Logic in Computer Science

For a given language $\mathcal{F}, \mathcal{P}$, a *first-order theory* is a set T of sentences (closed formulae) in this given language. The elements of T are also called *axioms* of T.

A model of T is a model $\mathcal{M}$ of the given language such that $\mathcal{M} \models \psi$ for all sentences ψ in T.

$T \vdash \varphi$ means that we can find $\psi_1, \dots, \psi_n$ in T such that $\psi_1, \dots, \psi_n \vdash \varphi$.
$T \models \varphi$ means that $\mathcal{M} \models \varphi$ for all models $\mathcal{M}$ of T.

The generalized form of *soundness* is that $T \vdash \varphi$ implies $T \models \varphi$ and *completness* is that $T \models \varphi$ implies $T \vdash \varphi$.

If T is a finite set $\psi_1, \dots, \psi_n$ this follows from the usual statement of soundness ($\vdash \delta$ implies $\models \delta$) and completness ($\models \delta$ implies $\vdash \delta$). Indeed, in this case, we have $T \vdash \varphi$ iff $\vdash (\psi_1 \wedge \dots \wedge \psi_n) \to \varphi$ and $T \models \varphi$ iff $\models (\psi_1 \wedge \dots \wedge \psi_n) \to \varphi$.

Compactness Theorem

Theorem 0.1 *A theory has a model iff any of its finite subtheory has a model*

Application 1: non-standard model

We recall that the theory of Peano arithmetic PA is a theory for the language $\mathcal{F} = \{\mathsf{zero}, \mathrm{S}, +, \cdot\}$ and with no predicate symbol apart from equality. We add the special constant u with the axioms

$$u \neq \mathsf{zero},\ u \neq \mathrm{S}(\mathsf{zero}),\ u \neq \mathrm{S}(\mathrm{S}(\mathsf{zero})),\ \dots$$

By the Compactness Theorem, this theory has a model. The domain of this model has to contain an element which is different from the semantics of zero, $\mathrm{S}(\mathsf{zero})$, $\mathrm{S}(\mathrm{S}(\mathsf{zero})), \dots$ This is a *non standard* model of arithmetic.

Application 2: transitive closure is not first-order definable

In the language with one binary relation symbol R and two constant a, b, we can state

Theorem 0.2 *There is no formula φ such that $\mathcal{M} \models \varphi$ iff there is a path from $a^{\mathcal{M}}$ to $b^{\mathcal{M}}$*

Indeed, if there was such a formula φ then the theory $\varphi,\ \neg\delta_0,\ \neg\delta_1, \dots$ would be consistent, by the Compactness Theorem, where δ_0 is $a = b$ and δ_{n+1} is $\delta_n \vee \exists z_1\ \dots\ z_n.R(a, z_1) \wedge \dots \wedge R(z_n, b)$. But this is a contraction.

Application 3: to be well-founded is not first-order definable

We recall that a relation S is well-founded iff there is no infinite sequence $x_0,\ x_1, \dots$ such that $S(x_0, x_1),\ S(x_1, x_2), \dots$. In the language with one binary relation symbol R we can state

Theorem 0.3 *There is no formula φ such that $\mathcal{M} \models \varphi$ iff $R^{\mathcal{M}}$ is well-founded.*

We add to the language infinitely many constants $a_0,\ a_1,\ a_2,\ \dots$ and, if there is such a formula φ, we consider the theory

$$\varphi, R(a_0, a_1),\ R(a_1, a_2),\ R(a_2, a_3), \dots$$

By the Compactness Theorem, this theory has a model, which is a contradiction.

Three traditions in logic

Before starting the presentation of Linear Temporal Logic, I started to recall the 3 traditions in logic, that are important for propositional logic (and temporal logics)

1. model theory
2. proof theory
3. algebraic logic

We present this in the case of propositional logic, where the syntax is

$$\varphi \ ::= \ p \mid \neg\varphi \mid \varphi \to \varphi$$

where p ranges over atoms. We can then define $\psi_0 \vee \psi_1 = \neg\psi_0 \to \psi_1$ and $\psi_0 \wedge \psi_1 = \neg(\psi_0 \to \neg\psi_1)$.

Model Theory

In the model theoretic approach, we start by defining what is a model α which is a function from the atomic formulae to $\{0, 1\}$. We then define $\alpha \models \varphi$ by induction on φ.

We write $\models \varphi$ iff $\alpha \models \varphi$ for all models α.

Proof Theory

In the proof theoretic approach, we define when φ is *derivable*, notation $\vdash \varphi$, and more generally, when φ is derivable from hypotheses $\psi_1, \dots, \psi_k$, notation $\psi_1, \dots, \psi_k \vdash \varphi$.

In this course, we presented this following the notion of *natural deduction*.

Another way to present the notion of derivability is via the so-called notion of *Hilbert-style* proof system (which was actually already in Frege). It consists in giving some axioms and to say that φ is derivable iff we can build a derivation tree using as the only derivation rule the *modus-ponens rule*

$$\frac{\psi \qquad \psi \to \delta}{\delta}$$

and the leaves are axioms.

For instance, for proposition a possible axiom system is the given by the 3 axiom schemas

- $\varphi \to \psi \to \varphi$
- $(\varphi \to \psi \to \delta) \to (\varphi \to \psi) \to \varphi \to \delta$
- $(\neg\varphi \to \psi) \to (\neg\varphi \to \neg\psi) \to \varphi$

With this presentation it is not at all obvious that, e.g. $p \to p$ is derivable!

Both presentations are actually equivalent, and we have $\vdash \varphi$ iff $\models \varphi$.

Algebraic logic

An important remark if that, if we define $\varphi \equiv \psi$ by $\alpha \models \varphi$ iff $\alpha \models \psi$ (or equivalently $\vdash \varphi \to \psi$ and $\vdash \psi \to \varphi$), then we have the rules

$$\frac{\varphi \equiv \psi}{\neg\varphi \equiv \neg\psi} \qquad\qquad \frac{\varphi_0 \equiv \psi_0 \qquad \varphi_1 \equiv \psi_1}{\varphi_0 \to \varphi_1 \equiv \psi_0 \to \psi_1}$$

It is then natural to write simply $\varphi = \psi$ instead of $\varphi \equiv \psi$ and to consider that we have two operations (negation and implication). It is also natural to write $\varphi \leqslant \psi$ instead of $\vdash \varphi \to \psi$.

We see then the set of formulae as a set equipped with some operations, satisfying some algebraic laws (e.g. $1 = p \to p$). The relation $\leqslant$ is a poset relation.

This was the view of logic coming from Boole (1815-1864). One can consider more generally algebras satisfying the same laws as the one of proposition formulae, and these are called Boolean algebras. In term of the operations $\neg, \vee$, one possible list of equational axioms for Boolean algebra is

$$x \vee (y \vee z) = (x \vee y) \vee z \qquad x \vee y = y \vee x \qquad x \vee 1 = 1 \qquad x \vee 0 = x$$

$$x \wedge (y \wedge z) = (x \wedge y) \wedge z \qquad x \wedge y = y \wedge x \qquad x \wedge 1 = x \qquad x \wedge 0 = 0$$

$$\neg(x \vee y) = \neg x \wedge (\neg y) \qquad 1 = \neg\, 0 \qquad 0 = \neg 1 \qquad \neg(\neg x) = x$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \qquad x \wedge (x \vee y) = x$$

In the algebraic approach, we can consider more general algebras than the algebras of propositional formulae.

In this approach, a natural question is how to solve equations. For instance, it can be shown (exercise) that the equation in x

$$(x \wedge b) \vee (\neg x \wedge (a \vee \neg b)) = 1$$

has exactly the solution $x = b \wedge (\neg a \vee u)$ where u is arbitrary.

For propositional logic, these three approaches, model theoretic, proof theoretic and algebraic are equivalent, but they provide very different intuitions.