Sets and propositions

Sets are important in mathematics, not least for computer scientists. It is sometimes said that set theory is a foundation for mathematics, but this is a rather dubious claim, even though set theory is an important discipline in its own right.

Mathematics doesn't need foundations; it has wings. Reuben Hersh

All of us deal with sets and operations on them each and every day ("My hairdresser is available Tuesday, Thursday and Friday. I'm free Monday, Tuesday and Friday. Which days am I definitely not getting a haircut?").

- That the element x belongs to the set A is denoted $x \in A$. That x does *not* belong to A is denoted $x \notin A$.
- The *intersection* of the sets A and B, denoted $A \cap B$, is the set of elements that belong to both A and B.
- The *union* of A and B, denoted $A \cup B$, is the set of elements that belong to A or B (or both).
- The set A is a *subset* of B, denoted $A \subseteq B$, if each element of A belongs to B.
- The *difference* of A and B, denoted $A \setminus B$, is the set of elements that belong to A but not B.
- If a set U is given, and A is a subset of U, then we define the *complement of A with respect to U* as the set of elements in U that do *not* belong to A. This set is denoted $\overline{A}$ so $\overline{A} = U \setminus A$. It is often clear from the circumstances what the set U is. If we talk about the complement of the set of real numbers greater than 5, then most of us would assume that U was the set of all real numbers. Often it doesn't matter what set U is, for example in (3) and (4) below.
- There is a set with no elements. There is only one such set, so it has a special name, *the empty set*, and it is denoted $\emptyset$. It may sound absurd, but every proposition (statement) about elements in the empty set is true. This is because in mathematics (and elsewhere) a proposition must be either true or false. Since a proposition can not be false for any element in the empty set (there are no elements for which the proposition can be false), such a proportion must be true.
- The *direct product*, or *Cartesian product*, of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. For example, we have

  $$\{1, 2, 3\} \times \{a, b\} = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}.$$

  Observe that in an ordered pair, the order of the elements matters. Thus, $(1, a)$ is not the same as $(a, 1)$.

Here are a few basic properties of sets:

$$A \cap (B \cup C) = (A \cap B) \cup (B \cap C) \tag{1}$$
$$A \cup (B \cap C) = (A \cup B) \cap (B \cup C) \tag{2}$$
$$\overline{(A \cap B)} = \overline{A} \cup \overline{B} \tag{3}$$
$$\overline{(A \cup B)} = \overline{A} \cap \overline{B} \tag{4}$$

In what follows there are a few problems on sets. You should make sure you understand the concepts and that you understand the answers to the exercises.

Ekki kíkja á svörin fyrr en þú ert búin(n) að svara öllum spurningunum and sannfæra þig um að svörin séu rétt!

1. Let A and B be two sets. Draw a Venn diagram for the following sets, describe them symbolically, say which ones of them are identical, explain *in words* why they are, and then prove your claims, using the identities above:
   (a) The intersection of the complement of A and the complement of B.
   (b) The union of the complement of A and the complement of B.
   (c) The complement of the intersection of A and B.
   (d) The complement of the union of A and B.
   (e) The union of C and the intersection of A and B.
   (f) The intersection of the union of A and C, on the one hand, and the union of B and C on the other.

Answer?: The edge graph of the n-dimensional unit cube is not planar if $n > 3$

2. Why do you rarely see Venn diagrams for more than three sets?
3. Express the hairdressing problem in the introduction in the language of sets.
4. That the proposition "there is an integer whose square is negative" is false is equivalent to saying that the following proposition is true: "Every integer has a square that is not negative." Express the *negation* of the following propositions in a nice way (don't just put a negation in front!).
   (a) There is an RU student who has blonde hair and brown eyes.
   (b) Every RU student has blond hair and brown eyes.
   (c) No RU student has brown eyes unless she/he also has blonde hair.
   (d) Can the following propositions both be true (simultaneously)?:
      i. Every blonde New Yorker is less than ten years old.
      ii. Every New Yorker with brown hair is more than ten years old.

      What could we say about the number of New Yorkers who are both blonde and brown eyed if both propositions are true?
5. What can we say about M if M is a subset of the set of all integers and the following statement is true?: The square of each number in M is negative.

Answers (not guaranteed to be right): 1. (a) and (d) are equal, (b) and (c) are equal, (e) and (f) are equal. 3. One possibility: NOT = $\overline{(\text{ME} \cup \text{SHE})}$. 4. (a) No RU student is both blonde and brown eyed. (b) There is an RU student who is neither blonde nor brown eyed. (c) There is an RU student who is brown eyed and not blonde. (d) If both statements are true, then the number of blonde, brown eyed New Yorkers is zero.