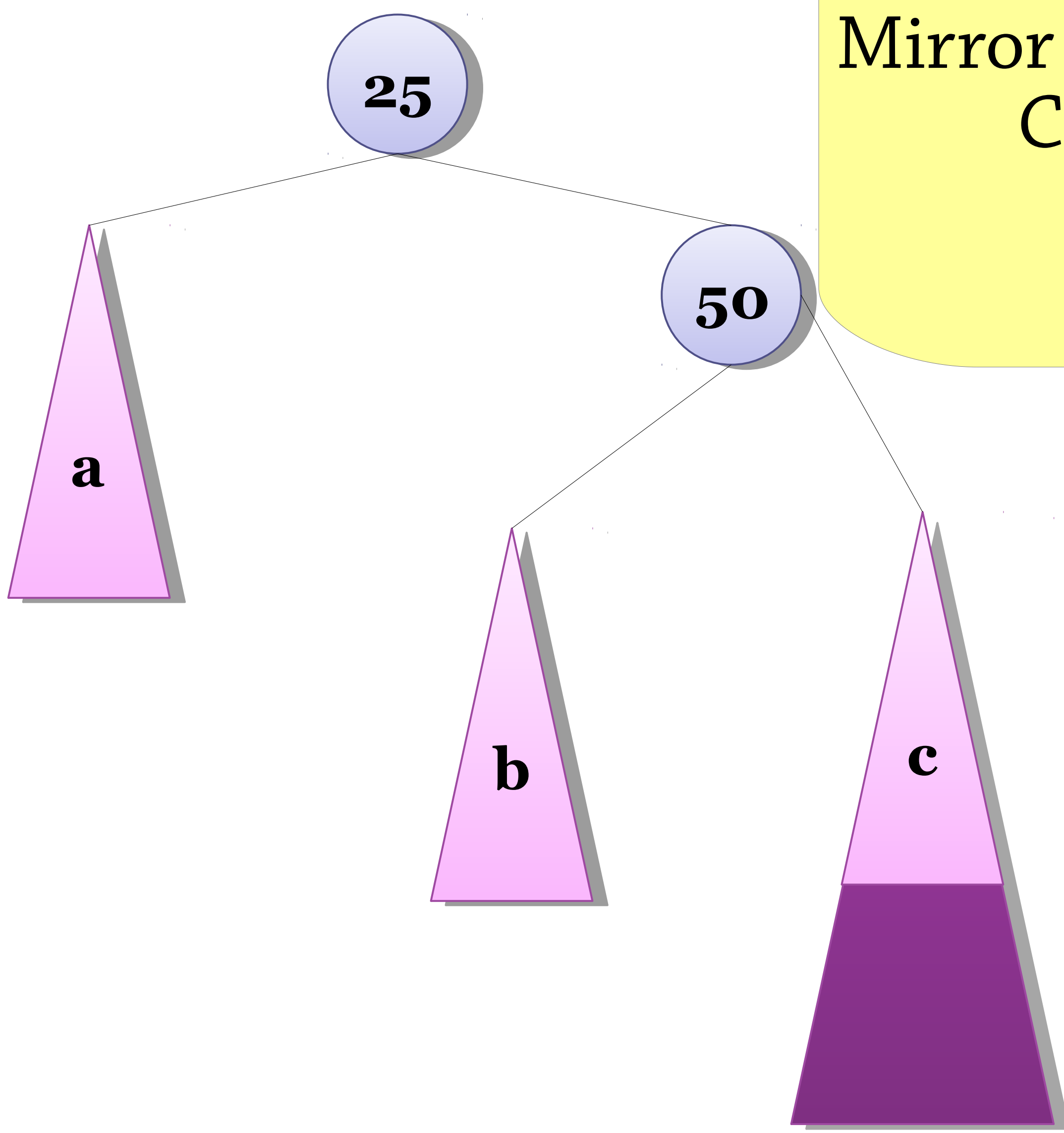
This is called a
left-left tree
because both the root and the left child are deeper on the left

To fix it we do a
right rotation

Balancing a left-left tree, afterwards

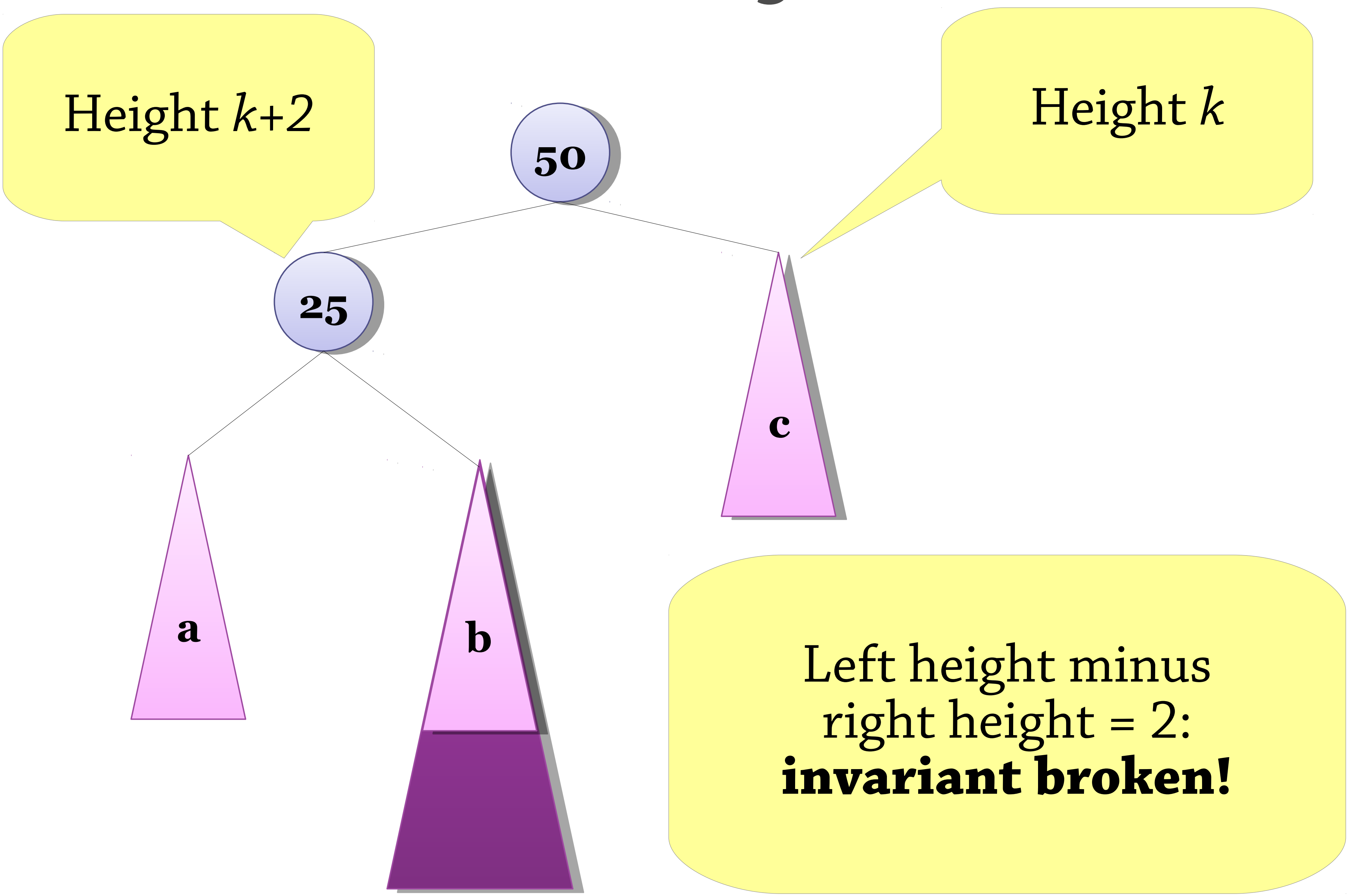


Case 2: a *right-right* tree

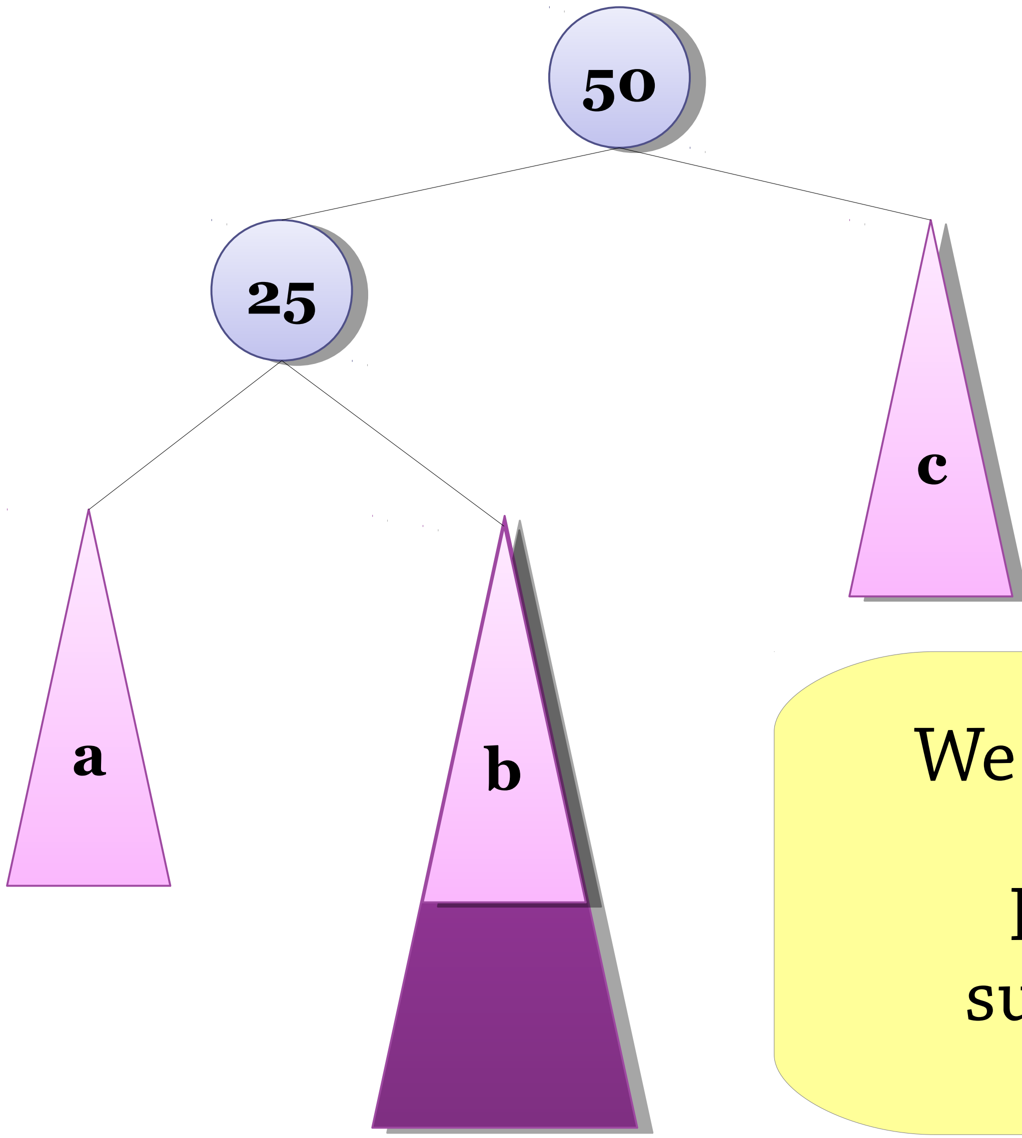


Mirror image of left-left tree
Can be fixed with
left rotation

Case 3: a *left-right* tree

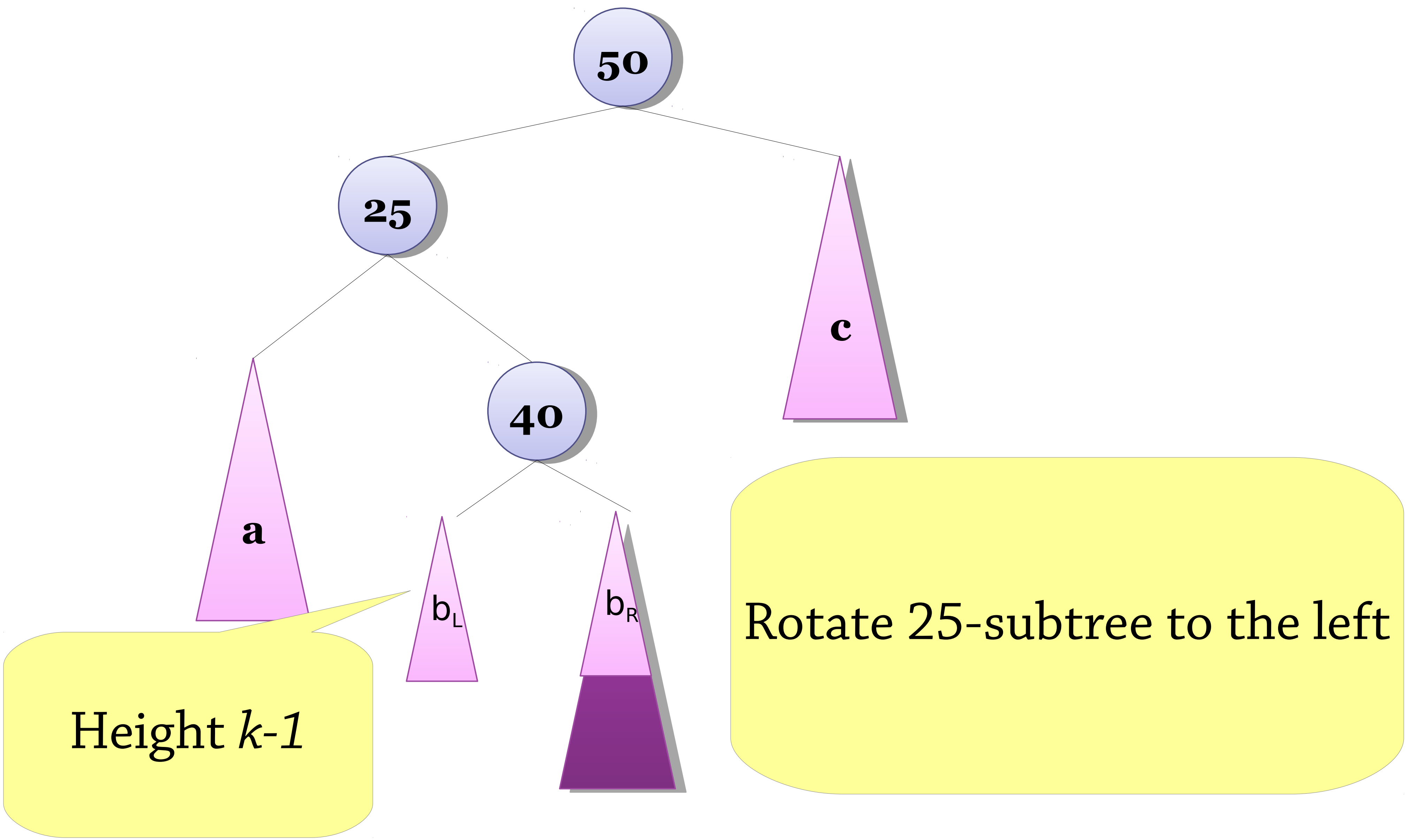


Case 3: a *left-right* tree

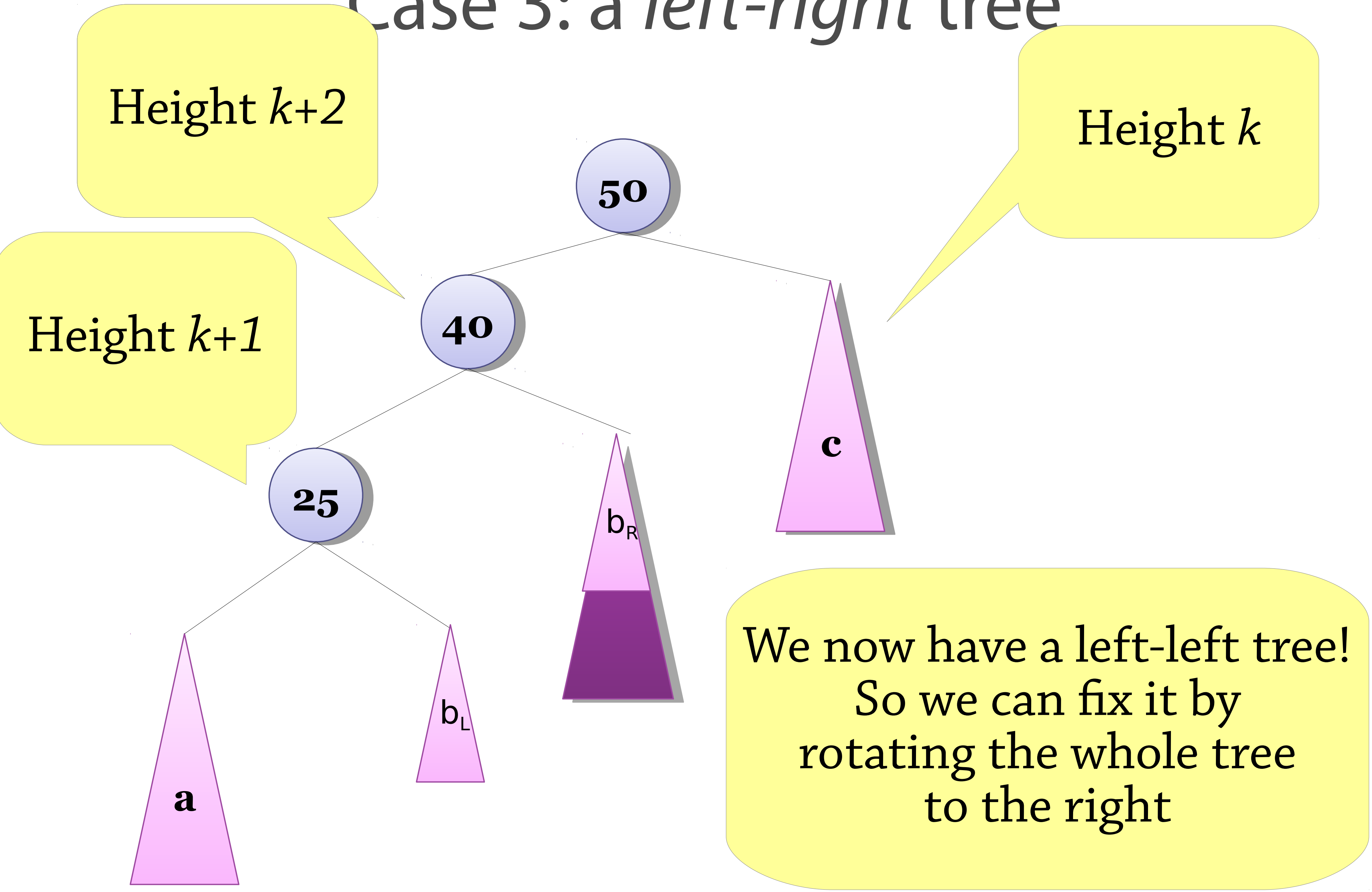


We can't fix this with
one rotation
Let's look at b's
subtrees b_L and b_R

Case 3: a *left-right* tree

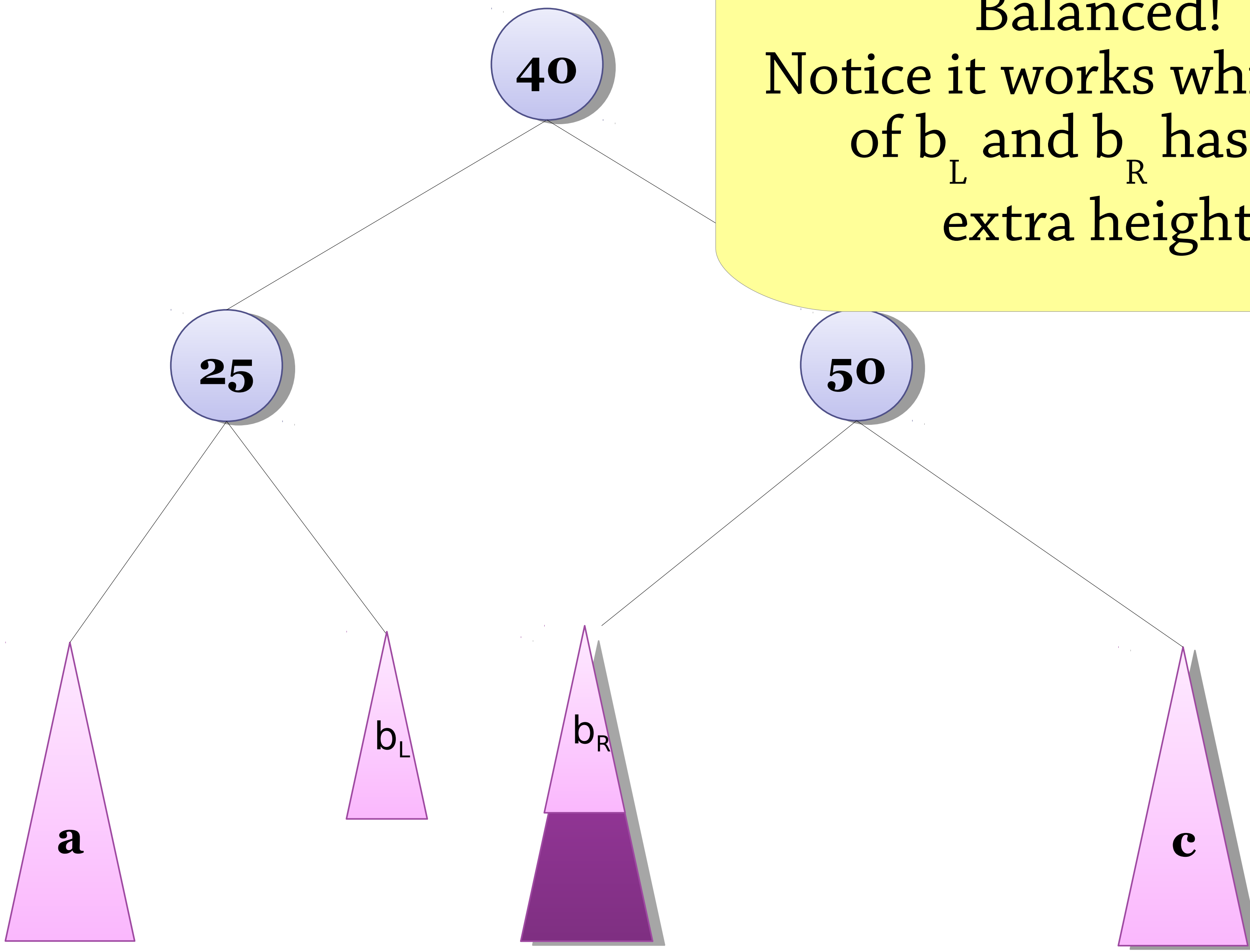


Case 3: a *left-right* tree

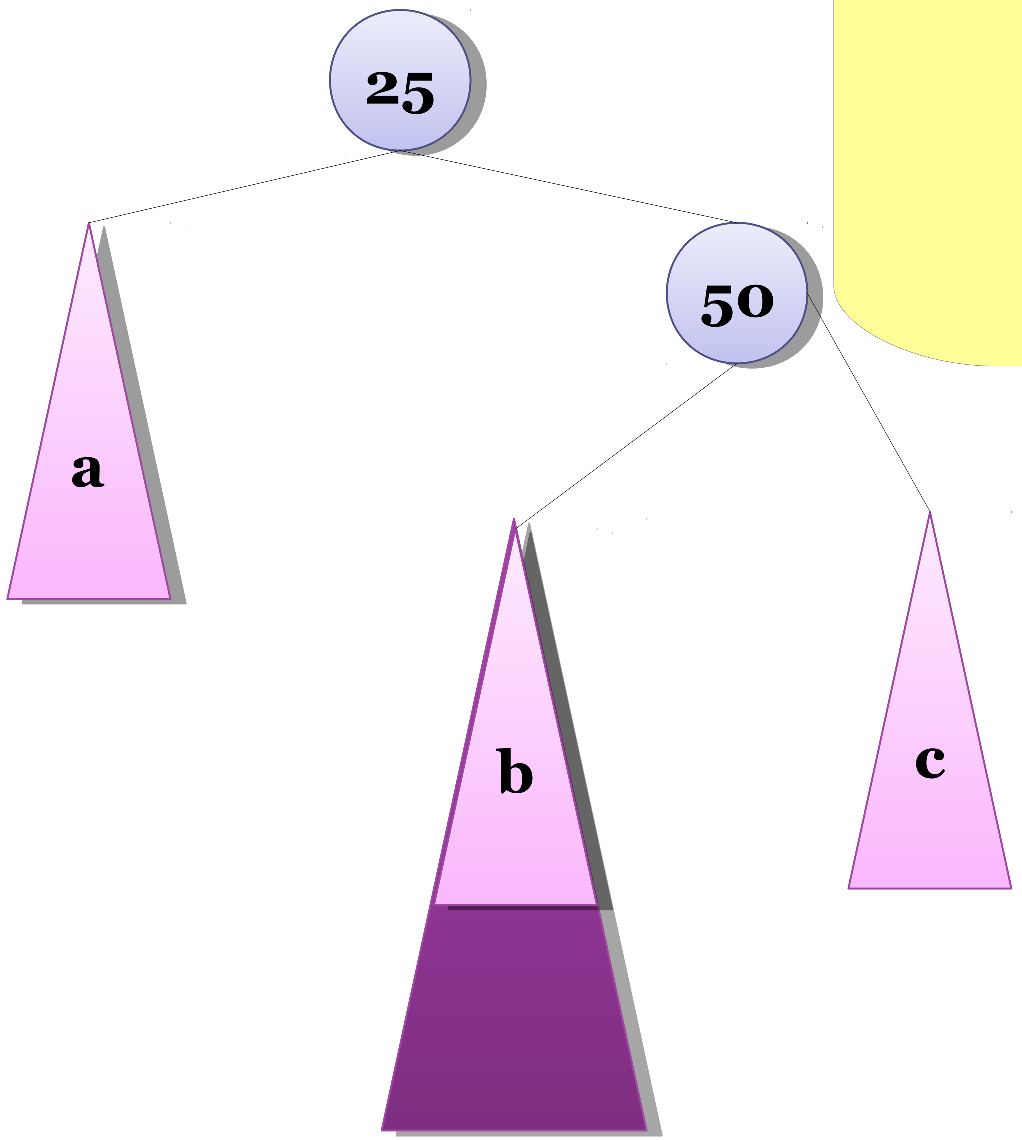


Case 3: a *left-right* tree

Balanced!
Notice it works whichever of b_L and b_R has the extra height



Case 4: a *right-left* tree



Mirror image of
left-right tree

Four sorts of unbalanced trees

Left-left (height of left-left grandchild = k+1, height of left child = k+2, height of right child = k)

- Rotate the whole tree to the right

Left-right (height of left-right grandchild = k+1, height of left child = k+2, height of right child = k)

- First rotate the left child to the left
- Then rotate the whole tree to the right

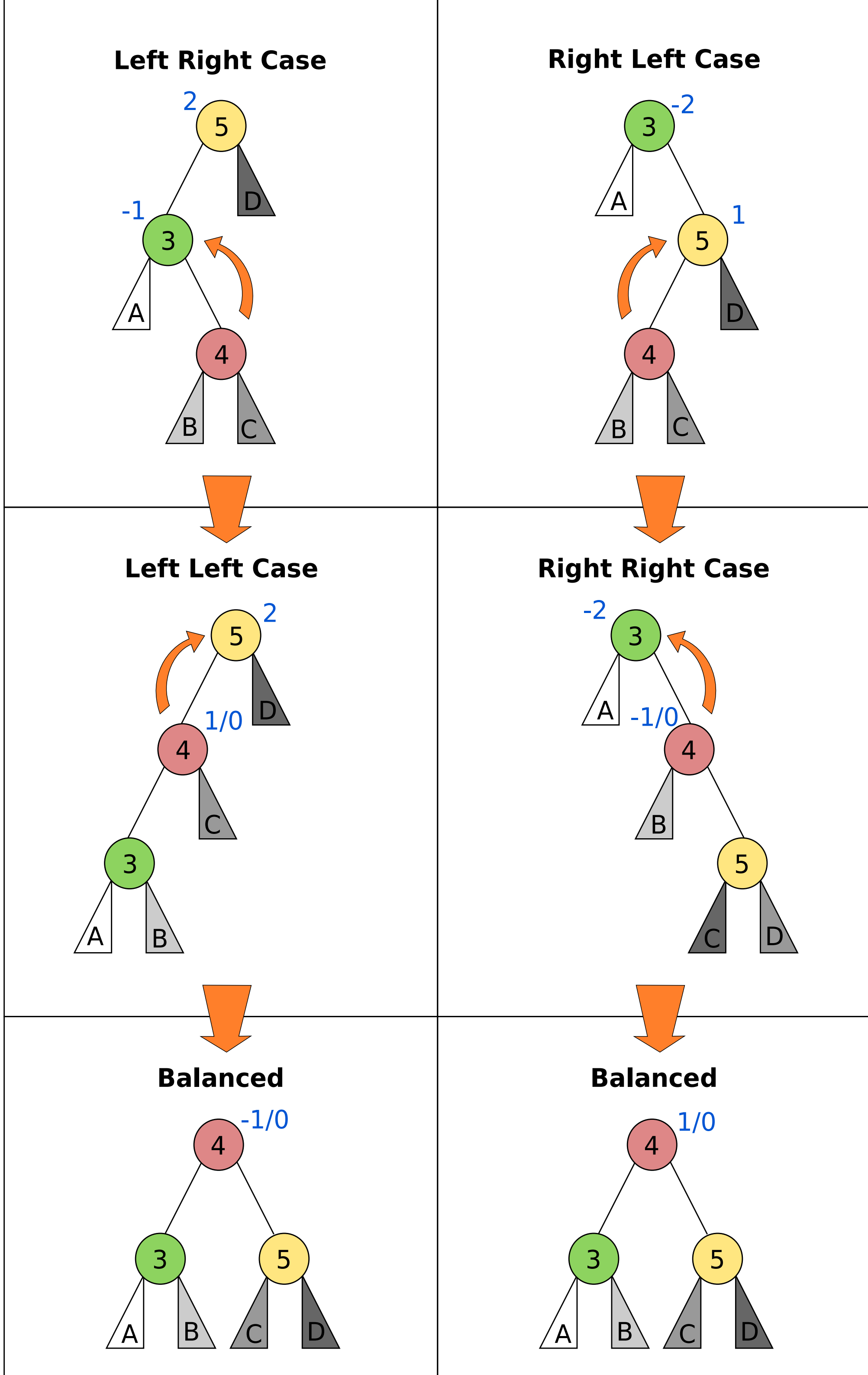
Right-left and right-right: symmetric

The four cases

(picture from Wikipedia)

The numbers in the diagram show the *balance* of the tree: left height minus right height

To implement this efficiently, record the balance in the nodes and look at it to work out which case you're in



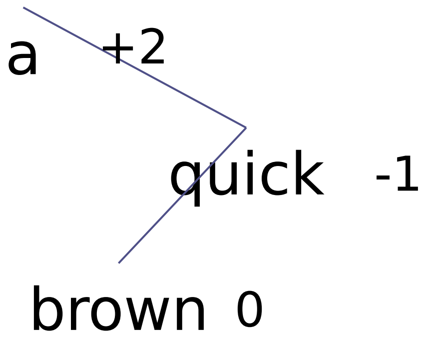
A bigger example
(slides from Peter Ljunglöf)

Let's build an AVL tree for the words in

”a quick brown fox jumps over the lazy dog”

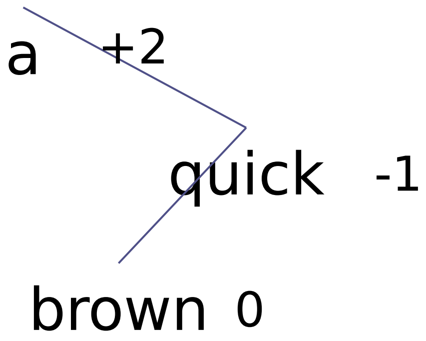
Try this example on
https://www.cs.usfca.edu/~galles/visualization/AVLtree.html

a quick brown…



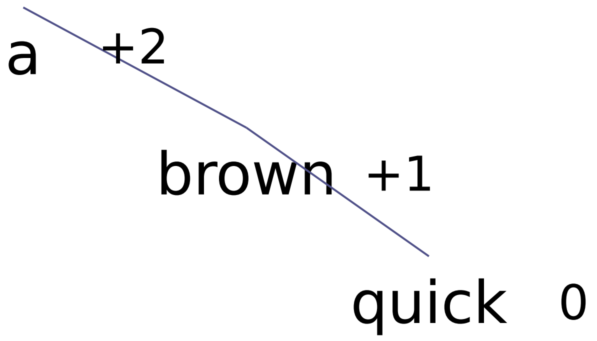
The overall tree is right-heavy (Right-Left)
parent balance = +2
right child balance = -1

a quick brown…



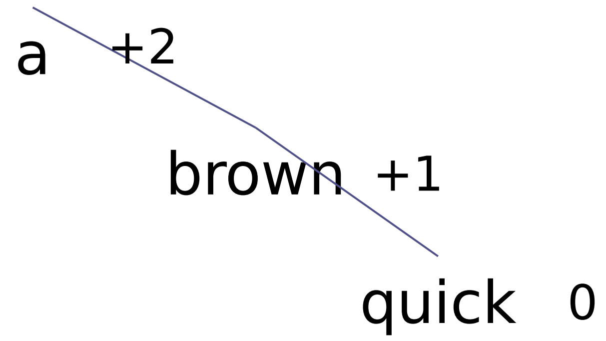
1. **Rotate right around the child**

a quick brown…



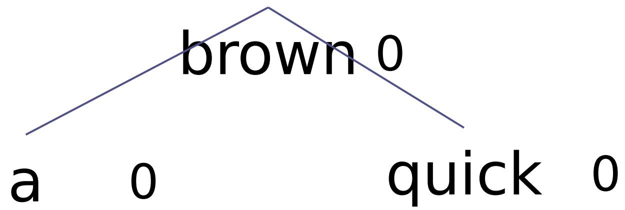
1. **Rotate right around the child**

a quick brown…



1. **Rotate right around the child**
2. **Rotate left around the parent**

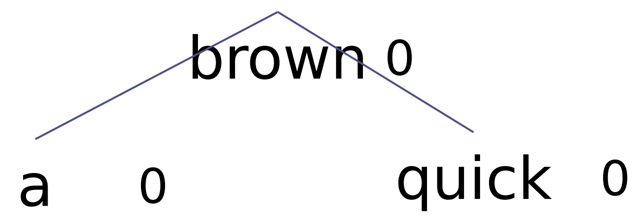
a quick brown…



1. **Rotate right around the child**

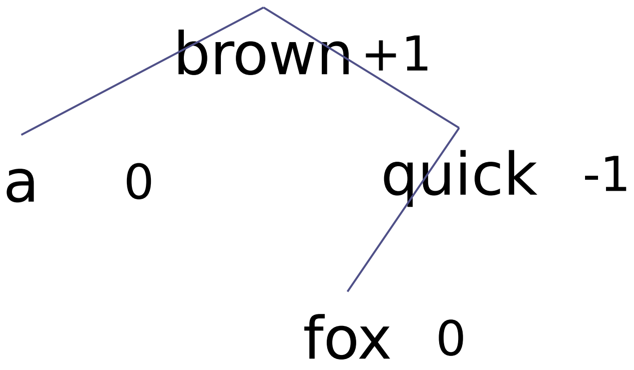
2. **Rotate left around the parent**

a quick brown fox…



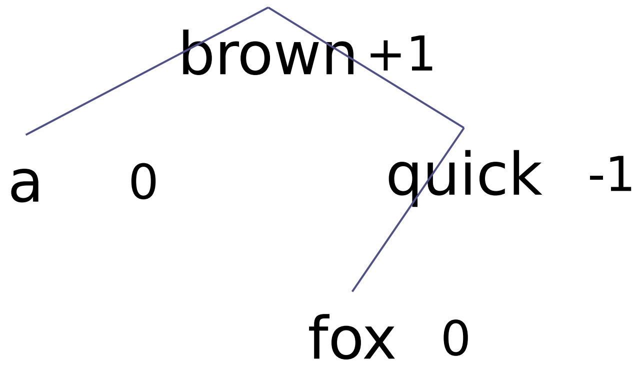
Insert *fox*

a quick brown fox…



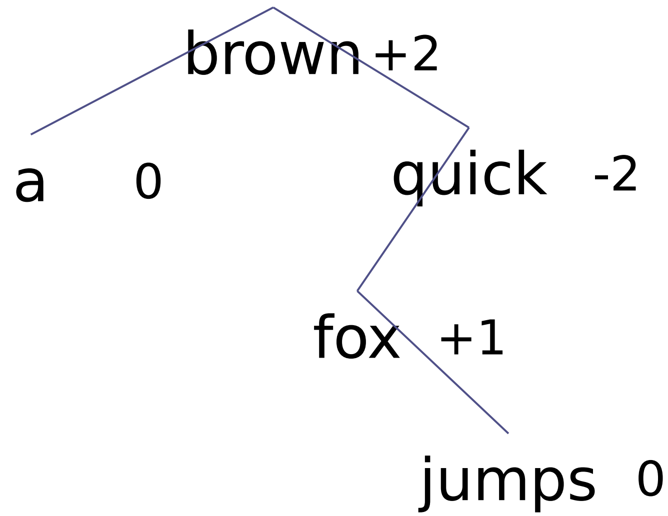
Insert *fox*

a quick brown fox jumps…



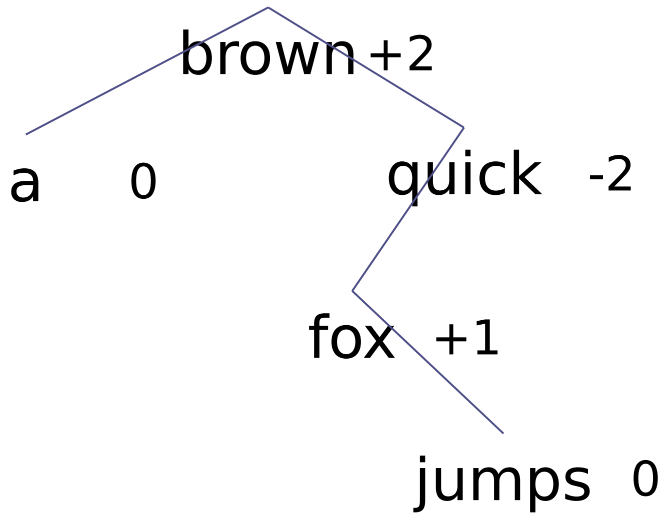
Insert *jumps*

a quick brown fox jumps…



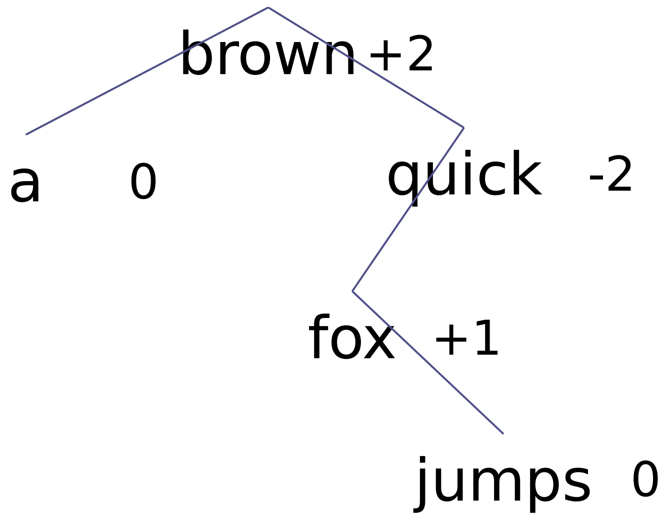
Insert *jumps*

a quick brown fox jumps…



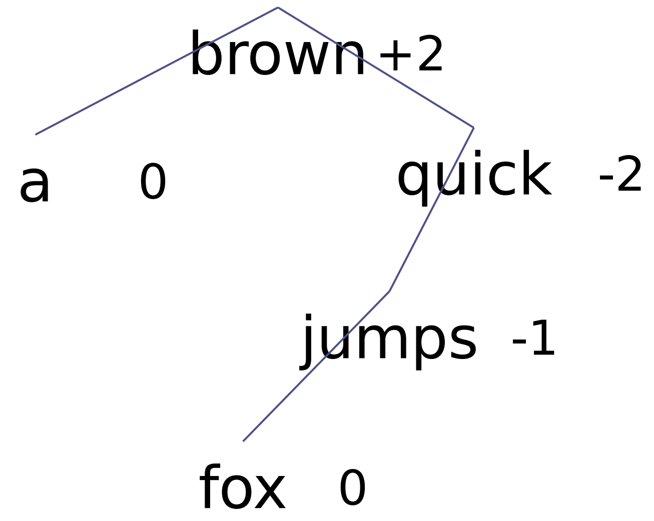
The tree is now left-heavy about *quick* (Left-Right case)

a quick brown fox jumps…



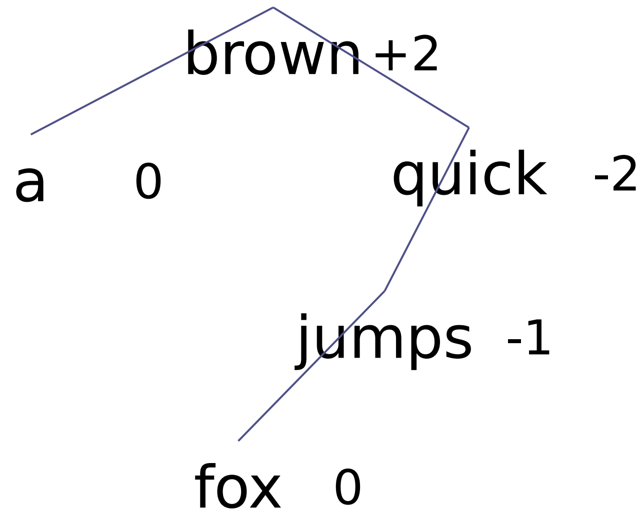
1. Rotate left around the child

a quick brown fox jumps…



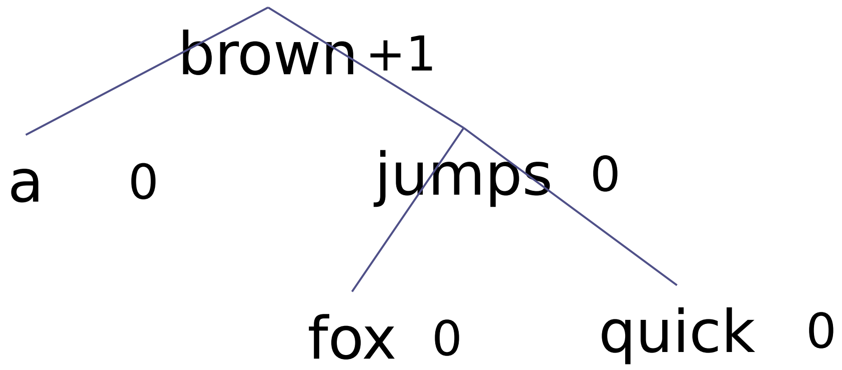
1. Rotate left around the child

a quick brown fox jumps…



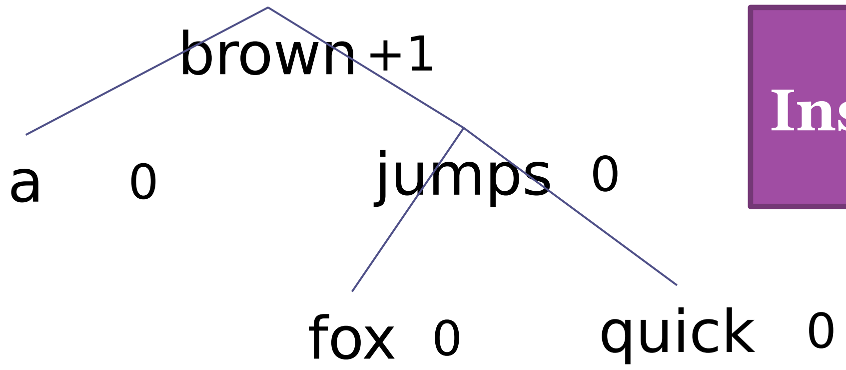
1. **Rotate left around the child**
2. **Rotate right around the parent**

a quick brown fox jumps…



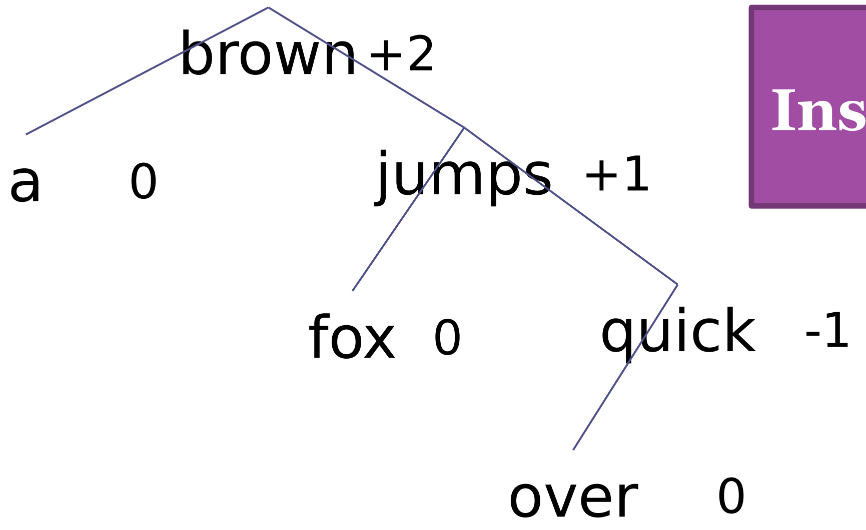
1. **Rotate left around the child**
2. **Rotate right around the parent**

a quick brown fox jumps over…



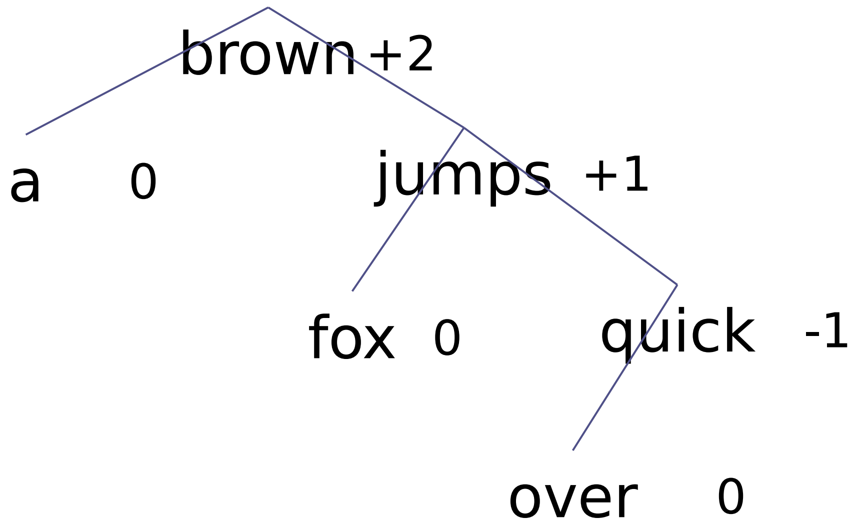
Insert *over*

a quick brown fox jumps over…



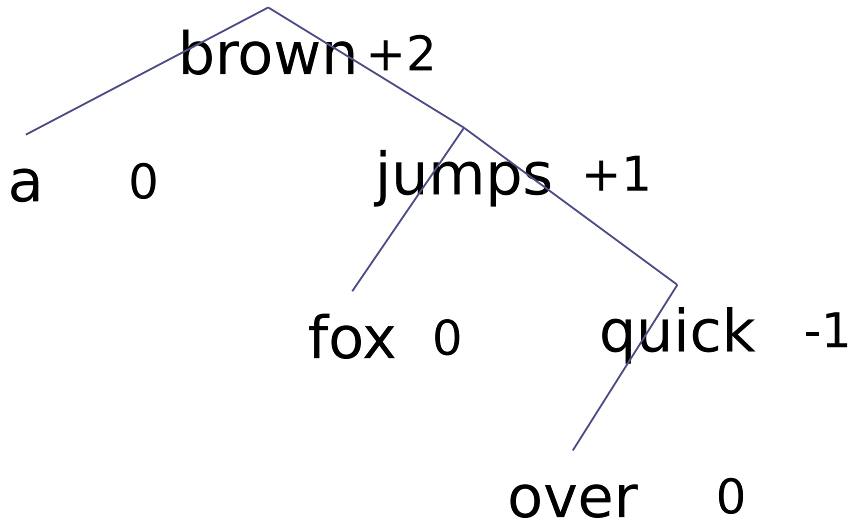
Insert *over*

a quick brown fox jumps over…



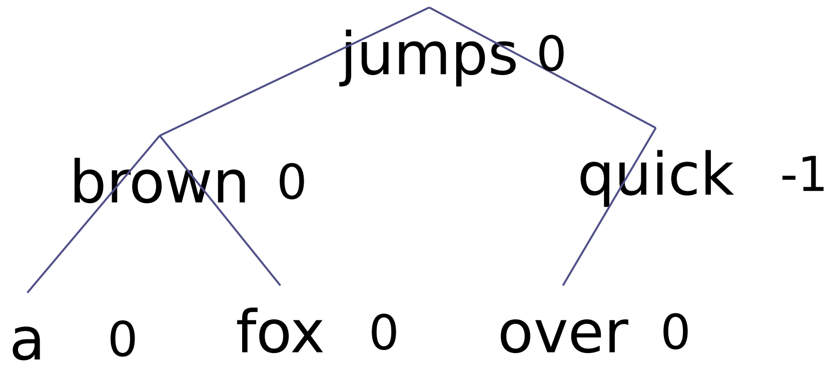
We now have a Right-Right imbalance

a quick brown fox jumps over…



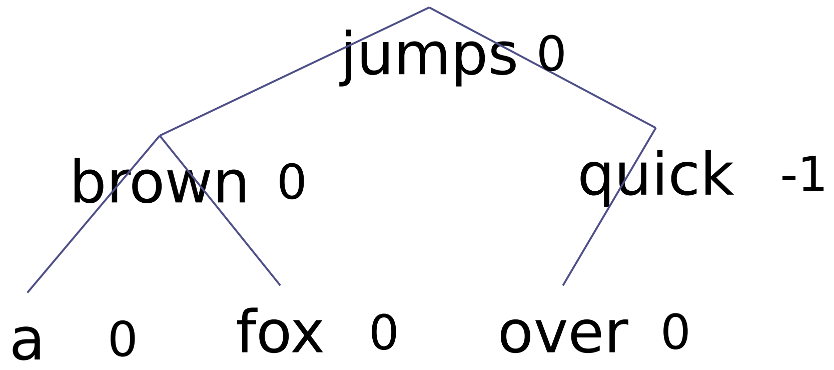
1. Rotate left around the parent

a quick brown fox jumps over…



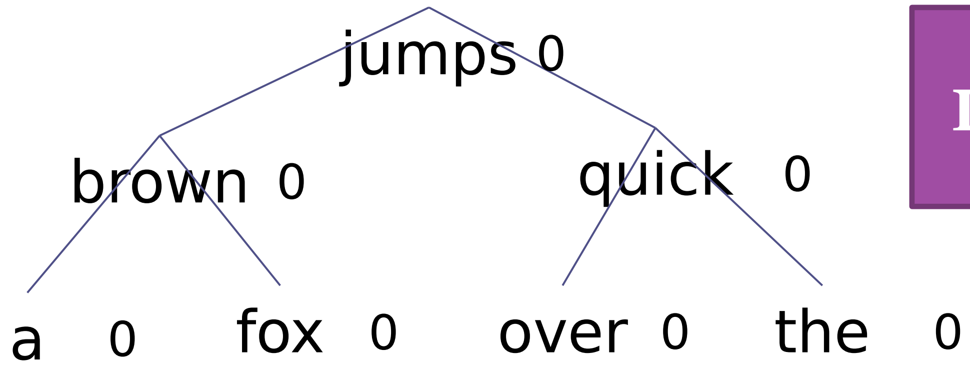
1. Rotate left around the parent

a quick brown fox jumps over the…



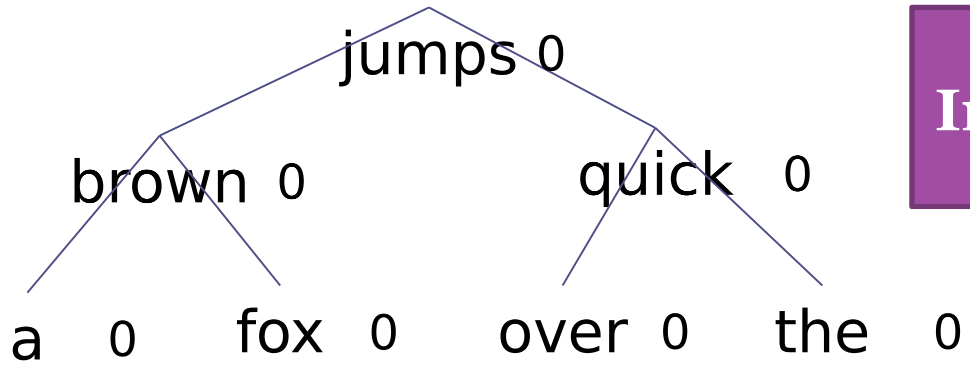
Insert *the*

a quick brown fox jumps over the…



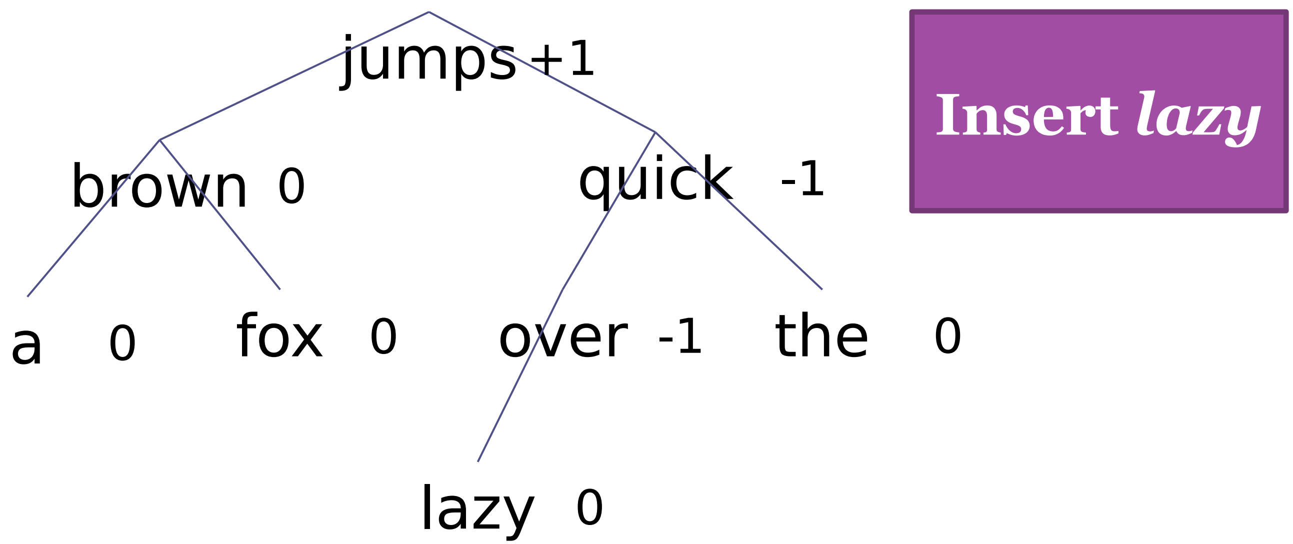
Insert *the*

a quick brown fox jumps over the lazy…

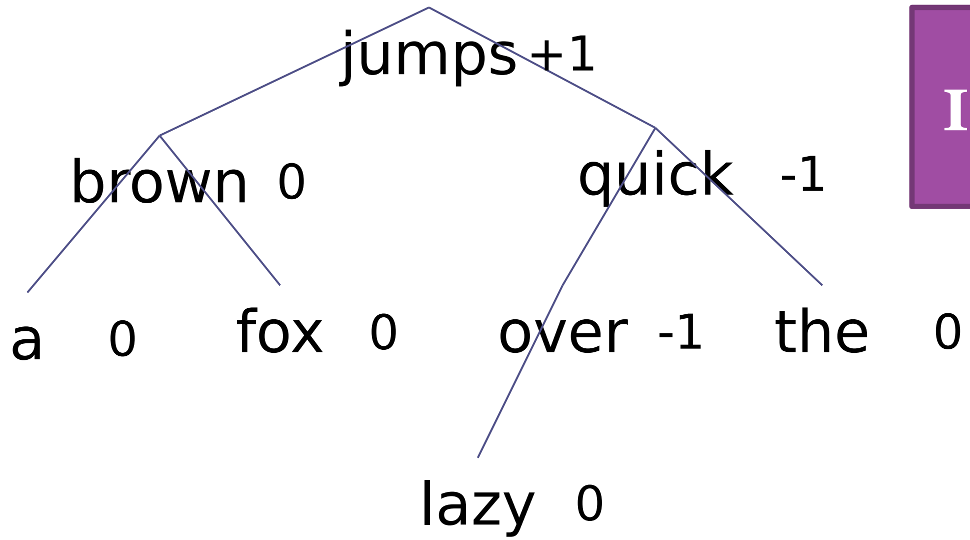


Insert *lazy*

a quick brown fox jumps over the lazy…

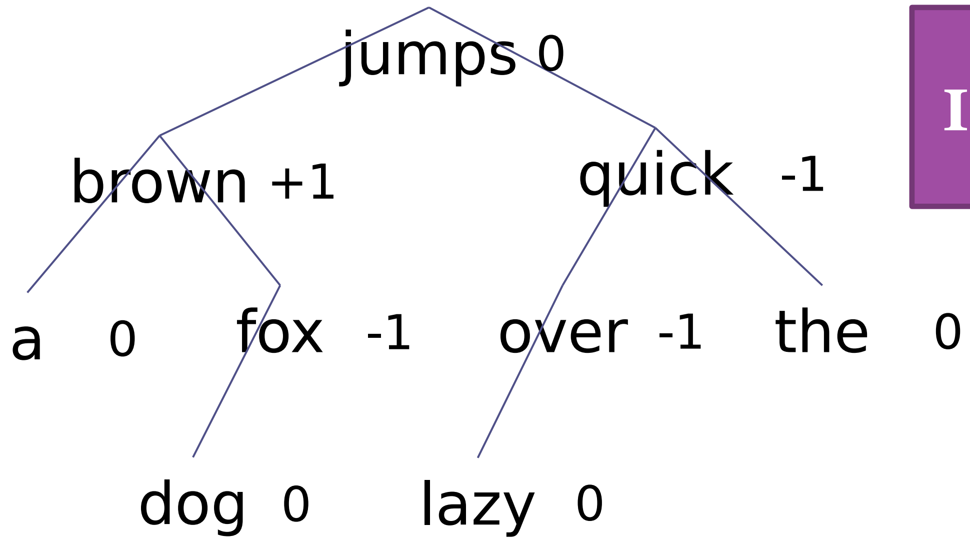


a quick brown fox jumps over the lazy dog



Insert *dog*

a quick brown fox jumps over the lazy dog!



Insert *dog*

AVL trees

A balanced BST that maintains balance by *rotating* the tree

- Insertion: insert as in a BST and move upwards from the inserted node, rotating unbalanced nodes
- Deletion (in book if you're interested): delete as in a BST and move upwards from the node that disappeared, rotating unbalanced nodes

Worst-case (it turns out) 1.44log n, typical log n comparisons for any operation – very balanced. This means lookups are quick.

- Insertion and deletion can be slower than in a naïve BST, because you have to do a bit of work to repair the invariant

Look in Haskell compendium (course website) for implementation

Visualisation: https://www.cs.usfca.edu/~galles/visualization/AVLtree.html