Software Engineering using Formal Methods

Reasoning about Programs with Loops and Method Calls

Wolfgang Ahrendt

18 October 2016

Program Logic Calculus – Repetition

Calculus realises symbolic interpreter:

- works on first active statement

$$\Gamma \Longrightarrow \langle \texttt{i=j++;}\mathbf{\texttt{if}}\texttt{(isValid)\{ok=}\mathbf{\texttt{true}}\texttt{;\}...}\rangle\phi$$

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- application of update computes weakest precondition of $\mathcal{U}'$ wrt. ϕ

$$
\begin{array}{c}
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\hline
\ldots \qquad\qquad \ldots \\
\hline
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\end{array}
$$

An Example

```
\javaSource "src/";

\programVariables{
 Person p;
 int j;
}

\problem {
  (\forall int i;
    (!p=null ->
      ({j := i}\<{p.setAge(j);}\>(p.age = i))))
}
```

Method Calls

Method Call with actual parameters $arg_0, \ldots, arg_n$

$$\langle \pi \ \texttt{o.m}(arg_0, \ldots, arg_n); \ \ \omega \rangle \phi$$

where m declared as $\textbf{void}\ \texttt{m}(\tau_0\ \texttt{p}_0, \ldots, \tau_\texttt{n}\ \texttt{p}_\texttt{n})$

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   split proof if implementation cannot be uniquely determined.

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1. Declare new local variables `p#i`, initialize them with actual parameter: τ_i `p#i` $= arg_i$;
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3. Replace method call with implementation invocation `o.m(p#0, ..., p#n)@`C

Method Calls Cont'd

Method Body Expand

1. Execute the (already generated) initialisers: $\tau_{\mathtt{i}}$ p#i $= arg_i$;
2. Call rule methodBodyExpand

$$\frac{\Gamma \Longrightarrow \langle \pi \; \mathtt{method\text{-}frame(source{=}C,\; this{=}o)\{} \; \mathit{body} \; \} \; \omega \rangle \phi, \Delta}{\Gamma \Longrightarrow \langle \pi \;\; \mathtt{o.m(p\#0,\ldots,p\#n)@C;} \; \omega \rangle \phi, \Delta}$$

  2.1 Replaces method invocation by method frame and method body
  2.2 Renames p_i in body to p#i

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Method frames:
Required in proof to represent call stack

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Demo

`methods/instanceMethodInlineSimple.key`

`methods/inlineDynamicDispatch.key`

Localisation of Fields and Method Implementations

JAVA has complex rules for localisation of fields and method implementations

- Polymorphism
- Late binding (dynamic dispatch)
- Scoping (class vs. instance)
- Visibility (private, protected, public)

Proof split into cases if implementation not statically determined

Object initialization

JAVA has complex rules for object initialization

- Chain of constructor calls until Object
- Implicit calls to **super**()
- Visibility issues
- Initialization sequence

Coding of initialization rules in methods `<createObject>()`, `<init>()`,...
which are then symbolically executed

Limitations of Method Inlining: methodBodyExpand

- Source code might be unavailable
  - Source code often unavailable for commercial APIs, even for some JAVA API methods (& implementation vendor-specific)
  - Method implementation deployment-specific
- Method is invoked multiple times in a program
  - Avoid multiple symbolic execution of identical code
- Cannot handle unbounded recursion
- Not modular:
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Use method contract instead of method implementation:

1. Show that requires clause is satisfied
2. Continue after method call,
   - 'ignoring' ealier values of modifiable locations
   - assuming ensures clause

Method Contract Rule: Normal Behavior Case

Warning: Simplified version

```
/*@ public normal_behavior
  @ requires preNormal;
  @ ensures postNormal;
  @ assignable mod;
  @*/ // implementation contract of m()
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- $\mathcal{F}(\cdot)$: translation from JML to Java DL

JML Method Contracts Revisited

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Implicit Preconditions and Postconditions

- The object referenced by **this** is not **null**: **this!=null** (precondition only; **this** cannot be changed by method)

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- Invariant for 'this': \invariant_for(**this**)

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- $\mathcal{V}_{\mathtt{mod}}$: anonymising update,
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- Anonymising updates $\mathcal{V}$ erase information about modified locations

Anonymising Heap Locations

Define anonymising function $\mathtt{anon}$: Heap $\times$ LocSet $\times$ Heap $\rightarrow$ Heap

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Definition:

$$\mathtt{select}(\mathtt{anon}(h1, locs, h2), o, f) = \begin{cases} \mathtt{select}(h2, o, f) & \text{if } (o, f) \in locs \\ \mathtt{select}(h1, o, f) & \text{otherwise} \end{cases}$$

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Effect: After $\mathcal{V}_{mod}$, modfied locations have unknown values

Anonymising Heap Locations: Example

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@ assignable o.a, this.*;
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To erase all knowledge about the values of the locations of the assignable expression:

- Anonymise the current heap on the designated locations:

$$\mathtt{anon}(\mathtt{heap}, \{(\mathtt{o}, \mathtt{a})\} \cup \mathtt{allFields}(\mathtt{this}), \mathtt{h}_{an})$$

- Make that anonymised current heap the new current heap.

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Method Contract Rule: Exceptional Behavior Case

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Method Contract Rule: Example

```
class Person {
 private /*@ spec_public @*/ int age;
 /*@ public normal_behavior
   @ requires age < 29;
   @ ensures age == \old(age) + 1;
   @ assignable age;
   @ also
   @ public exceptional_behavior
   @ requires age >= 29;
   @ signals_only ForeverYoungException;
   @ assignable \nothing;
   @//allows object creation (otherwise \strictly_nothing)
   @*/
 public void birthday() {
   if (age >= 29) throw new ForeverYoungException();
   age++;
 } }
```

Method Contract Rule: Example Cont'd

Demo

`methods/useContractForBirthday.key`

- Prove without contracts (all except object creation)
  - Method treatment: Expand
- Prove with contracts (until method contract application)
  - Method treatment: Contract
- Prove used contracts
  - Method treatment: Expand
  - Select contracts for `birthday()`in `src/Person.java`
  - Prove both specification cases

Verification of Loops

Symbolic execution of loops: unwind

$$\text{unwindLoop}\ \frac{\Gamma \Longrightarrow \mathcal{U}[\pi\,\mathtt{\textbf{if}(b)\{p;\ \textbf{while}(b)\ p\}}\,\omega]\phi, \Delta}{\Gamma \Longrightarrow \mathcal{U}[\pi\,\mathtt{\textbf{while}(b)\ p}\,\omega]\phi, \Delta}$$

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How to handle a loop with...

- 0 iterations?

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Symbolic execution of loops: unwind

$$\text{unwindLoop}\ \frac{\Gamma \Longrightarrow \mathcal{U}[\pi\, \mathtt{\textbf{if}(b)\{p;\ \textbf{while}(b)\ p\}}\ \omega]\phi, \Delta}{\Gamma \Longrightarrow \mathcal{U}[\pi\, \mathtt{\textbf{while}(b)\ p}\ \omega]\phi, \Delta}$$

How to handle a loop with. . .

- 0 iterations? Unwind $1\times$

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How to handle a loop with...

- 0 iterations? Unwind $1\times$
- 10 iterations?

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- 0 iterations? Unwind $1\times$
- 10 iterations? Unwind $11\times$

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- an unknown number of iterations?

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- an unknown number of iterations?

We need an invariant rule (or some form of induction)

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Idea behind loop invariants

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How to Derive Loop Invariants Systematically?

Example (First active statement of symbolic execution is loop)

```
n >= 0 & wellFormed(heap) ==>
{i := 0}\[{
  while (i < n) {
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Look at desired postcondition `(i = n)`

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Is (`i <= n`) established at beginning and preserved?

Yes! We have found a suitable loop invariant!

Demo `loops/simple.key` (auto after inv)

Obtaining Invariants by Strengthening

Example (Slightly changed problem)

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Can we use something from the precondition or the update?

- If we know `(n = m)` then `(i <= n)` suffices
- Strengthen the invariant candidate to: `(i <= n & n = m)`

Generalization

Example (Addition: `x,y` program variables, `x0,y0` rigid constants)

```
x = x0 & y = y0 & y0 >= 0 & wellFormed(heap) ==>
\[{
  while (y > 0) {
    x = x + 1;
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First attempt: use postcondition `x = x0 + y0`

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Finding the invariant

First attempt: use postcondition `x = x0 + y0`

- Not true at start whenever `y0 > 0`
- Not preserved by loop, because `x` is increased

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What stays invariant?

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```

Finding the invariant

What stays invariant?

- The sum of `x` and `y`: `x + y = x0 + y0` "Generalization"
- Can help to think of "δ" between `x` and `x0 + y0`

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Is `x + y = x0 + y0` a good invariant?

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- Holds in the beginning and is preserved by loop

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- Holds in the beginning and is preserved by loop
- But postcondition not achieved by `x + y = x0 + y0 & y <= 0`

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Strenghtening the invariant

Postcondition holds if `y = 0`

- Add `y >= 0` to invariant: `x + y = x0 + y0 & y >= 0`

Demo `loops/simple3.key`

Basic Loop Invariant: Context Loss

Basic Invariant Rule: a Problem

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- Context Γ, Δ, $\mathcal{U}$ must be omitted in 2nd and 3rd premise:

  Γ, Δ in general don't hold in state reached by $\mathcal{U}$

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- But: context contains (part of) precondition and class invariants
- Better to add context information to loop invariant *Inv*

Example

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
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Example

Precondition: a $\neq$ **null**

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- How to erase all values of **assignable** locations?
- Anonymising updates $\mathcal{V}$ erase information about modified locations

Anonymising JAVA Locations

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To erase all knowledge about the values of the locations of the assignable expression:

- introduce a new (not yet used) constant of type int, e.g., c
- introduce a new (not yet used) constant of type Heap, e.g., $\mathtt{h}_{an}$
  - anonymise the current heap: $\mathtt{anon}(\mathtt{heap}, \mathtt{allFields}(\mathtt{this.a}), \mathtt{h}_{an})$
- compute anonymizing update for assignable locations

$$\mathcal{V} = \{\mathtt{i} := \mathtt{c} \mid\mid \mathtt{heap} := \mathtt{anon}(\mathtt{heap}, \mathtt{allFields}(\mathtt{this.a}), \mathtt{h}_{an})\}$$

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For local program variables (e.g., `i`) KeY computes assignable clause automatically

Loop Invariants Cont'd

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- Context is kept as far as possible:
  $\mathcal{V}$ wipes out only information in locations assignable in loop
- Invariant *Inv* does not need to include unmodified locations
- For **`assignable \everything`** (the default):
  - $\mathtt{heap} := \mathtt{anon}(\mathtt{heap}, \mathtt{allLocs}, \mathtt{h}_{an})$ wipes out **all** heap information
  - Equivalent to basic invariant rule
  - Avoid this! Always give a specific **`assignable`** clause

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Postcondition: $\forall\, \mathbf{int}\ x;\ (0 \leq x\ \&\ x < \mathtt{a.length} \rightarrow \mathtt{a}[x] = 1)$

Loop invariant: $0 \leq \mathtt{i}\ \ \&\ \ \mathtt{i} \leq \mathtt{a.length}$

Example with Improved Invariant Rule

Precondition: $\texttt{a} \neq \texttt{\textbf{null}}$

```
int i = 0;
while(i < a.length) {
    a[i] = 1;
    i++;
}
```

Postcondition: $\forall\, \texttt{\textbf{int}}\ x;\ (0 \leq x \ \&\ x < \texttt{a.length} \rightarrow \texttt{a}[x] = 1)$

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Example with Improved Invariant Rule

Precondition: $\mathtt{a} \neq \mathbf{null}$ & *ClassInv*

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Loop invariant: $0 \leq \mathtt{i} \;\&\; \mathtt{i} \leq \mathtt{a.length}$
$\&\; \forall\, \mathbf{int}\ x; (0 \leq x \,\&\, x < \mathtt{i} \rightarrow \mathtt{a}[x] = 1)$

Example in JML/JAVA – Loop.java Demo

```
public int[] a;
/*@ public normal_behavior
  @  ensures (\forall int x; 0<=x && x<a.length; a[x]==1);
  @  diverges true;
  @*/
public void m() {
  int i = 0;
  /*@ loop_invariant
    @  0 <= i && i <= a.length &&
    @  (\forall int x; 0<=x && x<i; a[x]==1);
    @ assignable a[*];
    @*/
  while(i < a.length) {
    a[i] = 1;
    i++;
  }
}
```

Example from a Previous Lecture

$\forall$ **int** x;
  $(x = \mathtt{n} \wedge x >= 0 \rightarrow$

```
[ i = 0; r = 0;
  while (i<n) { i = i + 1; r = r + i;}
  r=r+r-n;
] (r = x * x)
```

How can we prove that the above formula is valid
(i.e., satisfied in all states)?

Example from a Previous Lecture

$\forall$ **int** x;
$(x = \mathtt{n} \land x >= 0 \rightarrow$
$[$
```
i = 0; r = 0;
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$]\,(\mathtt{r} = x * x)$

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Needed Invariant:

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$\forall$ **int** x;
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    $[$ `i = 0; r = 0;`
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      `r=r+r-n;`
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How can we prove that the above formula is valid
(i.e., satisfied in all states)?

Needed Invariant:

```
@ loop_invariant
@     i>=0  && i <= n && 2*r == i*(i + 1);
@ assignable \nothing; // no heap locations changed
```

Example from a Previous Lecture

$\forall$ **int** x;
$(x = \mathtt{n} \land x >= 0 \rightarrow$
$[$ `i = 0; r = 0;`
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Demo Loop2.java

Hints

Proving `assignable`

- Invariant rule above assumes that **`assignable`** is correct
  E.g., possible to prove nonsense with incorrect
  `assignable \nothing;`
- Invariant rule of KeY generates proof obligation that ensures correctness of **`assignable`**
  This proof obligation is part of 'Body Preserves Invariant' branch

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Proving `assignable`

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  This proof obligation is part of 'Body Preserves Invariant' branch

Setting in the KeY Prover when proving loops

- Loop treatment: Invariant
- Quantifier treatment: No Splits with Progs
- If program contains `*`, `/`: Arithmetic treatment: DefOps
- Is search limit high enough (time out, rule apps.)?
- When proving partial correctness, add **`diverges true;`**

What is still missing?

Is the sequent

$$\Longrightarrow [\texttt{i = -1; while (true)\{\}}]\texttt{i} = 4711$$

provable?

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@ loop_invariant true;
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Yes, e.g.,

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@ loop_invariant true;
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Possible to prove correctness of non-terminating loop

- Invariant trivially initially valid and preserved ⇒
  Initial Case and Preserved Case immediately closable
- Loop condition never false: Use case immediately closable

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Possible to prove correctness of non-terminating loop

- Invariant trivially initially valid and preserved ⇒
  Initial Case and Preserved Case immediately closable
- Loop condition never false: Use case immediately closable

But need a method to prove termination of loops

Mapping Loop Execution to Well-Founded Order

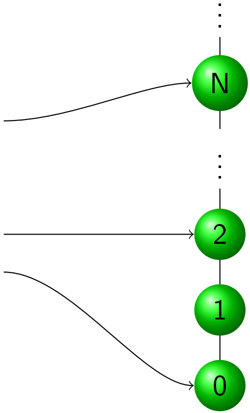
```
while (b) {
  body
}
```

if (b) { body $\}_1$

⋮

if (b) { body $\}_{17}$

if (b) { body $\}_{18}$



Need to find expression getting smaller wrt $\mathbb{N}$ in each iteration

Such an expression is called a decreasing term or variant

Total Correctness: Decreasing Term (Variant)

Find a decreasing integer term v (called variant)

Add the following premisses to the invariant rule:

- $v \geq 0$ is initially valid
- $v \geq 0$ is preserved by the loop body
- v is *strictly* decreased by the loop body

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Proving termination in JML/Java

- Remove directive `diverges true;` from contract
- Add directive `decreasing v;` to loop invariant
- KeY creates suitable invariant rule and PO (with $\langle \ldots \rangle \phi$)

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Example (The `array` loop)

```
@ decreasing
```

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```
@ decreasing  a.length - i;
```

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Example (The `array` loop)

```
@ decreasing  a.length - i;
```

Files:

- `LoopT.java`
- `Loop2T.java`

Final Example: Computing the GCD

```
public class Gcd {
 /*@ public normal_behavior
   @ requires _small>=0 && _big>=_small;
   @ ensures _big!=0 ==>
   @  (_big % \result == 0 && _small % \result == 0 &&
   @    (\forall int x; x>0 && _big % x == 0
   @       && _small % x == 0; \result % x == 0));
   @ assignable \nothing;
   @*/
 private static int gcdHelp(int _big, int _small) {
   int big = _big; int small = _small;
   while (small != 0) {
     final int t = big % small;
     big = small;
     small = t;
   }
   return big;
 }
}
```

Computing the GCD: Method Specification

```
public class Gcd {
 /*@ public normal_behavior
   @ requires _small>=0 && _big>=_small;
   @ ensures _big!=0 ==>
   @ (_big % \result == 0 && _small % \result == 0 &&
   @   (\forall int x; x>0 && _big % x == 0
   @      && _small % x == 0; \result % x == 0));
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 private static int gcdHelp(int _big, int _small) {...}
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requires normalization assumptions on method parameters (both non-negative and `_big` $\geq$ `_small`)

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ensures if `_big` positive, then

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ensures if `_big` positive, then

- the return value `\result` is a divider of both arguments

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public class Gcd {
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   @   (\forall int x; x>0 && _big % x == 0
   @     && _small % x == 0; \result % x == 0));
   @ assignable \nothing;
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 private static int gcdHelp(int _big, int _small) {...}
```

requires normalization assumptions on method parameters (both non-negative and `_big` $\geq$ `_small`)

ensures if `_big` positive, then

- the return value `\result` is a divider of both arguments
- all other dividers `x` of the arguments are also dividers of `\result` and thus smaller or equal to `\result`

Computing the GCD: Specify the Loop Body

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Which locations are changed (at most)?

Computing the GCD: Specify the Loop Body

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int big = _big; int small = _small;
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Which locations are changed (at most)?

```
@ assignable \nothing; // no heap locations changed
```

What is the variant?

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int big = _big; int small = _small;
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}
return big;
```

Which locations are changed (at most)?

```
@ assignable \nothing; // no heap locations changed
```

What is the variant?

```
@ decreases small;
```

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
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}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`

Computing the GCD: Specify the Loop Body Cont'd

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int big = _big; int small = _small;
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- Order between `small` and `big` preserved by loop: `big>=small`
- Possible for `big` to become 0 in a loop iteration?

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
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  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Possible for `big` to become 0 in a loop iteration? No.

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Adding `big>0` to loop invariant?

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Adding `big>0` to loop invariant? No. Not initially valid.

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
while (small != 0) {
  final int t = big % small;
  big = small;
  small = t;
}
return big;
```

Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Weaker condition necessary: `big==0 ==> _big==0`

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
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- Order between `small` and `big` preserved by loop: `big>=small`
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- What does the loop preserve?

Computing the GCD: Specify the Loop Body Cont'd

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Loop Invariant

- Order between `small` and `big` preserved by loop: `big>=small`
- Weaker condition necessary: `big==0 ==> _big==0`
- What does the loop preserve? The set of dividers!
  All common dividers of `_big, _small` are also dividers of `big, small`

Computing the GCD: Specify the Loop Body Cont'd

```
int big = _big; int small = _small;
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- Order between `small` and `big` preserved by loop: `big>=small`
- Weaker condition necessary: `big==0 ==> _big==0`
- What does the loop preserve? The set of dividers!

  All common dividers of `_big`, `_small` are also dividers of `big`, `small`

```
(\forall int x; x > 0;
                (_big%x == 0 && _small%x == 0)
                <==>
                (big%x == 0 && small%x == 0));
```

Computing the GCD: Final Specification

```
int big = _big; int small = _small;
/*@ loop_invariant small >= 0 && big >= small &&
  @   (big == 0 ==> _big == 0) &&
  @   (\forall int x; x > 0; (_big % x == 0 && _small % x == 0)
  @                          <==>
  @                          (big % x == 0 && small % x == 0));
  @ decreases small;
  @ assignable \nothing;
  @*/
  while (small != 0) {
    final int t = big % small;
    big = small;
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  }
  return big; // assigned to \result
```

Computing the GCD: Final Specification

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Why does `big` divides `_small` and `_big` follow from the loop invariant?

Computing the GCD: Final Specification

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    big = small;
    small = t;
  }
  return big; // assigned to \result
```

Why does `big` divides `_small` and `_big` follow from the loop invariant?

If `big` is positive, one can instantiate `x` with it, and use `small == 0`

Computing the GCD: Demo

Demo `loops/Gcd.java`

1. Show `Gcd.java` and `gcd(a,b)`
2. Ensure that "DefOps" and "Contracts" is selected, $\geq$ 10,000 steps
3. Proof contract of `gcd()`, using contract of `gcdHelp()`
4. Note KeY check sign in parentheses:
   - 4.1 Click "Proof Management"
   - 4.2 Choose tab "By Proof"
   - 4.3 Select proof of `gcd()`
   - 4.4 Select used method contract of `gcdHelp()`
   - 4.5 Click "Start Proof"
5. After finishing proof obligations of `gcdHelp()` parentheses are gone

Some Hints On Finding Invariants

General Advice

- Invariants must be developed, they don't come out of thin air!
- Be as systematic in deriving invariants as when debugging a program

Some Hints On Finding Invariants, Cont'd

Technical Hints

- The desired postcondition is a good starting point
  - What, in addition to negated loop guard, is needed for it to hold?

Some Hints On Finding Invariants, Cont'd

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- The desired postcondition is a good starting point
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  - Can you add stuff from the precondition?
  - Does it need strengthening?
  - Try to express the relation between partial and final result

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Some Hints On Finding Invariants, Cont'd

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- Simulate a few loop body executions to discover invariant patterns
- If the invariant is not initially valid:
  - Can it be weakened such that the postcondition still follows?
  - Did you forget an assumption in the requires clause?
- Several "rounds" of weakening/strengthening might be required
- Use the KeY tool for each premiss of invariant rule
  - After each change of the invariant make sure all cases are ok
  - Interactive dialogue: previous invariants available in "Alt" tabs

Understanding Unclosed Proofs

Reasons why a proof may not close

- Buggy or incomplete specification
- Bug in program
- Maximal number of steps reached: restart or increase # of steps
- Automatic proof search fails: manual rule applications necessary

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Understanding open proof goals

- Follow the control flow from the proof root to the open goal
- Branch labels give useful hints
- Identify unprovable part of post condition or invariant
- Sequent remains always in "pre-state"
  Constraints on program variables refer to value at start of program
  (exception: formula is behind update or modality)
- NB: $\Gamma \Longrightarrow o = \texttt{null}, \Delta$ is equivalent to $\Gamma, o \neq \texttt{null} \Longrightarrow \Delta$

Literature for this Lecture

- W. Ahrendt, S. Grebing, Using the KeY Prover
  to appear in the new KeY Book, end 2016
  (available via Google group or personal request)
- B. Beckert, V. Klebanov, and B. Weiß, Dynamic Logic for Java
  to appear in the new KeY Book, end 2016
  (available via Google group or personal request)
  Section 3.7