Examination, Models of Computation (DIT310/TDA183/TDA184)

- Date and time: 2017-01-10, 14:00–18:00.
- Author[1]/responsible/examiner: Nils Anders Danielsson. Telephone number: 1680. Visits to the examination rooms: ∼15:00 and ∼17:00.
- Authorised aids (except for aids that are always permitted): None.
- The GU grades Pass (G) and Pass with Distinction (VG) correspond to the Chalmers grades 3 and 5, respectively.
- To get grade n on the exam you have to be awarded grade n or higher on at least n exercises.
- A completely correct solution of one exercise is awarded the grade 5. Solutions with minor mistakes *might* get the grade 5, and solutions with larger mistakes might get lower grades.
- Exercises can contain parts and/or requirements that are only required for a certain grade (or higher). To get grade n on such an exercise you have to get grade n or higher on every part marked with grade n or lower, and you have to fulfil every requirement marked with grade n or lower.
- Do not hand in solutions for several exercises on the same sheet.
- Write your examination code on each sheet.
- Solutions can be rejected if they are hard to read, unstructured, or poorly motivated.
- After correction the graded exams are available in the student office in room 4482 of the EDIT building. If you want to discuss the grading you can, within three weeks after the result has been reported, contact the examiner and set up a time for a meeting (in which case you should not remove the exam from the student office).

[1]Thanks to Daniel Schoepe for feedback.

1. (a) *For grade 3:* Give an example of a set A for which $A \to A$ is countable, and give an example of a set B for which $B \to B$ is not countable. You do not need to provide proofs.

   (b) *For grade 4:* Either prove that the set

   $$\{ f \in \mathbb{N} \to \mathbb{N} \mid f \text{ is } \chi\text{-computable} \}$$

   is countable, or that it is not countable.

2. Give the standard χ encoding, as presented in the lectures, of the following χ expression: $\lambda x.\ x\ x$. Assume that the variable x corresponds to the number 0. Use concrete syntax in your answer.

3. If $f \in \mathbb{N} \to \mathbb{N}$ and $g \in \mathbb{N} \to \mathbb{N}$ are both χ-computable, is the composition $g \circ f$, i.e. the function mapping natural numbers n to $g\ (f\ n)$, always χ-computable?

   For grade 3: Motivate your answer.

   For grade 4: Provide a proof.

   For grade 5: You may not use Rice's theorem.

4. Is the following function χ-decidable?

   $$\begin{aligned} &f \in \mathit{CExp} \to \mathit{Bool} \\ &f\ e = \textbf{if}\ [\![e]\!] = \ulcorner 35 \urcorner\ \textbf{then}\ \mathsf{true}\ \textbf{else}\ \mathsf{false} \end{aligned}$$

   Here *CExp* is a set containing the abstract syntax of every closed χ expression, and $\ulcorner 35 \urcorner$ is the standard encoding of the natural number 35.

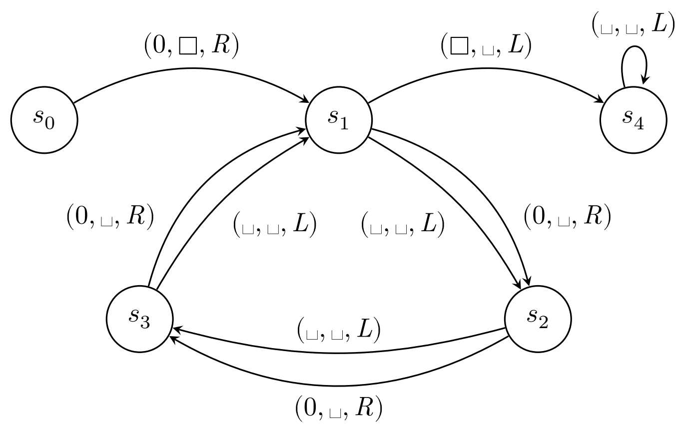
   For grade 3: Motivate your answer.

   For grade 4: Provide a proof.

   For grade 5: You may not use Rice's theorem.

5. Consider the following Turing machine:

- Input alphabet: $\{ 0 \}$.
- Tape alphabet: $\{ 0, \square, ␣ \}$.
- States: $\{ s_0, s_1, s_2, s_3, s_4 \}$.
- Initial state: s_0.
- Transition function:



(a) *For grade 3:* If this Turing machine is run with 000 as the input string, then it halts successfully. What is the resulting string?

(b) *For grade 4:* Does this Turing machine halt successfully (with the head positioned over the leftmost square) for every possible input? Provide a proof.

6. Define an extension of χ, let us call it $\overline{\chi}$, with an additional expression former that solves the halting problem for the unmodified language χ. (You may note that it is hard to implement this language using regular hardware, but your task is not to implement the language.)

   The abstract syntax of χ is extended with the constructor halts:

$$\frac{e \in \mathit{Exp}}{\mathsf{halts}\ e \in \mathit{Exp}}$$

   Substitution is defined in the following way for the new constructor:

$$\mathsf{halts}\ e\ [x \leftarrow e'] = \mathsf{halts}\ (e\ [x \leftarrow e'])$$

   Your task is to extend the operational semantics of χ with one or more inference rules for the constructor halts, in such a way that the following properties hold:

   - *For grade 3:* The extended semantics must be deterministic. You do not have to prove that this is the case.
   - *For grade 3:* The halting problem for χ must be $\overline{\chi}$-decidable, i.e., there must be a closed $\overline{\chi}$ expression $\underline{\mathit{halts}}$ such that, for every closed χ expression e, if e terminates with a value (according to the unmodified semantics), then

$$[\![\mathsf{apply}\ \underline{\mathit{halts}}\ \ulcorner e \urcorner]\!] = \ulcorner \mathsf{true} \urcorner,$$

     and otherwise

$$[\![\mathsf{apply}\ \underline{\mathit{halts}}\ \ulcorner e \urcorner]\!] = \ulcorner \mathsf{false} \urcorner.$$

     For grade 4: Prove that this property holds.
   - *For grade 5:* The halting problem, formulated for the extended language $\overline{\chi}$, must not be $\overline{\chi}$-decidable. Provide a convincing argument (not necessarily a detailed proof) for why this is the case.