Homework 3

Exercise 1: Find closed λ-terms F such that

1. $F\ x = F$ (called the "eater")
2. $F\ x = x\ F$

Exercise 2: We consider a type T with a constant $a : T$. Find two pairs (t, u) of terms $t : T \to T$ and $u : T$ such that $t\ u = a$.

Exercise 3: We recall the nameless presentation of typed lambda-calculus with

$$t\ ::=\ n \mid \lambda T.t \mid t\ t \mid bv \qquad bv\ ::=\ \mathsf{true} \mid \mathsf{false}$$

$$n\ ::=\ 0 \mid n+1 \qquad T\ ::=\ \mathsf{Bool} \mid T \to T$$

We use also sequences of terms $ts\ ::=\ () \mid (ts, t)$ and contexts $\Gamma, \Delta\ ::=\ () \mid \Gamma.T$.

Define in Agda the typing relation $\Gamma \vdash t : T$. From this we can define the relation $\Delta \vdash ts : \Gamma$ by $\Delta \vdash () : ()$ and $\Delta \vdash (ts, t) : \Gamma.T$ if $\Delta \vdash ts : \Gamma$ and $\Delta \vdash t : T$.

Define in Agda a substitution operation $u[ts]$ such that $() \vdash u[ts] : T$ given $\Gamma \vdash t : T$ and $() \vdash ts : \Gamma$. (Hint: One can define first the concatenation Γ, Δ of two contexts and define more generally $\Delta \vdash u[ts] : T$ if $() \vdash ts : \Gamma$ and $\Gamma, \Delta \vdash u : T$.)

Exercise 4: Show that a lambda term in normal form can be written $\lambda x_1 : T_1. \dots \lambda x_k : T_k.\ x\ M_1 \dots M_l$ where we can have $k = 0$ or $l = 0$ and $M_1, \dots, M_l$ are in normal form. If $k = 0$ the term is of the form $x\ M_1\ \dots\ M_l$ and if $l = 0$ the term is of the form $\lambda x_1 \dots \lambda x_k\ x$. Another way to state this is that we have the following grammar for terms in normal form

$$N\ ::=\ \lambda x : T.N \mid K \qquad K\ ::=\ x \mid K\ N$$

Use this to enumerate the closed terms of the following types (ι is a ground type)

1. $\iota \to \iota$
2. $\iota \to \iota \to \iota$
3. $(\iota \to \iota) \to \iota \to \iota$
4. $\iota \to (\iota \to \iota) \to \iota$
5. $(\iota \to \iota) \to \iota$
6. $((\iota \to \iota) \to \iota) \to \iota$