

# Finite Automata and Formal Languages

## TMV026/DIT321 – LP4 2011

### Formal Proofs, Alphabets and Words

#### Week 1

In these exercises, book sections and pages refer to those in the third edition of the course book.

Let  $\mathbb{N}$  is the set of all positive integers  $\{0, 1, 2, \dots\}$  (see page 22 of the text book: “Integers as recursively defined concepts”).

1. Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be defined by recursion as

$$f(0) = 0 \quad f(n + 1) = f(n) + n$$

What are the values of  $f(2)$  and  $f(3)$ ?

Use mathematical induction to show that for all  $n \in \mathbb{N}$  we have

$$2f(n) = n^2 - n$$

2. Suppose that we have stamps of 4 kr and 3 kr. Show that any amount of postage over 5 kr can be made with some combinations of these stamps.
3. Let us define by recursion the following function:

$$0! = 1 \quad (n + 1)! = (n + 1) \times n!$$

Show that  $n! \geq 2^n$  for  $n \geq 4$  by analogy with the proof of example 1.17, page 21 of the text book.

4. Let us define by recursion the following two functions  $f, g : \mathbb{N} \rightarrow \mathbb{N}$

$$\begin{aligned} f(0) &= 0 & g(0) &= 1 \\ f(n + 1) &= g(n) & g(n + 1) &= f(n) \end{aligned}$$

What are the values of  $g(2)$  and  $f(4)$ ? Show by mathematical induction that for all  $n \in \mathbb{N}$  we have

$$f(n) + g(n) = 1 \quad f(n)g(n) = 0$$

Show by *mutual* induction that  $f(n) = 0$  iff  $g(n) = 1$  iff  $n$  is even and that  $f(n) = 1$  iff  $g(n) = 0$  iff  $n$  is odd, in analogy to the proof in pages 26–28 in the text book.

5. Let us define the Fibonacci function:

$$f(0) = 0 \quad f(1) = 1 \quad f(n+2) = f(n+1) + f(n)$$

We then define  $s(0) = 0$ ,  $s(n+1) = s(n) + f(n+1)$ .

Prove by induction that we have

$$\forall n. s(n) = f(n+2) - 1.$$

Now we define

$$l(0) = 2, \quad l(1) = 1, \quad l(n+2) = l(n+1) + l(n)$$

Prove by induction that we have  $l(n+1) = f(n) + f(n+2)$ .

6. If  $\Sigma = \{a, b, c\}$ , what are  $\Sigma^1$ ,  $\Sigma^2$  and  $\Sigma^0$ ?

7. If  $\Sigma = \{0, 1\}$ , find a counterexample to the following alleged theorem:  $\forall x, y \in \Sigma^*$  we have

$$x^2y = xyx$$

(cf. section 1.3.4)

8. Let  $\Sigma = \{0, 1\}$ . We define  $\phi : \Sigma^* \rightarrow \Sigma^*$  by recursion as follows

$$\phi(\epsilon) = \epsilon \quad \phi(w0) = \phi(w)1 \quad \phi(w1) = \phi(w)0$$

What are  $\phi(1011)$  and  $\phi(1101)$ ?

Show by induction on  $|w|$  that

$$|\phi(w)| = |w|.$$

9. Let  $\Sigma = \{0, 1\}$ . We define the reverse function on  $\Sigma^*$  by the equations

$$\text{rev}(\epsilon) = \epsilon \quad \text{rev}(ax) = \text{rev}(x)a$$

What are  $\text{rev}(010)$  and  $\text{rev}(10)$ ?

Show by induction on  $y$  that we have

$$\text{rev}(yx) = \text{rev}(x)\text{rev}(y).$$

Show by induction on  $n \in \mathbb{N}$  that we have

$$\text{rev}(x^n) = (\text{rev}(x))^n.$$

10. Given a finite alphabet  $\Sigma$ , when can we have  $x^2 = y^3$  with  $x, y \in \Sigma^*$ ?