TIN092 — Algorithms

Quiz: Preliminaries

Willard Rafnsson & Bassel Mannaa

August 25, 2011

Solve as much of this quiz as you can without *any* form of help. Make note of everything you look up / struggle with / do not understand, and approach us if it turns out to be a lot, so we can help you identify what to recap (don't be shy/embarrassed). Starred exercises are more challenging (struggling with those is okay).

1 Sets and Functions

1. Let A, B, C be subsets of a set U, Which of the following hold, and why?
   a) $U \setminus (A \cap B) = (U \setminus A) \cup (U \setminus B)$
   b) $A \setminus B = A \cap (U \setminus B)$

2. Let A, B be sets. Using only set notation, complete the following definitions.
   a) "A function f from A to B is..."
   b) "A binary relation R between A and B is..."

3. Let A, B, C be sets. $|A| = m$ and $|B| = n$. ($|S|$ is the number of elements in S)
   a) How many elements does $A \times B$ have?
   b) How many elements does $A \cup B$ have?
   c) Does $A \times (B \times C) = (A \times B) \times C$ hold? Justify your answer.
   *d) Give an isomorphism between $A \times (B \times C)$ and $(A \times B) \times C$.

4. Consider the following function for computing the nth number in the Fibonacci sequence.

   $$f(0) = 0 \qquad f(1) = 1 \qquad f(n+2) = f(n+1) + f(n)$$

   What is $f(5)$? Draw the call tree for this computation.
   *: Try to think of a way to make the computation more efficient.

5. The following imperative procedure yields the biggest integer in an array.

```
procedure maxA(int[] A){
  int max = A[0];
  for ( int i = 1 ; i < A.length ; i++) {
    if ( A[i] > max ){ max = A[i]; }
  }
}
```

With integer lists defined inductively (like in Haskell) as

- [] is a list (the empty one).
- if X is a list and x an integer, then $x :: X$ is a list.

write a function `maxL` which yields the biggest integer in a list.

*6. Some programming language constructs (methods in Java/C#, functions in C/C++), which are sometimes called functions in-literature/by-programmers, are *not* functions in the mathematical sense. Which property of mathematical functions do these constructs violate?

2 Data Structures

7. In exercise 5, we saw an inductive definition of lists of integers. Give an inductive definition of binary trees of integers.

8. What is a heap (the data structure)?

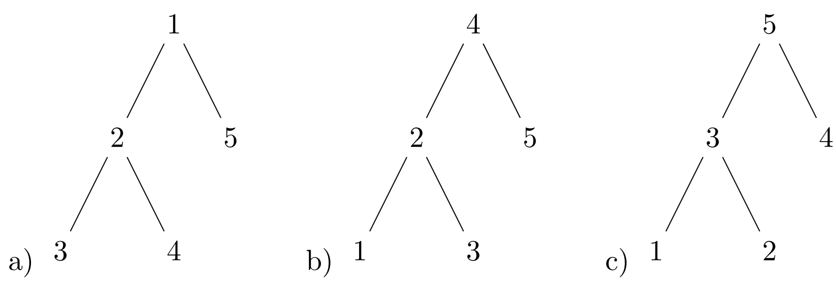
9. Write pseudocode which applies the function `foo` (discarding the result) to each node of a given tree. You should write three versions of this algorithm, one for each of the three standard ways of traversing a tree.

   a) preorder

   b) inorder

   c) postorder

   As a reminder, here is the order in which these traversal algorithms traverse an example binary tree.



10. Using only set notation (when needed), complete the following definitions.
    a) "A graph G is..."
    b) "A directed graph G is..."
    c) "A clique is..."

11. Give a data representation (using data structures) of graphs. Try implementing it.

12. Using your data representation for a graph, write pseudocode (or actual code) which applies the function `foo` (discarding the result) to each vertex of a given graph. You should write two versions of this algorithm, one for each of the two standard ways of traversing a graph.
    a) breadth-first
    b) depth-first

3 Prelude to Asymptotics

13. A polynomial is a function of its variables. Below there are 7 functions (polynomials), one of each of constant, logarithmic, linear, quadratic, cubic, and exponential. Determine which function has which of these properties.
    a) $11x + 189$
    b) $4x^2 + 3x + 2$
    c) $4\frac{2x}{x}$
    d) $2 \log 2x$
    e) $2^{x+1} + 5$
    f) $2\frac{x^4+2x}{2x}$

14. We say a function f (eventually) dominates g if, for some x_0, it holds that for all $x \geq x_0$, $|f(x)| \geq |g(x)|$. Visually, when plotting $|f|$ and $|g|$ ($|\cdot|$ is the absolute value function), $|f|$ will, at some point, go "over" $|g|$, and stay over $|g|$ forever.
    a) does $64x$ dominate x^2?
    b) does x dominate 1024?
    c) does $\log x$ dominate $\sqrt{x}$?
    d) does $4x^2 + 42$ dominate 1.1^x?

15. Consider the simple imperative programming language, Imp, given by the grammar

$$
\begin{array}{lcl}
e & ::= & n \\
  & | & x \\
  & | & (e) \\
  & | & e \odot e \\
\odot & ::= & + \mid - \mid * \mid / \mid \%
\end{array}
\qquad
\begin{array}{lcl}
s & ::= & \texttt{skip} \\
  & | & x := e \\
  & | & s \,\texttt{;}\, s \\
  & | & \texttt{if } e \texttt{ then } s \texttt{ else } s \texttt{ end} \\
  & | & \texttt{while } e \texttt{ do } s \texttt{ end}
\end{array}
$$

n is an integer, e an expression, $\odot$ an operator, and s a statement. Integer expression evaluation follows the standard rules of arithmetic (% is the modulus operator (remainder of division)). If a variable has not been assigned a value in memory, reading it yields the default value 0. For conditionals in `if...` and `while...`, 0 counts as Boolean false, and any other integer value as Boolean true.

A program in Imp can be seen as a function which maps a given initial memory, to a final memory. The semantics of an Imp program can thus be given as a function $\mathsf{eval}(s, m) = m'$, where m is the initial memory for s, and m' the final memory. The memories themselves are also functions mapping variables to their values.

For instance, running

```
result := 1 ; tmp := 1 ; i := 0 ;
while input − i + 1 do
  if 2 ∗ tmp − i then
    skip
  else
    result := result + 1 ;
    tmp := i
  end ;
  i := i + 1
end
```

on any memory m with $m(input) = 9$ yields a memory m' with $m'(result) = 4$.

a) Explain in words what this program is computing when $input \geq 0$ initially (try following how the program computes with initial *input* set to 0..10).

b) An algorithm is efficient when it uses few machine instructions to obtain its goal. Instead of counting machine instructions in compiled programs, we can get a good estimate of efficiency by counting high-level "operations". For instance, we can define $\mathsf{ops}(e)$, the number of operations occurring when evaluating e, as

$$\begin{aligned} \mathsf{ops}(n) &= 0 \\ \mathsf{ops}(x) &= 1 \\ \mathsf{ops}((e)) &= \mathsf{ops}(e) \\ \mathsf{ops}(e_1 \odot e_2) &= \mathsf{ops}(e_1) + \mathsf{ops}(e_2) + 1 \end{aligned}$$

For instance, $\mathsf{ops}(2 * x + 3) = 3$.
Define $\mathsf{ops}(s, m)$. (hint: use eval)

c) Using your definition of ops, count the number of operations performed by the above program for initial value of *input* *i)* 4, *ii)* 7 and *iii)* 8.

*d) Give a function of n which will dominate the number operations performed by the above program with input set to n. Try to make this function as "small" as possible (don't pick 2^n, for instance).

*e) The program given above is suboptimal. Write a more efficient version of it, and redo *d), such that the new function is dominated by the old one.

4 Logic and Proofs

16. Use mathematical induction to prove the following
    a) $1^2 + 2^2 + ... + n^2 = n(n+1)(2n+1)/6$ where n is a positive integer.
    b) 2 divides $n^2 + n$ whenever n is a positive integer.

17. Let us define by recursion the following two functions $f, g : \mathbb{N} \to \mathbb{N}$ as.

$$f(0) = 0 \qquad\qquad g(0) = 1$$
$$f(n+1) = g(n) \qquad\qquad g(n+1) = f(n).$$

    a) Show by mathematical induction that for all n, we have

$$f(n) + g(n) = 1 \qquad\qquad f(n)g(n) = 0$$

    b) Show by mutual induction that $f(n) = 0$ iff $g(n) = 1$ iff n is even and that $f(n) = 1$ iff $g(n) = 0$ iff n is odd.

18. Let $P(n)$ be the proposition "all horses in a set of n horses are of the same color."
Basis Step: clearly $P(1)$ is true.
Inductive Step: Assume that $P(k)$ is true, so that all the horses in any set of k horses are the same color and consider any $k+1$ horses. Number these horses $1, 2, 3, ..., k, k+1$. The first k of these horses, $1, 2, ..., k$, all must have the same color by induction hypothesis. Also by induction hypothesis the set of k horses numbered $2, 3, ..., K+1$ have the same color. These two sets overlap and thus $P(k+1)$ is true.
What is wrong with this proof?

19. "All the elephants on the moon are orange." True or false?

*20. The quantified Boolean formula "$\forall q.\, \exists p.\, p > q \wedge \forall x, y > 1.\, xy \neq p$" ($p, q, x, y \in \mathbb{N}$) expresses a well-known mathematical truth. Which?

5 Combinatorics

21. A bit string is a string of $0's$ and $1's$. How many bit strings of length n are there?

22. How many functions are there from the set $\{1, 2, ..., n\}$ to the set $\{0, 1\}$, n is a positive integer?

23. A photographer at a wedding is arranging 6 people (including the bride and groom) for a photo. In how many ways can he arrange them if
    a) The bride must be next to the groom.
    b) The bride must not be next to the groom.
    c) The bride is somewhere (not necessarily next to) to the left of the groom.

24. How many distinct subsets...
    a) ...does a set of 5 elements have?
    b) ...does $\emptyset$ have?
    c) ...of 2 elements does a set of 5 elements have?
    d) ...does a set of n elements have?
    e) ...of k elements does a set of n elements have?

25. How many different ways can a set of 3 student representatives be selected from a class of 90 students?

26. A department contains 7 women and 9 men; How many ways are there to select a committee of five members of the department if at least one woman must be on the committee.

6 Programming

27. `http://www.spoj.pl/problems/RESN04/`

28. `http://www.spoj.pl/problems/MISERMAN/`

29. `http://www.spoj.pl/problems/FAVDICE/`

7 Puzzles

**30. There are 5 rational pirates, A, B, C, D and E. They find 100 gold coins. They must decide how to distribute them.

The pirates have a strict order of seniority: A is superior to B, who is superior to C, who is superior to D, who is superior to E.

The rules of distribution in the pirate world are the following: the most senior pirate should propose a distribution of coins. The pirates, including the proposer, then vote on whether to accept this distribution. If the proposed allocation is approved by a majority or a tie vote, it happens. If not, the proposer is thrown overboard from the pirate ship and dies, and the next most senior pirate makes a new proposal to begin the system again.

Pirates base their decisions on three factors. First of all, each pirate wants to survive. Second, each pirate wants to maximize the number of gold coins he receives. Third, each pirate would prefer to throw another overboard, if all other results would otherwise be equal.

You are pirate A. Propose a distribution which maximizes the amount of gold you get without costing you your life. Explain why the other pirates are contempt with this distribution.