Algorithms Exam [1]

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kl 1400 – 1800
"vag och vatten"
salar

Ansvarig:
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Points:	60	
Passing criteria:	Chalmers	5:48, 4:36, 3:24
	GU	VG:48, G:28
	Doktorander	G:36
Helping material:	Textbook, notes, course page stuff.	

- Recommended: First look through all questions and make sure that you understand them properly. In case of doubt, do not hesitate to ask.
- **Answer all questions in the given space on the question paper (the stapled sheets of paper you are looking at). The question paper will be collected from you after the exam. Only the solutions written in the space provided on the question paper will count for your points.**
- Use extra sheets only for your own rough work and then write the final answers on the question paper.
- Answer concisely and to the point. (English if you can and Swedish if you must!)
- Code strictly forbidden! Motivated pseudocode or plain but clear English/Swedish description is fine.

Lycka till! Good Luck!

[1] 2009 LP 1, INN200 (GU) / TIN090 (CTH).

Problem 1 Sorting Revisited [10]

(a) Give a linear time algorithm to sort an array that is known to consist only of 0s and 1s.

(b) Suppose it is known that an array $A[1 \ldots n]$ contains at most k distinct values (which are not known). Describe a data structure and algoritahm to sort the array in time $O(nk)$. For what range of values of k is this an asymptotically better algorithm than Mergesort?

(c) For the same situation as in (b), describe a data structure and algorithm to sort in time $O(n \log k)$. For what range of values of k is this an asymptotically bettre algorithm than Mergesort?

(d) Show that the algorithm in (c) is asymptotically best possible.

Problem 2 Greedy Tape Storage [10] Let $P_1, P_2, \ldots, P_n$ be n programs to be stored on a tape. Program P_i requires s_i kilobytes of storage; the tape has enough capacity to store all programs. We also know how often each program is used: a fraction π_i of requests concern program P_i (thus $\sum_i \pi_i = 1$). Information is recorded along the tape at constant density and the speed of the tape drive is also constant. After a program is loaded, the tape is rewound to the beginning. So, if the programs are stored in the order $i_1, i_2, \ldots, i_n$, the time needed to load program P_j when it is requested is $\sum_{1 \le k \le j} s_{i_k}$ and the average time to load a program (computed over all program requests) is thus:

$$\overline{T} = c \sum_{1 \le j \le n} \left[\pi_{i_j} \sum_{1 \le k \le j} s_{i_k} \right],$$

where the constant c depends on the recording density and the speed of the drive. We want to minimise $\overline{T}$ using a greedy algorithm. Prove, or give a counter-example for each of the following: to minimize $\overline{T}$, we can select the programs

(a) in order of non-decreasing s_i,

(b) in order of non-increasing π_i,

(c) in order of non-increasing $\frac{\pi_i}{s_i}$.

(For a counter-example, you have to specify program sizes and frequencies and show that the selected order does not give the optimum.)

Problem 3 Buying Stocks [10] You're consulting for a small computation-intensive investment company and they have the following type of problem that they want to solve over and over every day. They're doing a simulation in which they look at n consecutive days of a given stock, at some point in the past. Let's number the days $1, 2, \ldots, n$. For each day i, they have a price $p(i)$ per share for the stock on that day. Suppose that during this time period, they want to buy 1000 shares on some day and sell all those shares on some later date. They want to know: when should they have bought and when should they have sold to make as much money as possible?
For example, suppose $n = 3$, $p(1) = 9$, $p(2) = 1$, $p(3) = 5$. Then you should "*buy on day 2 and sell on day 3*". This would make a profit of SEK 4 on each share, the maximum in that period. (This was
This was Solved Exercise 2 in [KT, Chapter 5] where a Divide-and-Conquer strategy was used to develop a $O(n \log n)$ algorithm. Here your goal is to design a faster algorithm. Let $OPT(i)$ denote the optimal solution considering transactions are possible only on days $1 \cdots i$.

(a) What is the value we are trying to compute in terms of this notation?

(b) What is the value $OPT(1)$?

(c) Write a recurrence for $OPT(i)$ based on what is done on day i.

(d) Implement the recurrence efficiently in pseudo-code.

(e) What is the time and space complexity of your algorithm?

(f) Can your algorithm also tell you which day to buy and sell and if so, how?

Problem 4 Closest Points [10] Given n points in the plane, develop a Divide-and-Conquer algorithm to count the number of pairs of points such that the distance between the points is at most twice the distance between the closest pair of points. Explain the conquer step clearly and briefly justify it. (HINT: First compute the minimum distance between any two points.)

Problem 5 Bilkoperative Picnic [10] Majorna's Bilkoperative (car pool society) is organising a picnic. There are n families with family i having f_i members, and there are m cars, with car$j$ having capacity to seat c_j people. Also family i has a list L_i of cars which they are willing to to travel in (for reasons of safety). Not all members of a family need be in the same car. Give an efficient algorithm to determine if everyone can be packed into the available cars or whether they will have to rent more cars. Analyse the running time of your algorithm.

Problem 6 Cybercommunities [10] An interesting problem in internet algorithmics is to find a collection of densely connected web sites i.e. set of web sites which have a lot of hyperlinks between each other. Such collections are called *cybercommunities* - for example, the set of sites discussing *Harry Potter* films is one such cybercommnunity. In graph theory terms, the concept is captured by the notion of a *clique*: a clique is an undirected graph $G = (V, E)$ is a subset $U \subseteq V$ such that for any two vertices $u, v \in U$, $(u, v) \in E$. Consider the optimization problem of finding the size of the largest clique in a given graph.

(a) Formulate a decision problem corresponding to whether or not a graph has a clique of a certain size. Show that the decision problem and the optimization problem are polynomial-time equivalent i.e. if one can be solved in polynomial time, so can the other.

(b) Show that the decision problem is in $\mathcal{NP}$.

(c) Show that the decision problem is $\mathcal{NP}$-complete by giving a reduction from the *Independent Set* problem (which is know to be $\mathcal{NP}$-complete).