CS 473: Algorithms

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Fall 2008

Part I

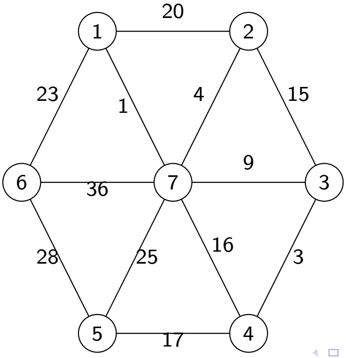
Greedy Algorithms: Minimum Spanning Tree

Minimum Spanning Tree

Input Connected graph $G = (V, E)$ with edge costs

Goal Find $T \subseteq E$ such that (V, T) is connected and total cost of all edges in T is smallest

- T is the minimum spanning tree (MST) of G



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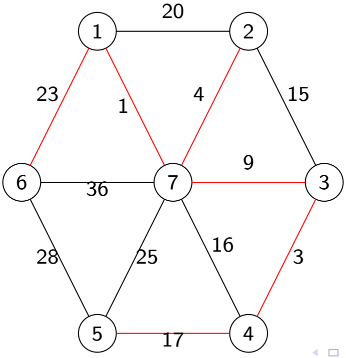


Applications

- Network Design
  - Designing networks with minimum cost but maximum connectivity
- Approximation algorithms
  - Can be used to bound the optimality of algorithms to approximate Travelling Salesman Problem, Steiner Trees, etc.
- Cluster Analysis

Greedy Template

```
Initially E is the set of all edges in G
T is empty (* T will store edges of a MST *)
while E is not empty
    choose i ∈ E
    if (i satisfies condition)
        add i to T
return the set T
```

Main Task: In what order should edges be processed? When should we add edge to spanning tree?

KA PA RD

Kruskal's Algorithm

Process edges in the order of their costs (starting from the least) and add edges to T as long as they don't form a cycle.

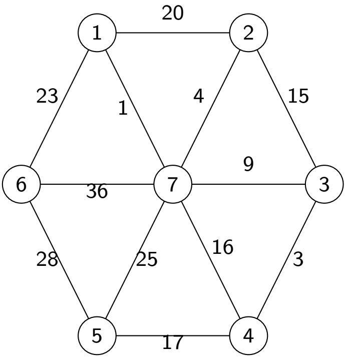
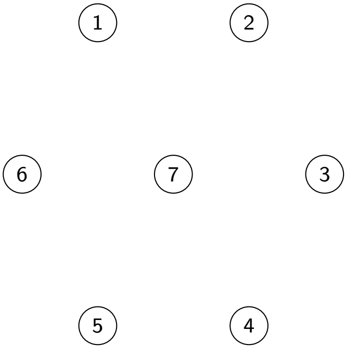


Figure: Graph G



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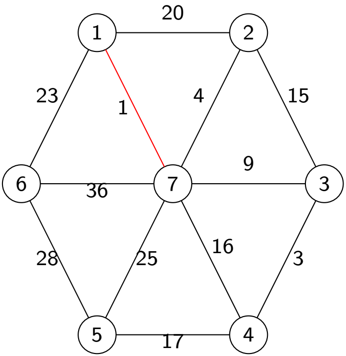

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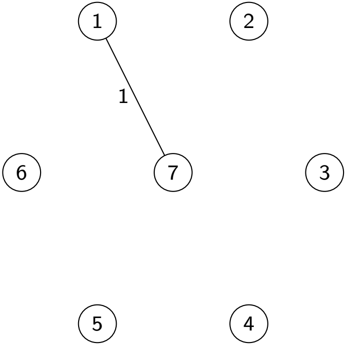

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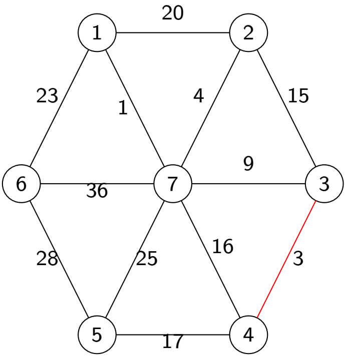

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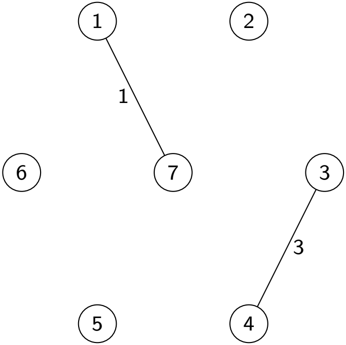

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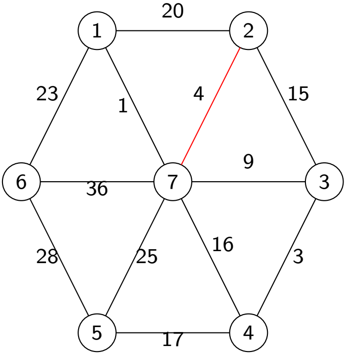

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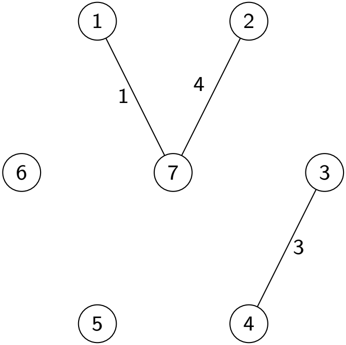

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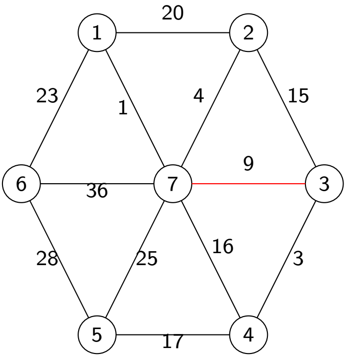

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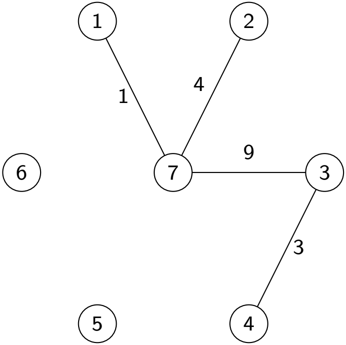

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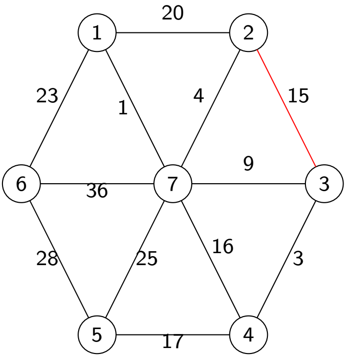

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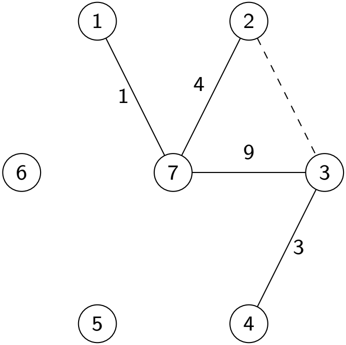

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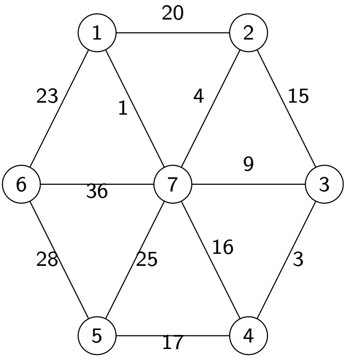

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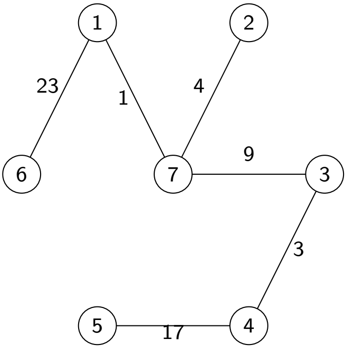

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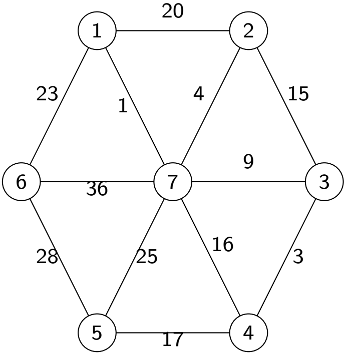
Prim's Algorithm

T maintained by algorithm will be a tree. Start with a node in T.
In each iteration, pick edge with least attachment cost to T.



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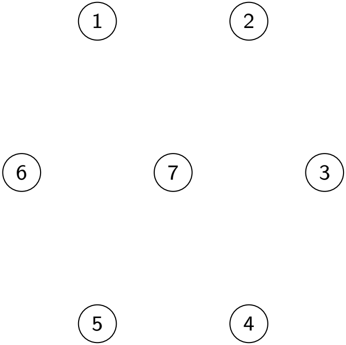

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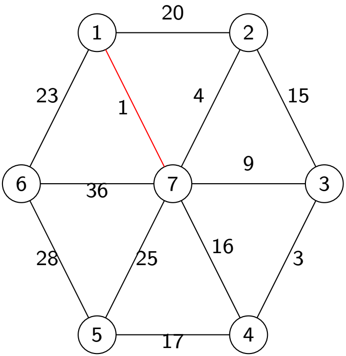

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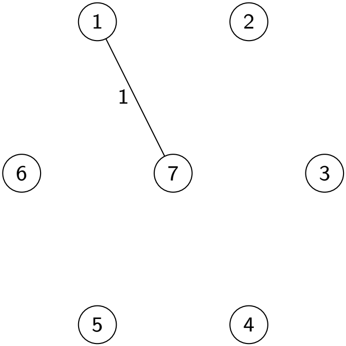

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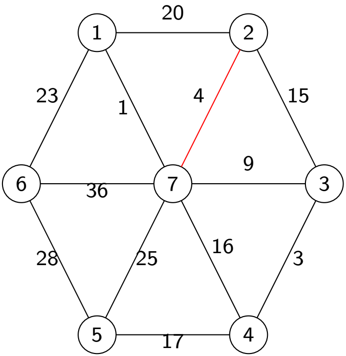

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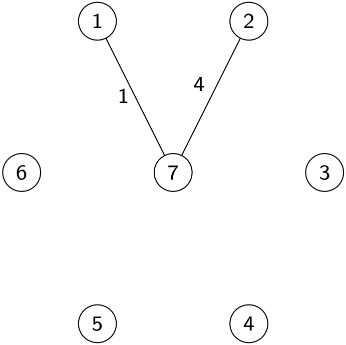

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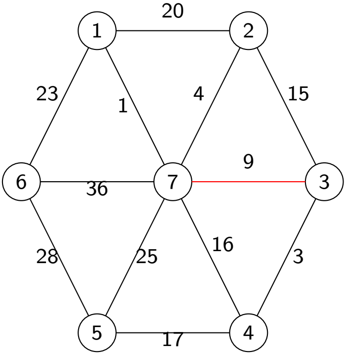

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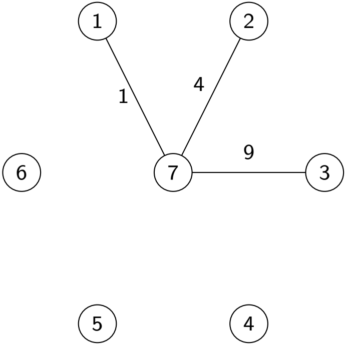

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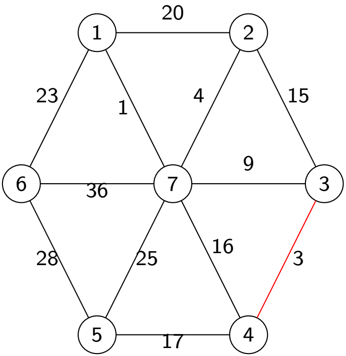

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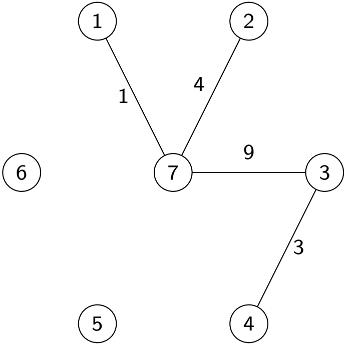

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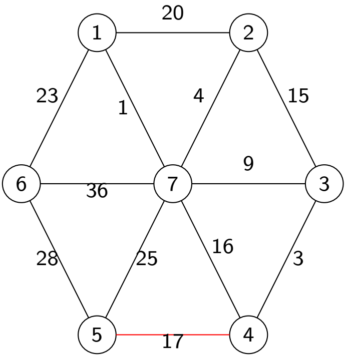

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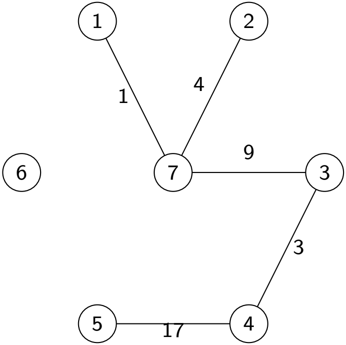

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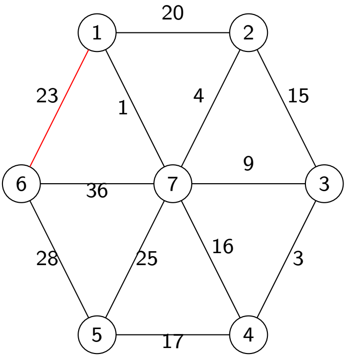

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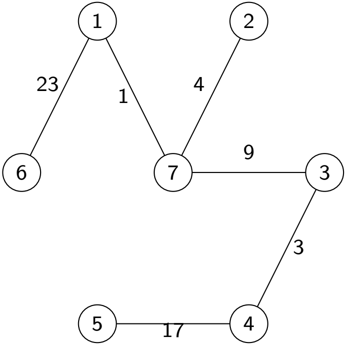

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Reverse Delete Algorithm

```
Initially E is the set of all edges in G
T is E (* T will store edges of a MST *)
while E is not empty
    choose i ∈ E of largest cost
    if removing i does not disconnect T
        remove i from T
return the set T
```

Returns a minimum spanning tree.

Back

Correctness of MST Algorithms

- Many different MST algorithms
- All of them rely on some basic properties of MSTs, in particular the *Cut Property* to be seen shortly.

And for now . . .

Assumption

Edge costs are distinct, that is no two edge costs are equal.

Safe and Unsafe Edges

Definition

An edge $e = (u, v)$ is a safe edge if there is some partition of V into S and $V \setminus S$ and e is the unique minimum cost edge crossing S (one end in S and the other in $V \setminus S$).

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Proposition

If edge costs are distinct then every edge is either safe or unsafe.

Proof.

Exercise. □

Example

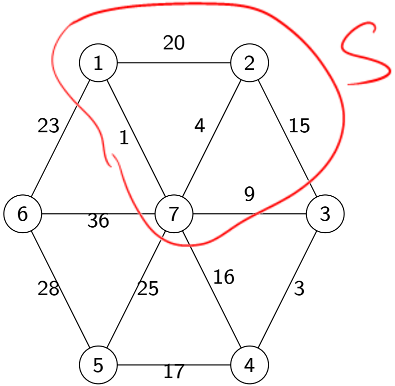

Figure: Graph with unique edge costs.

Example

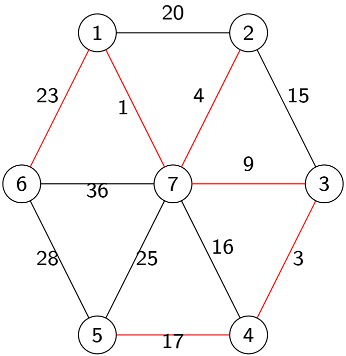

Figure: Graph with unique edge costs. Safe edges are red, rest are unsafe.

Key Observation: Cut Property

Lemma

If e is a safe edge then every minimum spanning tree contains e.

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- Suppose (for contradiction) e is not in MST T.
- Since e is safe there is an $S \subset V$ such that e is the unique min cost edge crossing S.

□

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- Suppose (for contradiction) e is not in MST T.
- Since e is safe there is an $S \subset V$ such that e is the unique min cost edge crossing S.
- Since T is connected, there must be some edge f with one end in S and the other in $V \setminus S$

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- Since e is safe there is an $S \subset V$ such that e is the unique min cost edge crossing S.
- Since T is connected, there must be some edge f with one end in S and the other in $V \setminus S$
- Since $c_f > c_e$, $T' = (T \setminus \{f\}) \cup \{e\}$ is a spanning tree of lower cost!

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- Since $c_f > c_e$, $T' = (T \setminus \{f\}) \cup \{e\}$ is a spanning tree of lower cost! Error: T' may not be a spanning tree!!

□

Error in Proof: Example

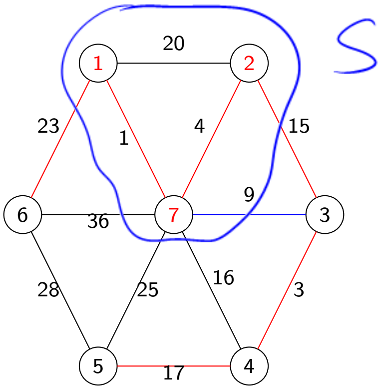

Figure: Problematic example. $S = \{1, 2, 7\}$, $e = (7, 3)$, $f = (1, 6)$

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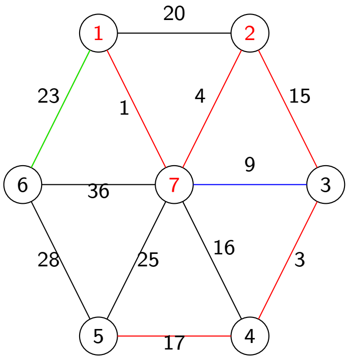

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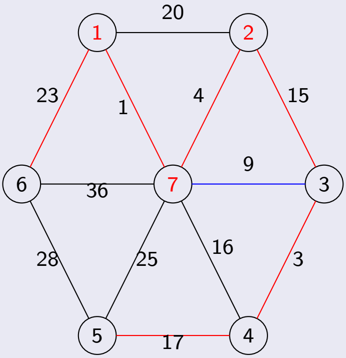
Proof of Cut Property

Proof.



- Suppose minimum $(S, V \setminus S)$-cut edge $e = (v, w)$ is not in MST T.

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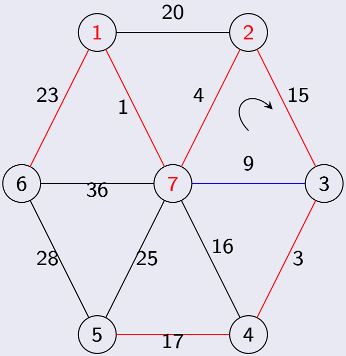

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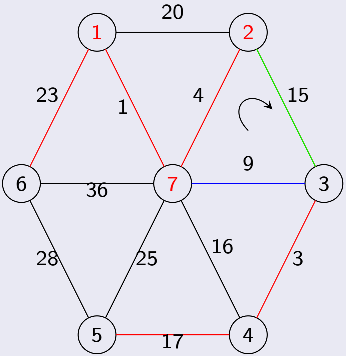

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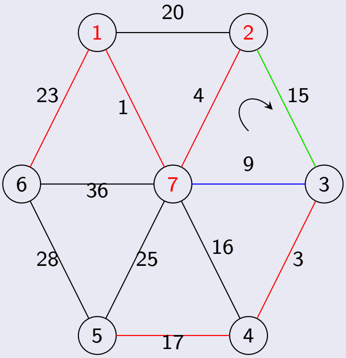
- Let w' be the first vertex in P belonging to $V \setminus S$; let v' be the vertex just before it on P, and let $e' = (v', w')$

Proof of Cut Property

Proof.



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- $T' = (T \setminus \{e'\}) \cup \{e\}$ is spanning tree of lower cost

Proof of Cut Property (contd)

Observation

$T' = (T \setminus \{e'\}) \cup \{e\}$ *is a spanning tree.*

Proof.

T' is connected.

T' is acyclic

□

Proof of Cut Property (contd)

Observation

$T' = (T \setminus \{e'\}) \cup \{e\}$ *is a spanning tree.*

Proof.

T' is connected.

If path uses $e' = (v', w')$, then go from v' to v, use edge (v, w) and then go from w to w' in T'

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Proof of Cut Property (contd)

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- If path uses $e' = (v', w')$, then go from v' to v, use edge (v, w) and then go from w to w' in T'

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- Only one cycle in $T' \cup \{e'\}$, namely, one involving e and e', which is not present in T'

Proof of Cut Property (contd)

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- Only one cycle in $T' \cup \{e'\}$, namely, one involving e and e', which is not present in T'
- Alternatively: T' is connected and has $n - 1$ edges (since T had $n - 1$ edges) and hence T is a tree

□

Safe Edges form a Tree

Lemma

Let G be a connected graph with distinct edge costs, then the set of safe edges form a connected graph.

Proof.

- Suppose not. Let S be a connected component in the safe edges.
- Consider the edges crossing S, there must be a safe edge among them since edge costs are distinct.

□

Safe Edges form an MST

Corollary

Let G be a connected graph with distinct edge costs, then set of safe edges form the unique MST of G.

Safe Edges form an MST

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Let G be a connected graph with distinct edge costs, then set of safe edges form the unique MST of G.

Consquence: Every correct MST algorithm when G has unique edge costs includes exactly the safe edges.

Cycle Property

Lemma

If e is an unsafe edge then no MST of G contains e.

Proof.

Exercise. See text book. □

Note: Cut and Cycle properties hold even when edge costs are not distinct. Safe and unsafe definitions do not rely on distinct cost assumption.

Correctness of Prim's Algorithm

Prim's Algorithm

Pick edge with minimum attachment cost to current tree, and add to current tree.

Proof of correctness.

- If e is added to tree, then e is safe and belongs to every MST.
- Set of edges output is a spanning tree

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    - Let S be the vertices connected by edges in T when e is added.
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  - Let S be the vertices connected by edges in T when e is added.
  - e is edge of lowest cost with one end in S and the other in $V \setminus S$ and hence e is safe.
- Set of edges output is a spanning tree

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  - Set of edges output forms a connected graph: by induction, S is connected in each iteration and eventually $S = V$.

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  - Let S be the vertices connected by edges in T when e is added.
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- Set of edges output is a spanning tree
  - Set of edges output forms a connected graph: by induction, S is connected in each iteration and eventually $S = V$.
  - Only safe edges added and they do not have a cycle □

Correctness of Kruskal's Algorithm

Kruskal's Algorithm

Pick edge of lowest cost and add if it does not form a cycle with existing edges.

Proof of correctness.

- If $e = (u, v)$ is added to tree, then e is safe
- Set of edges output is a spanning tree : exercise

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Pick edge of lowest cost and add if it does not form a cycle with existing edges.

Proof of correctness.

- If $e = (u, v)$ is added to tree, then e is safe
  - When algorithm adds e let S and S' be the connected components containing u and v respectively
- Set of edges output is a spanning tree : exercise

□

Correctness of Kruskal's Algorithm

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Pick edge of lowest cost and add if it does not form a cycle with existing edges.

Proof of correctness.

- If $e = (u, v)$ is added to tree, then e is safe
  - When algorithm adds e let S and S' be the connected components containing u and v respectively
  - e is the lowest cost edge crossing S (and also S').
- Set of edges output is a spanning tree : exercise

□

Correctness of Kruskal's Algorithm

Kruskal's Algorithm

Pick edge of lowest cost and add if it does not form a cycle with existing edges.

Proof of correctness.

- If $e = (u, v)$ is added to tree, then e is safe
  - When algorithm adds e let S and S' be the connected components containing u and v respectively
  - e is the lowest cost edge crossing S (and also S').
  - If there is an edge e' crossing S and has lower cost than e, then e' would come before e in the sorted order and would be added by the algorithm to T
- Set of edges output is a spanning tree : exercise

□

Correctness of Reverse Delete Algorithm

Reverse Delete Algorithm

Consider edges in decreasing cost and remove an edge if it does not disconnect the graph

Proof of correctness.

Argue that only unsafe edges are removed (see text book). □

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- $e_i \prec e_j$ if either $c(e_i) < c(e_j)$ or $(c(e_i) = c(e_j)$ and $i < j)$
- Lexicographic ordering extends to sets of edges. If $A, B \subseteq E$, $A \neq B$ then $A \prec B$ if either $c(A) < c(B)$ or $(c(A) = c(B)$ and $A \setminus B$ has a lower indexed edge than $B \setminus A)$

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- Can order all spanning trees according to lexicographic order of their edge sets. Hence there is a unique MST.

Prim's, Kruskal, and Reverse Delete Algorithms are optimal with respect to lexicographic ordering.

Edge Costs: Postive and Negative

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Part II

Data Structures for MST: Priority Queues and Union-Find

Implementing Prim's Algorithm

```
E is the set of all edges in G
S = {1}
T is empty (* T will store edges of a MST *)
while S =/= V
    pick e = (v,w) in E such that
        v ∈ S and w ∈ V - S
        e has minimum cost
    T = T U e
    S = S U w
return the set T
```

Analysis

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Analysis

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- Picking e is $O(m)$ where m is the number of edges
- Total time $O(nm)$

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Maintain vertices in $V \setminus S$ in a priority queue with key $a(v)$

Priority Queues

Data structure to store a set S of n elements where each element $v \in S$ has an associated real/integer key $k(v)$ such that the following operations

- `makeQ`: create an empty queue
- `findMin`: find the minimum key in S
- `extractMin`: Remove $v \in S$ with smallest key and return it
- `add(v, k(v))`: Add new element v with key $k(v)$ to S
- `delete(v)`: Remove element v from S
- `decreaseKey(v, k'(v))`: *decrease* key of v from $k(v)$ (current key) to $k'(v)$ (new key). Assumption: $k'(v) \leq k(v)$
- `meld`: merge two separate priority queues into one

Prim's using priority queues

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Running time of Prim's Algorithm

$O(n)$ `extractMin` operations and $O(m)$ `decreaseKey` operations

- Using standard Heaps, `extractMin` and `decreaseKey` take $O(\log n)$ time. Total: $O((m+n)\log n)$
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Prim's algorithm and Dijkstra's algorithms are similar. Where is the difference?

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Initially E is the set of all edges in G
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- Total time $O(m \log m) + O(m \cdot (n + m))$

Implementing Kruskal's Algorithm Efficiently

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Need a data structure to check if two elements belong to same set and to merge two sets.

Union-Find Data Structure

Data Structure

Store disjoint sets of elements that supports the following operations

- `makeUnionFind`(S) returns a data structure where each element of S is in a separate set

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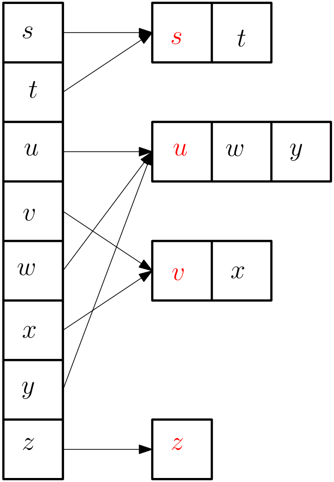
- `union`(A, B) merges two sets A and B.
  Typically: `union(find(u), find(v))`

Implementing Union-Find using Arrays and Lists

Using lists

- Each set stored as list with a name associated with the list.
- For each element $u \in S$ a pointer to the its set. Array for pointers: `component[u]` is pointer for u.
- `makeUnionFind(S)` takes $O(n)$ time and space.

Example



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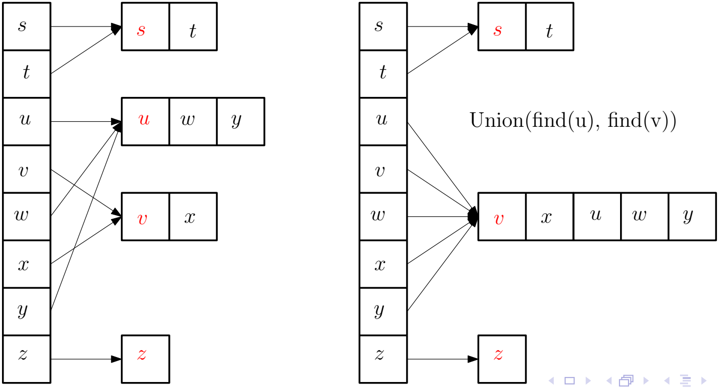
Implementing Union-Find using Arrays and Lists

- `find(u)` reads the entry `component[u]`: $O(1)$ time
- `union(A,B)` involves updating the entries component$[u]$ for all elements u in A and B: $O(|A| + |B|)$ which is $O(n)$



Improving the List Implementation for Union

New Implementation

As before use `component`$[u]$ to store set of u.
Change to `union(A,B)`:

- with each set, keep track of its size
- assume $|A| \leq |B|$ for now
- Merge the list of A into that of B: $O(1)$ time (linked lists)
- Update `component[u]` only for elements in the smaller set A
- Total $O(|A|)$ time.

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`find` still takes $O(1)$ time

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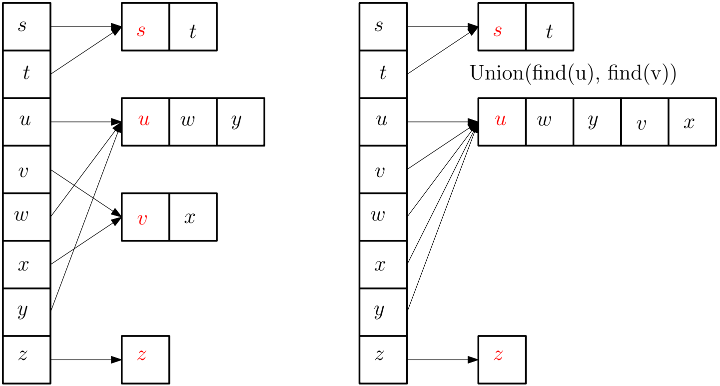

The smaller set (list) is appended to the largest set (list)

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Is the improved implementation provably better or is it simply a nice heuristic?

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Any sequence of k `union` *operations, starting from* `makeUnionFind`(S) *on set S of size n, takes at most $O(k \log k)$.*

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Corollary

Kruskal's algorithm can be implemented in $O(m \log m)$ time.

Sorting takes $O(m \log m)$ time, $O(m)$ finds take $O(m)$ time and $O(n)$ unions take $O(n \log n)$ time.

Average Case or Amortized Analysis

Why does theorem work?

Key Observation

`union(A,B)` takes $O(|A|)$ time where $|A| \leq |B|$. Size of new set is $\geq 2|A|$. Cannot double too many times.

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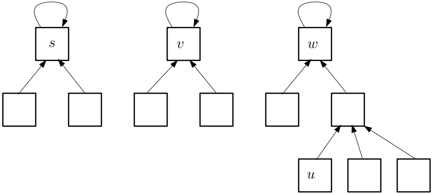
- Total number of updates is $2k \log 2k = O(k \log k)$ □

Improving Worst Case Time



Better Data Structure

Maintain elements in a forest of *in-trees*; all elements in one tree belong to a set with root's name.

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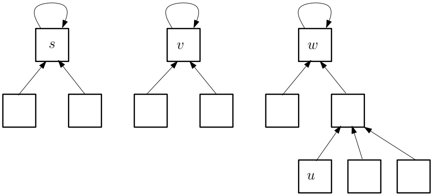

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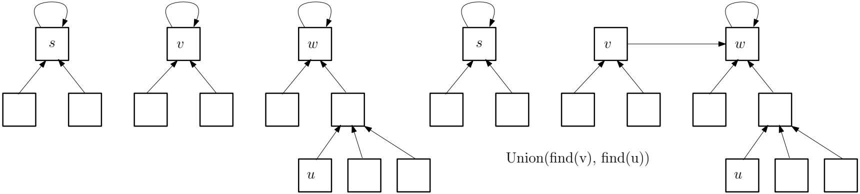

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- `find`(u): Traverse from u to the root
- `union`(A, B): Make root of A (smaller set) point to root of B. Takes $O(1)$ time.

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Each element $u \in S$ has a pointer `parent(u)` to its ancestor.

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union(component(u), component(v)) (* parent(u) = u & parent(v) = v *)
    if (|component(u)| ≤ |component(v)|)
        parent(u) = v
    else
        parent(v) = u
    update new component size to be |component(u)| + |component(v)|
```

Analysis

Theorem

The forest based implementation for a set of size n, has the following complexity for the various operations: `makeUnionFind` *takes* $O(n)$, `union` *takes* $O(1)$, *and* `find` *takes* $O(\log n)$.

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- Height of u increases by at most 1 only when the set containing u changes its name
- If height of u increases then size of the set containing u (at least) doubles

Analysis

Theorem

The forest based implementation for a set of size n, has the following complexity for the various operations: `makeUnionFind` *takes* $O(n)$, `union` *takes* $O(1)$, *and* `find` *takes* $O(\log n)$.

Proof.

- `find`(u) depends on the height of tree containing u
- Height of u increases by at most 1 only when the set containing u changes its name
- If height of u increases then size of the set containing u (at least) doubles
- Maximum set size is n; so height of any tree is at most $O(\log n)$

□

Further Improvements: Path Compression

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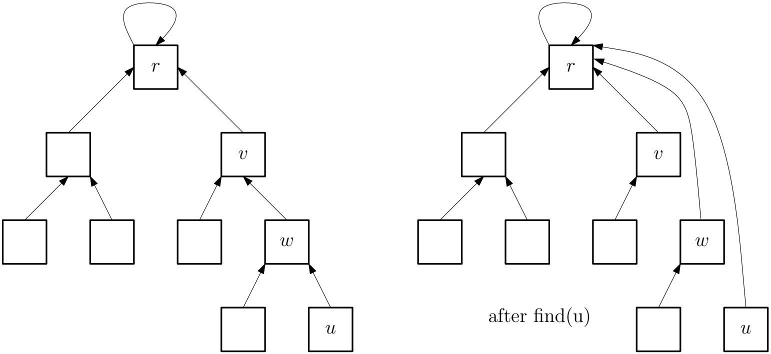
Idea: Path Compression

Make all nodes encountered in the `find`(u) point to root.

Path Compression: Example



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find(u):
    if (parent(u) ≠ u)
        parent(u) = find(parent(u))
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Yes!

Theorem

With Path Compression, k operations (`find` *and/or* `union`*) take* $O(k\alpha(k, \min\{k, n\}))$ *time where* α *is the* inverse Ackermann function.

Ackerman and Inverse Ackerman Functions

Ackerman function $A(m, n)$ defined for $m, n \geq 0$ recursively

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

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$A(3, n) = 2^{n+3} - 3$
$A(4, 3) = 2^{65536} - 3$

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$$\alpha(m,n) = \min\{i \mid A(i, \lfloor m/n \rfloor) \geq \log_2 n\}$$

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For **all practical** purposes $\alpha(m,n) \leq 5$

Lower Bound for Union-Find Data Structure

Amazing result:

Theorem (Tarjan)

For UnionFind, any data structure in the pointer model requires $O(m\alpha(m, n))$ time for m operations.

Running time of Kruskal's Algorithm

Using Union-Find data structure:

- $O(m)$ `find` operations (two for each edge)
- $O(n)$ `union` operations (one for each edge added to T)
- Total time: $O(m \log m)$ for sorting plus $O(m\alpha(n))$ for union-find operations. Thus $O(m \log m)$ time despite the improved Union-Find data structure.

Best Known Asymptotic Running Times for MST

Prim's algorithm using Fibonacci heaps: $O(n \log n + m)$.
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- $O(m \log^* m)$ time [Fredman and Tarjan '1986]
- $O(m)$ time using bit operations in RAM model [Fredman and Willard 1993]
- $O(m)$ expected time (randomized algorithm) [Karger, Klein and Tarjan '1985]
- $O(m\alpha(m, n))$ time [Chazelle '97]
- Still open: is there an $O(m)$ time deterministic algorithm in the comparison model?