Software Engineering using Formal Methods

Reasoning about Programs with Loops and Method Calls

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Program Logic Calculus – Repetition

Calculus realises symbolic interpreter:

- works on first active statement
- decomposition of complex statements into simpler ones
- simple assignments to updates
- accumulated update captures changed program state (abbr. w. $\mathcal{U}$)
- control flow branching induces proof splitting
- application of update computes weakest precondition of $\mathcal{U}'$ wrt. ϕ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma' \Longrightarrow \{\mathcal{U}'\}\phi \qquad \ldots}{\ldots \qquad \ldots}}{\begin{array}{ll}\textit{`branch1'} & \Gamma, \{\mathcal{U}\}(\mathtt{isValid} = \mathtt{TRUE}) \Longrightarrow \{\mathcal{U}\}\langle\{\mathtt{ok=\mathbf{true};}\}\ldots\rangle\phi \\ \textit{`branch2'} & \Gamma, \{\mathcal{U}\}(\mathtt{isValid} = \mathtt{FALSE}) \Longrightarrow \{\mathcal{U}\}\langle\ldots\rangle\phi\end{array}}}{\Gamma \Longrightarrow \{\mathtt{t} := \mathtt{j} \| \mathtt{j} := \mathtt{j} + 1 \| \mathtt{i} := \mathtt{j}\}\{\mathcal{U}\}\{\mathcal{U}\}\langle\mathtt{\mathbf{if}(isValid)\{ok=\mathbf{true};\}\ldots}\rangle\phi}}{\ldots}}{\Gamma \Longrightarrow \{\mathtt{t} := \mathtt{j}\}\langle\mathtt{j=j+1;i=t;\mathbf{if}(isValid)\{ok=\mathbf{true};\}\ldots}\rangle\phi}}{\dfrac{\Gamma \Longrightarrow \langle\mathtt{t=j;j=j+1;i=t;\mathbf{if}(isValid)\{ok=\mathbf{true};\}\ldots}\rangle\phi}{\Gamma \Longrightarrow \langle\mathtt{i=j++;\mathbf{if}(isValid)\{ok=\mathbf{true};\}\ldots}\rangle\phi}}$$

An Example

```
\javaSource "src/";

\programVariables{
 Person p;
 int j;
}

\problem {
  (\forall int i;
    (!p=null ->
      ({j := i}\<{p.setAge(j);}\>(p.age = i))))
}
```

Method Calls

Method Call with actual parameters $arg_0, \ldots, arg_n$

$$\langle \pi \; \texttt{o.m}(arg_0, \ldots, arg_n); \;\; \omega \rangle \phi$$

where $\texttt{m}$ declared as **void** $\texttt{m}(\tau_0 \; \texttt{p}_0, \ldots, \tau_\texttt{n} \; \texttt{p}_\texttt{n})$

Actions of rule methodCall

1. Declare new local variables $\texttt{p\#i}$, initialize them with actual parameter: $\tau_i \; \texttt{p\#i} = arg_i;$
2. Look-up implementing class C of $\texttt{m}$; split proof if implementation cannot be uniquely determined.
3. Replace method call with implementation invocation $\texttt{o.m}(\texttt{p\#0}, \ldots, \texttt{p\#n})\texttt{@}C$

Method Calls Cont'd

Method Body Expand

1. Execute the (already generated) initialisers: $\tau_{\mathtt{i}}$ p#i =arg_i;
2. Call rule methodBodyExpand

$$\frac{\Gamma \Longrightarrow \langle \pi\ \mathtt{method\text{-}frame(source{=}C,\ this{=}o)\{}\ body\ \mathtt{\}}\ \omega \rangle \phi, \Delta}{\Gamma \Longrightarrow \langle \pi\ \mathtt{o.m(p\#0,\ldots,p\#n)@C;}\ \omega \rangle \phi, \Delta}$$

   2.1 Replaces method invocation by method frame and method body
   2.2 Renames p_i in body to p#i

Method frames:
Required in proof to represent call stack

Demo

```
methods/instanceMethodInlineSimple.key
methods/inlineDynamicDispatch.key
```

Localisation of Fields and Method Implementations

JAVA has complex rules for localisation of fields and method implementations

- Polymorphism
- Late binding (dynamic dispatch)
- Scoping (class vs. instance)
- Visibility (private, protected, public)

Proof split into cases if implementation not statically determined

Object initialization

JAVA has complex rules for object initialization

- Chain of constructor calls until Object
- Implicit calls to **super**()
- Visibility issues
- Initialization sequence

Coding of initialization rules in methods `<createObject>()`, `<init>()`,...
which are then symbolically executed

Limitations of Method Inlining: methodBodyExpand

- Source code might be unavailable
  - Source code often unavailable for commercial APIs, even for some JAVA API methods (& implementation vendor-specific)
  - Method implementation deployment-specific
- Method is invoked multiple times in a program
  - Avoid multiple symbolic execution of identical code
- Cannot handle unbounded recursion
- Not modular:
  Changing a method requires re-verification of all callers

Use method contract instead of method implementation:

1. Show that requires clause is satisfied
2. Continue after method call,
   - 'ignoring' ealier values of modifiable locations
   - assuming ensures clause

Method Contract Rule: Normal Behavior Case

Warning: Simplified version

```
/*@ public normal_behavior
  @ requires preNormal;
  @ ensures postNormal;
  @ assignable mod;
  @*/ // implementation contract of m()
```

$$\frac{\Gamma \Longrightarrow \mathcal{U}\mathcal{F}(\texttt{preNormal}), \Delta \quad \text{(precondition)} \qquad \Gamma \Longrightarrow \mathcal{U}\mathcal{V}_{\texttt{mod}}(\mathcal{F}(\texttt{postNormal}) \rightarrow \langle \pi\ \omega \rangle \phi), \Delta \quad \text{(normal)}}{\Gamma \Longrightarrow \mathcal{U}\langle \pi\, \texttt{result} = \texttt{m}(\texttt{a}_1, \ldots, \texttt{a}_\texttt{n});\ \omega \rangle \phi, \Delta}$$

- $\mathcal{F}(\cdot)$: translation from JML to Java DL
- $\mathcal{V}_{\texttt{mod}}$: anonymising update,
  forgetting prevalues of modifiable locations

JML Method Contracts Revisited

```
/*@ public normal_behavior
  @ requires preNormal;
  @ ensures postNormal;
  @ assignable mod;
  @*/
T m(T1 a1, ..., Tn an) { ... }
```

Implicit Preconditions and Postconditions

- The object referenced by **this** is not **null**: **this**!=**null** (precondition only; **this** cannot be changed by method)
- The heap is wellformed: `wellFormed(heap)` (precondition only)
- Invariant for 'this': `\invariant_for(`**this**`)`

Method Contract Rule: Normal Behavior Case

Warning: Simplified version

```
/*@ public normal_behavior
  @ requires preNormal;
  @ ensures postNormal;
  @ assignable mod;
  @*/ // implementation contract of m()
```

$$\frac{\begin{array}{ll}\Gamma \Longrightarrow \mathcal{U}\mathcal{F}(\mathtt{preNormal}), \Delta & \text{(precondition)} \\ \Gamma \Longrightarrow \mathcal{U}\mathcal{V}_{\mathtt{mod}}(\mathcal{F}(\mathtt{postNormal}) \rightarrow \langle \pi\ \omega \rangle \phi), \Delta & \text{(normal)}\end{array}}{\Gamma \Longrightarrow \mathcal{U}\langle \pi\, \mathtt{result} = \mathtt{m}(\mathtt{a_1}, \ldots, \mathtt{a_n});\ \omega \rangle \phi, \Delta}$$

- $\mathcal{F}(\cdot)$: translation from JML to Java DL
- $\mathcal{V}_{\mathtt{mod}}$: anonymising update,
  forgetting prevalues of modifiable locations

Keeping the Context

- Want to keep part of prestate $\mathcal{U}$ that is unmodified by called method
- **assignable** clause of contract tells what can possibly be modified

```
@ assignable mod;
```

- How to erase all values of **assignable** locations in state $\mathcal{U}$?
- Anonymising updates $\mathcal{V}$ erase information about modified locations

Anonymising Heap Locations

Define anonymising function $\texttt{anon}$: Heap × LocSet × Heap → Heap

The resulting heap anon(...) coincides with the first heap on all locations except for those specified in the location set. Those locations attain the value specified by the second heap.

Definition:

$$\texttt{select}(\texttt{anon}(h1, locs, h2), o, f) = \begin{cases} \texttt{select}(h2, o, f) & \text{if } (o, f) \in locs \\ \texttt{select}(h1, o, f) & \text{otherwise} \end{cases}$$

Usage:

$$\mathcal{V}_{mod} = \{\texttt{heap} := \texttt{anon}(\texttt{heap}, locs_{mod}, \texttt{h}_{an})\}$$

where $\texttt{h}_{an}$ a new (not yet used) constant of type Heap

Effect: After $\mathcal{V}_{mod}$, modfied locations have unknown values

Anonymising Heap Locations: Example

```
@ assignable o.a, this.*;
```

To erase all knowledge about the values of the locations of the assignable expression:

- Anonymise the current heap on the designated locations:

$$\mathtt{anon}(\mathtt{heap}, \{(\mathtt{o}, \mathtt{a})\} \cup \mathtt{allFields}(\mathtt{this}), \mathtt{h}_{an})$$

- Make that anonymised current heap the new current heap.

$$\mathcal{V}_{mod} = \{\mathtt{heap} := \mathtt{anon}(\mathtt{heap}, \{(\mathtt{o}, \mathtt{a})\} \cup \mathtt{allFields}(\mathtt{this}), \mathtt{h}_{an})\}$$

Method Contract Rule: Exceptional Behavior Case

Warning: Simplified version

```
/*@ public exceptional_behavior
  @ requires preExc;
  @ signals (Exception exc) postExc;
  @ assignable mod;
  @*/
```

$$\frac{\begin{array}{l}\Gamma \Longrightarrow \mathcal{U}\mathcal{F}(\mathtt{preExc}), \Delta \quad \text{(precondition)} \\ \Gamma \Longrightarrow \mathcal{U}\mathcal{V}_{mod}((\mathcal{F}(\mathtt{postExc}) \land \mathtt{exc} \neq \mathbf{null}) \\ \qquad\qquad \rightarrow \langle \pi\, \mathbf{throw}\ \mathtt{exc};\ \omega\rangle\phi), \Delta \quad \text{(exceptional)}\end{array}}{\Gamma \Longrightarrow \mathcal{U}\langle \pi\, \mathtt{result} = \mathtt{m}(\mathtt{a}_1, \ldots, \mathtt{a}_\mathtt{n});\ \omega\rangle\phi, \Delta}$$

- $\mathcal{F}(\cdot)$: translation from JML to Java DL
- $\mathcal{V}_{mod}$: anonymising update

Method Contract Rule – Combined

Warning: Simplified version

KeY uses actually only one rule for both kinds of cases.

Therefore translation of postcondition ϕ_{post} as follows (simplified):

$$\begin{array}{lcl} \phi_{post\_n} & \equiv & \mathcal{F}(\texttt{\textbackslash old}(\texttt{normalPre})) \land \mathcal{F}(\texttt{normalPost}) \\ \phi_{post\_e} & \equiv & \mathcal{F}(\texttt{\textbackslash old}(\texttt{excPre})) \land \mathcal{F}(\texttt{excPost}) \end{array}$$

$$\frac{\begin{array}{ll} \Gamma \Longrightarrow \mathcal{U}(\mathcal{F}(\texttt{normalPre}) \lor \mathcal{F}(\texttt{excPre})), \Delta & \text{(precondition)} \\ \Gamma \Longrightarrow \mathcal{U}\mathcal{V}_{mod_{normal}}(\phi_{post\_n} \rightarrow \langle \pi\, \omega \rangle \phi), \Delta & \text{(normal)} \\ \Gamma \Longrightarrow \mathcal{U}\mathcal{V}_{mod_{exc}}((\phi_{post\_e} \land \texttt{exc} \neq \texttt{null}) & \\ \qquad \rightarrow \langle \pi\, \texttt{throw exc;}\ \omega \rangle \phi), \Delta & \text{(exceptional)} \end{array}}{\Gamma \Longrightarrow \mathcal{U}\langle \pi\, \texttt{result} = \texttt{m}(\texttt{a}_1, \ldots, \texttt{a}_\texttt{n});\ \omega \rangle \phi, \Delta}$$

- $\mathcal{F}(\cdot)$: translation to Java DL
- $\mathcal{V}_{mod}$: anonymising update (similar to loops)

Method Contract Rule: Example

```
class Person {
 private /*@ spec_public @*/ int age;
 /*@ public normal_behavior
   @ requires age < 29;
   @ ensures age == \old(age) + 1;
   @ assignable age;
   @ also
   @ public exceptional_behavior
   @ requires age >= 29;
   @ signals_only ForeverYoungException;
   @ assignable \nothing;
   @//allows object creation (otherwise \strictly_nothing)
   @*/
 public void birthday() {
   if (age >= 29) throw new ForeverYoungException();
   age++;
 } }
```

Method Contract Rule: Example Cont'd

Demo

`methods/useContractForBirthday.key`

- Prove without contracts (all except object creation)
  - Method treatment: Expand
- Prove with contracts (until method contract application)
  - Method treatment: Contract
- Prove used contracts
  - Method treatment: Expand
  - Select contracts for `birthday()`in `src/Person.java`
  - Prove both specification cases

Verification of Loops

Symbolic execution of loops: unwind

$$\text{unwindLoop}\ \frac{\Gamma \Longrightarrow \mathcal{U}[\pi\,\texttt{\textbf{if}}\,(\texttt{b})\{\texttt{p};\ \texttt{\textbf{while}}(\texttt{b})\ \texttt{p}\}\ \omega]\phi, \Delta}{\Gamma \Longrightarrow \mathcal{U}[\pi\,\texttt{\textbf{while}}(\texttt{b})\ \texttt{p}\ \omega]\phi, \Delta}$$

How to handle a loop with...

- 0 iterations? Unwind $1\times$
- 10 iterations? Unwind $11\times$
- 10000 iterations? Unwind $10001\times$
- an unknown number of iterations?

We need an invariant rule (or some form of induction)

Loop Invariants

Idea behind loop invariants

- A formula *Inv* whose validity is preserved by loop body whenever the loop guard is true
- Consequence: if *Inv* was valid at start of the loop, then it still holds after arbitrarily many loop iterations
- In particular, if the loop terminates, then *Inv* holds afterwards
- Construct *Inv* such that, *together with loop exit condition*, it implies postcondition of loop

Basic Invariant Rule

$$\text{loopInvariant}\ \frac{\begin{array}{ll} \Gamma \Longrightarrow \mathcal{U}\,\mathit{Inv}, \Delta & \text{(valid when entering loop)} \\ \mathit{Inv},\ b = \mathtt{TRUE} \Longrightarrow [\mathtt{p}]\mathit{Inv} & \text{(preserved by p)} \\ \mathit{Inv},\ b = \mathtt{FALSE} \Longrightarrow [\pi\ \omega]\phi & \text{(assumed after exit)} \end{array}}{\Gamma \Longrightarrow \mathcal{U}[\pi\ \mathtt{while(b)\ p}\ \omega]\phi, \Delta}$$

How to Derive Loop Invariants Systematically?

Example (First active statement of symbolic execution is loop)

```
n >= 0 & wellFormed(heap) ==>
{i := 0}\[{
  while (i < n) {
    i = i + 1;
  }
}\] (i = n)
```

Look at desired postcondition `(i = n)`

What, in addition to negated guard `(i >= n)`, is needed? `(i <= n)`

Is `(i <= n)` established at beginning and preserved?

Yes! We have found a suitable loop invariant!

Demo `loops/simple.key` (auto after inv)

Obtaining Invariants by Strengthening

Example (Slightly changed problem)

```
n >= 0 & n = m & wellFormed(heap) ==>
{i := 0}\[{
  while (i < n) {
    i = i + 1;
  }
}\] (i = m)
```

Look at desired postcondition (`i = m`)

What, in addition to negated guard (`i >= n`), is needed? (`i = m`)

Is (`i = m`) established at beginning and preserved? Neither!

Can we use something from the precondition or the update?

- If we know (`n = m`) then (`i <= n`) suffices
- Strengthen the invariant candidate to: (`i <= n & n = m`)

Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

```
x = x0 & y = y0 & y0 >= 0 & wellFormed(heap) ==>
\[{
  while (y > 0) {
    x = x + 1;
    y = y - 1;
  }
}\] (x = x0 + y0)
```

Finding the invariant

First attempt: use postcondition `x = x0 + y0`

- Not true at start whenever `y0 > 0`
- Not preserved by loop, because `x` is increased

Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

```
x = x0 & y = y0 & y0 >= 0 & wellFormed(heap) ==>
\[{
  while (y > 0) {
    x = x + 1;
    y = y - 1;
  }
}\] (x = x0 + y0)
```

Finding the invariant

What stays invariant?

- The sum of `x` and `y`: `x + y = x0 + y0` "Generalization"
- Can help to think of "δ" between `x` and `x0 + y0`

Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

```
x = x0 & y = y0 & y0 >= 0 & wellFormed(heap) ==>
\[{
  while (y > 0) {
    x = x + 1;
    y = y - 1;
  }
}\] (x = x0 + y0)
```

Checking the invariant

Is `x + y = x0 + y0` a good invariant?

- Holds in the beginning and is preserved by loop
- But postcondition not achieved by `x + y = x0 + y0 & y <= 0`

Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

```
x = x0 & y = y0 & y0 >= 0 & wellFormed(heap) ==>
\[{
  while (y > 0) {
    x = x + 1;
    y = y - 1;
  }
}\] (x = x0 + y0)
```

Strenghtening the invariant

Postcondition holds if `y = 0`

- Add `y >= 0` to invariant: `x + y = x0 + y0 & y >= 0`

Demo `loops/simple3.key`

Basic Loop Invariant: Context Loss

Basic Invariant Rule: a Problem

$$\text{loopInvariant}\quad \frac{\begin{array}{ll} \Gamma \Longrightarrow \mathcal{U}\mathit{Inv}, \Delta & \text{(initially valid)} \\ \mathit{Inv},\ b = \mathtt{TRUE} \Longrightarrow [\mathtt{p}]\mathit{Inv} & \text{(preserved)} \\ \mathit{Inv},\ b = \mathtt{FALSE} \Longrightarrow [\pi\ \omega]\phi & \text{(use case)} \end{array}}{\Gamma \Longrightarrow \mathcal{U}[\pi\ \mathtt{while(b)\ p}\ \omega]\phi, \Delta}$$

- Context Γ, Δ, $\mathcal{U}$ must be omitted in 2nd and 3rd premise:

  Γ, Δ in general don't hold in state reached by $\mathcal{U}$

  2nd premise *Inv* must be invariant for any state, not only $\mathcal{U}$

  3rd premise We don't know the state after the loop exits
- But: context contains (part of) precondition and class invariants
- Better to add context information to loop invariant *Inv*

Example

Precondition: $a \neq$ **null** & *ClassInv*

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
    i++;
}
```

Postcondition: $\forall$ **int** x; $(0 \leq x \;\&\; x < \texttt{a.length} \rightarrow \texttt{a}[x] = 1)$

Loop invariant: $0 \leq \texttt{i} \;\&\; \texttt{i} \leq \texttt{a.length}$
$\&\; \forall$ **int** x; $(0 \leq x \;\&\; x < \texttt{i} \rightarrow \texttt{a}[x] = 1)$
$\&\; \texttt{a} \neq$ **null**
& *ClassInv*

Keeping the Context (As In Method Contract Rule)

- Want to keep part of the context that is unmodified by loop
- **assignable** clauses for loops tell what can possibly be modified

```
@ assignable i, a[*];
```

- How to erase all values of **assignable** locations?
- Anonymising updates $\mathcal{V}$ erase information about modified locations

Anonymising JAVA Locations

```
@ assignable i, a[*];
```

To erase all knowledge about the values of the locations of the assignable expression:

- introduce a new (not yet used) constant of type `int`, e.g., c
- introduce a new (not yet used) constant of type Heap, e.g., $\mathtt{h}_{an}$
  - anonymise the current heap: $\mathtt{anon}(\mathtt{heap}, \mathtt{allFields}(\mathtt{this.a}), \mathtt{h}_{an})$
- compute anonymizing update for assignable locations

$$\mathcal{V} = \{\mathtt{i} := \mathtt{c} \mid\mid \mathtt{heap} := \mathtt{anon}(\mathtt{heap}, \mathtt{allFields}(\mathtt{this.a}), \mathtt{h}_{an})\}$$

For local program variables (e.g., `i`) KeY computes assignable clause automatically

Loop Invariants Cont'd

Improved Invariant Rule

$$\frac{\begin{array}{ll} \Gamma \Longrightarrow \mathcal{U}\mathit{Inv}, \Delta & \text{(initially valid)} \\ \Gamma \Longrightarrow \mathcal{U}\mathcal{V}(\mathit{Inv} \ \& \ b = \mathtt{TRUE} \rightarrow [\mathtt{p}]\mathit{Inv}), \Delta & \text{(preserved)} \\ \Gamma \Longrightarrow \mathcal{U}\mathcal{V}(\mathit{Inv} \ \& \ b = \mathtt{FALSE} \rightarrow [\pi\ \omega]\phi), \Delta & \text{(use case)} \end{array}}{\Gamma \Longrightarrow \mathcal{U}[\pi\ \mathbf{while}(\mathtt{b})\ \mathtt{p}\ \omega]\phi, \Delta}$$

- Context is kept as far as possible:
  $\mathcal{V}$ wipes out only information in locations assignable in loop
- Invariant *Inv* does not need to include unmodified locations
- For **`assignable \everything`** (the default):
  - $\mathtt{heap} := \mathtt{anon}(\mathtt{heap}, \mathtt{allLocs}, \mathtt{h}_{an})$ wipes out **all** heap information
  - Equivalent to basic invariant rule
  - Avoid this! Always give a specific **`assignable`** clause

Example with Improved Invariant Rule

Precondition: $\texttt{a} \neq \texttt{null}\ \&\ \mathit{ClassInv}$

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
    i++;
}
```

Postcondition: $\forall\, \texttt{int}\ x;\ (0 \leq x\ \&\ x < \texttt{a.length} \rightarrow \texttt{a}[x] = 1)$

Loop invariant: $0 \leq \texttt{i}\ \ \&\ \ \texttt{i} \leq \texttt{a.length}$
$\&\ \ \forall\, \texttt{int}\ x;\ (0 \leq x\ \&\ x < \texttt{i} \rightarrow \texttt{a}[x] = 1)$

Example in JML/Java – Loop.java

Demo

```
public int[] a;
/*@ public normal_behavior
  @  ensures (\forall int x; 0<=x && x<a.length; a[x]==1);
  @  diverges true;
  @*/
public void m() {
  int i = 0;
  /*@ loop_invariant
    @  0 <= i && i <= a.length &&
    @  (\forall int x; 0<=x && x<i; a[x]==1);
    @ assignable a[*];
    @*/
  while(i < a.length) {
    a[i] = 1;
    i++;
  }
}
```

Example from a Previous Lecture

$\forall$ **int** x;
  $(x = \mathtt{n} \land x >= 0 \rightarrow$
    $[$ `i = 0; r = 0;`
      `while (i<n) { i = i + 1; r = r + i;}`
      `r=r+r-n;`
    $]\, (\mathtt{r} = x * x)$

How can we prove that the above formula is valid
(i.e., satisfied in all states)?

Needed Invariant:

```
@ loop_invariant
@     i>=0  && i <= n && 2*r == i*(i + 1);
@ assignable \nothing; // no heap locations changed
```

Demo Loop2.java

Hints

Proving `assignable`

- Invariant rule above assumes that **`assignable`** is correct
  E.g., possible to prove nonsense with incorrect
  `assignable \nothing;`
- Invariant rule of KeY generates proof obligation that ensures correctness of **`assignable`**
  This proof obligation is part of 'Body Preserves Invariant' branch

Setting in the KeY Prover when proving loops

- Loop treatment: Invariant
- Quantifier treatment: No Splits with Progs
- If program contains `*`, `/`: Arithmetic treatment: DefOps
- Is search limit high enough (time out, rule apps.)?
- When proving partial correctness, add **`diverges true;`**

What is still missing?

Is the sequent

$$\Longrightarrow [\texttt{i = -1; while (true)\{\}}]\texttt{i} = 4711$$

provable?

Yes, e.g.,

```
@ loop_invariant true;
@ assignable \nothing;
```

Possible to prove correctness of non-terminating loop

- Invariant trivially initially valid and preserved ⇒
  Initial Case and Preserved Case immediately closable
- Loop condition never false: Use case immediately closable

But need a method to prove termination of loops

Mapping Loop Execution to Well-Founded Order

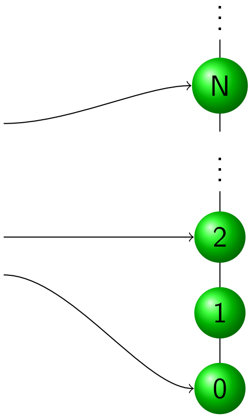
```
while (b) {
  body
}
```

`if (b) { body }`$_1$

⋮

`if (b) { body }`$_{17}$

`if (b) { body }`$_{18}$



Need to find expression getting smaller wrt $\mathbb{N}$ in each iteration

Such an expression is called a decreasing term or variant

Total Correctness: Decreasing Term (Variant)

Find a decreasing integer term v (called variant)

Add the following premisses to the invariant rule:

- $v \geq 0$ is initially valid
- $v \geq 0$ is preserved by the loop body
- v is *strictly* decreased by the loop body

Proving termination in JML/Java

- Remove directive `diverges true;` from contract
- Add directive `decreasing v;` to loop invariant
- KeY creates suitable invariant rule and PO (with $\langle \ldots \rangle \phi$)

Example (The `array` loop)

```
@ decreasing  a.length - i;
```

Files:

- `LoopT.java`
- `Loop2T.java`

Final Example: Computing the GCD

```
public class Gcd {
 /*@ public normal_behavior
   @ requires _small>=0 && _big>=_small;
   @ ensures _big!=0 ==>
   @  (_big % \result == 0 && _small % \result == 0 &&
   @    (\forall int x; x>0 && _big % x == 0
   @       && _small % x == 0; \result % x == 0));
   @ assignable \nothing;
   @*/
 private static int gcdHelp(int _big, int _small) {
   int big = _big; int small = _small;
   while (small != 0) {
     final int t = big % small;
     big = small;
     small = t;
   }
   return big;
 }
}
```

Computing the GCD: Method Specification

```
public class Gcd {
 /*@ public normal_behavior
   @ requires _small>=0 && _big>=_small;
   @ ensures _big!=0 ==>
   @ (_big % \result == 0 && _small % \result == 0 &&
   @   (\forall int x; x>0 && _big % x == 0
   @      && _small % x == 0; \result % x == 0));
   @ assignable \nothing;
   @*/
 private static int gcdHelp(int _big, int _small) {...}
```

requires normalization assumptions on method parameters (both non-negative and `_big` $\geq$ `_small`)

ensures if `_big` positive, then

- the return value `\result` is a divider of both arguments
- all other dividers `x` of the arguments are also dividers of `\result` and thus smaller or equal to `\result`

Computing the GCD: Specify the Loop Body

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Which locations are changed (at most)?

```
@ assignable \nothing; // no heap locations changed
```

What is the variant?

```
@ decreases small;
```

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Possible for `big` to become 0 in a loop iteration? No.
- Adding `big>0` to loop invariant? No. Not initially valid.
- Weaker condition necessary: `big==0 ==> _big==0`

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Weaker condition necessary: `big==0 ==> _big==0`
- What does the loop preserve? The set of dividers!
  All common dividers of `_big`, `_small` are also dividers of `big`, `small`

```
(\forall int x; x > 0;
                 (_big%x == 0 && _small%x == 0)
                 <==>
                 (big%x == 0 && small%x == 0));
```

Computing the GCD: Final Specification

```
int big = _big; int small = _small;
/*@ loop_invariant small >= 0 && big >= small &&
  @   (big == 0 ==> _big == 0) &&
  @   (\forall int x; x > 0; (_big % x == 0 && _small % x == 0)
  @                          <==>
  @                          (big % x == 0 && small % x == 0));
  @ decreases small;
  @ assignable \nothing;
  @*/
  while (small != 0) {
    final int t = big % small;
    big = small;
    small = t;
  }
  return big; // assigned to \result
```

Why does `big` divides `_small` and `_big` follow from the loop invariant?

If `big` is positive, one can instantiate x with it, and use `small == 0`

Computing the GCD: Demo

Demo `loops/Gcd.java`

1. Show `Gcd.java` and `gcd(a,b)`
2. Ensure that "DefOps" and "Contracts" is selected, $\geq$ 10,000 steps
3. Proof contract of `gcd()`, using contract of `gcdHelp()`
4. Note KeY check sign in parentheses:
   - 4.1 Click "Proof Management"
   - 4.2 Choose tab "By Proof"
   - 4.3 Select proof of `gcd()`
   - 4.4 Select used method contract of `gcdHelp()`
   - 4.5 Click "Start Proof"
5. After finishing proof obligations of `gcdHelp()` parentheses are gone

Some Hints On Finding Invariants

General Advice

- Invariants must be developed, they don't come out of thin air!
- Be as systematic in deriving invariants as when debugging a program

Some Hints On Finding Invariants, Cont'd

Technical Hints

- The desired postcondition is a good starting point
  - What, in addition to negated loop guard, is needed for it to hold?
- If the invariant candidate is not preserved by the loop body:
  - Can you add stuff from the precondition?
  - Does it need strengthening?
  - Try to express the relation between partial and final result
- Simulate a few loop body executions to discover invariant patterns
- If the invariant is not initially valid:
  - Can it be weakened such that the postcondition still follows?
  - Did you forget an assumption in the requires clause?
- Several "rounds" of weakening/strengthening might be required
- Use the KeY tool for each premiss of invariant rule
  - After each change of the invariant make sure all cases are ok
  - Interactive dialogue: previous invariants available in "Alt" tabs

Understanding Unclosed Proofs

Reasons why a proof may not close

- Buggy or incomplete specification
- Bug in program
- Maximal number of steps reached: restart or increase # of steps
- Automatic proof search fails: manual rule applications necessary

Understanding open proof goals

- Follow the control flow from the proof root to the open goal
- Branch labels give useful hints
- Identify unprovable part of post condition or invariant
- Sequent remains always in "pre-state"
  Constraints on program variables refer to value at start of program
  (exception: formula is behind update or modality)
- NB: $\Gamma \Longrightarrow o = \mathtt{null}, \Delta$ is equivalent to $\Gamma, o \neq \mathtt{null} \Longrightarrow \Delta$

Literature for this Lecture

- W. Ahrendt, S. Grebing, Using the KeY Prover
  to appear in the new KeY Book, end 2016
  (available via Google group or personal request)
- B. Beckert, V. Klebanov, and B. Weiß, Dynamic Logic for Java
  to appear in the new KeY Book, end 2016
  (available via Google group or personal request)
  Section 3.7