CS 473: Algorithms

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Part I

Information Transmission

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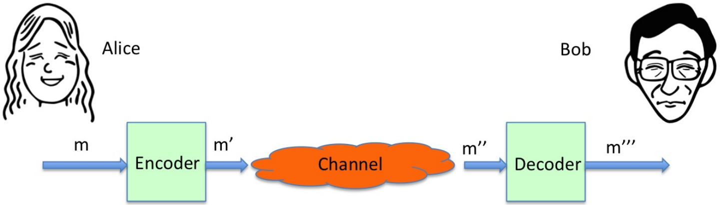

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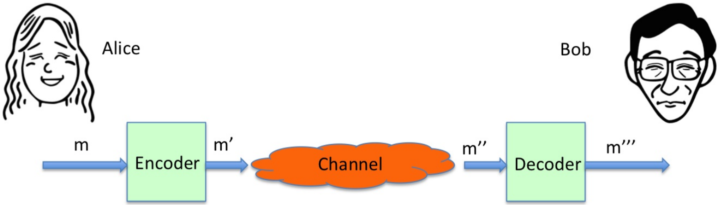

- compression
- error correction
- cryptography/security

(En)Coding and Decoding

- input alphabet Σ (letters)
- output/channel alphabet Δ
- message m: string in Σ^*

(En)Coding

A function that maps strings $m \in \Sigma^*$ to strings $m' \in \Delta$:
$C : \Sigma^* \rightarrow \Delta^*$.

Decoding

A function that maps strings in Δ to strings in Σ: $D : \Delta^* \rightarrow \Sigma^*$.

Error Correction

- input message m, coded message $m' = C(m)$
- m' corrupted by channel, received message is m''
- Decoded message is $D(m'')$
- Goal: want $D(m'') = m$ if not too many errors (different models)
    - maximum k errors
    - maximum α fraction of errors
    - each bit randomly flipped with some probability
    - some bits not received (erasures)

Requires length of $C(m)$ to be longer than m.

Cryptography

- input message m, coded message $m' = C(m)$
- Decoded message is $D(m')$
- Goal: want $D(m') = m$ *and* eavesdropper should not be able to infer m from m'. Many different scenarios.

Typically requires length of $C(m)$ to be longer than m.

Compression

- input message m, coded message $m' = C(m)$
- Decoded message is $D(m')$
- Goal: want $D(m') = m$ *and* m' is as "short" as possible

Single Use Compression

Comression of a file: example Unix compress, gzip, WinZip, pkzip ...

- m is (usually) very large
- tailor made code C that works only for m
- the endecoding mechanism/decoding algorithm stored as part of m'!
- m is large enough that above does not increase size of m' too much.

Compression in Information Transmission

- m may not be very big
- many different messages sent over time
- sender and receiver may have to agree on C apriori

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Requirement: some assumption on distribution of messages
Example: messages are English text (emails)
Knowledge: frequencies of various letters, words, phrases etc.

A Simple Distributional Model

Knowledge about *typical* frequency of letters from Σ.

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- let $|\Sigma| = n$
- know probability of occurence of each letter: $p_1, p_2, \ldots, p_n$
- for $1 \leq i \leq n$, $p_i \in [0, 1]$ and $\sum_{i=1}^{n} p_i = 1$

A Simple Coding Strategy

- Map each letter in Σ to a string in Δ^*, that is $C : \Sigma \to \Delta^*$
- Suppose message $m = a_1 a_2 \ldots a_k$ where $a_i \in \Sigma$. Then $C(a_1 a_2 \ldots a_k) = C(a_1)C(a_2)\ldots C(a_k)$

Fixed Length Codes

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Have same length encoding for each symbol in Σ. That is $|C(a)| = |C(b)|$ for each $a, b \in \Sigma$.

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$\Delta = \{0, 1\}$
Decoding: break output string into chunks of 7 bits and map them back to letters.

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Fixed length codes ignore different frequencies of letters and hence essentially achieve no compression. They are used for information representation.

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What is the text for 0101?

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Ambiguity removed by adding pauses between letters.

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- But then encoding is not over 0,1 but over 0,1,2.

Prefix Codes

Definition

A prefix code for a set Σ is function γ such that

1. For $x \in \Sigma$, $\gamma(x)$ is a bit-string
2. For distinct x and y, it is not the case that $\gamma(x)$ is a prefix of $\gamma(y)$, or vice versa.

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Example

Consider $\Sigma = \{a, b, c, d, e\}$ with encoding γ as follows:

$$\begin{array}{cc} \gamma(a) = 11 & \gamma(b) = 01 \\ \gamma(c) = 001 & \gamma(d) = 10 \\ \multicolumn{2}{c}{\gamma(e) = 000} \end{array}$$

String "bad" encoded as 01 11 10

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$$c \quad e \quad c \quad a \quad b$$

Part II

Huffman Codes

Average Bits per Letter

Given:

- input alphabet Σ with $|\Sigma| = n$ and
- letter probabilities $p_1, p_2, \ldots, p_n$
- $\Delta = \{0, 1\}$: binary

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- letter probabilities $p_1, p_2, \ldots, p_n$
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Definition

For an alphabet Σ, with probability p_x for symbol x ($\sum_{x \in \Sigma} p_x = 1$), the average number of bits required per letter under the encoding γ

$$\mathrm{ABL}(\gamma) = \sum_{x \in \Sigma} p_x |\gamma(x)|.$$

ABL: Example

Example

For $\Sigma = \{a, b, c, d, e\}$, with probabilities

$$p_a = 0.32 \qquad p_b = 0.25 \qquad p_c = 0.20 \qquad p_d = 0.18 \qquad p_e = 0.05$$

Consider

$\gamma(a) = 11,\ \gamma(b) = 01,\ \gamma(c) = 001,\ \gamma(d) = 10,\ \gamma(e) = 000$

$$\mathrm{ABL}(\gamma) = 2 \times 0.32 + 2 \times 0.25 + 3 \times 0.2 + 2 \times 0.18 + 3 \times 0.05 = 2.25$$

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Consider
$\gamma'(a) = 11,\ \gamma'(b) = 10,\ \gamma'(c) = 01,\ \gamma'(d) = 001,\ \gamma'(e) = 000$
Then $\mathrm{ABL}(\gamma') = 2.23$

Optimal Prefix Codes

Input Given a set Σ and probabilities p_x for each $x \in \Sigma$

Goal Find a prefix code γ for Σ over $\Delta = \{0, 1\}$ such that $\mathrm{ABL}(\gamma)$ is minimum.

Prefix Codes and Binary Trees

Proposition

There is a 1-to-1 onto correspondence between prefix codes in Σ and binary trees whose leaves are labelled by $x \in \Sigma$

Proof.

$\gamma(x)$ will be path from root to leaf labelled x in tree, where left child is 0 and right child is 1.

$$\gamma'(a) = 11$$
$$\gamma'(b) = 10$$
$$\gamma'(c) = 01$$
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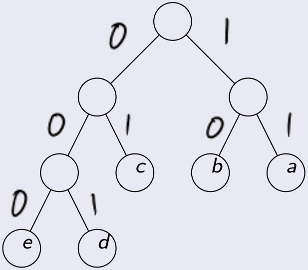
$\longleftrightarrow$



Prefix Codes and Binary Trees

Lemma

If T is a rooted binary tree and there is a bijection between the leaves L of T and Σ, then there is a prefix-code $\gamma : \Sigma \to \{0,1\}^*$ where $\gamma(a)$ is given by the path from root of T to a.

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Proof Sketch.

- Define $\gamma(a)$ for each a by walking from root to a: output a 0 if the path uses a left child and a 1 if path uses right child. Creates a string of 0's and 1's.
- γ is a prefix code. Why?

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- Define $\gamma(a)$ for each a by walking from root to a: output a 0 if the path uses a left child and a 1 if path uses right child. Creates a string of 0's and 1's.
- γ is a prefix code. Why? If $\gamma(a)$ is a prefix of $\gamma(b)$ then from construction a must be on the path from root to b. But all letters are at leaves of T.

□

Prefix Codes and Binary Trees

Lemma

If $\gamma : \Sigma \to \{0,1\}^*$ *is a prefix-code then there is a rooted binary tree* T *and a bijection from* Σ *to the leaves* L *of* T*.*

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If $\gamma : \Sigma \rightarrow \{0,1\}^*$ is a prefix-code then there is a rooted binary tree T and a bijection from Σ to the leaves L of T.

Proof Sketch.

- Given γ, create T as follows.
- Let $\Sigma_0 \subset \Sigma$ where $a \in \Sigma_0$ iff $\gamma(a)$ starts with 0. $\Sigma_1 = \Sigma - \Sigma_0$.
- Recursively create tree T_0 for Σ_0 with $\gamma'(a)$ is obtained from $\gamma(a)$ by removing the leading 0. Note: γ' is prefix-code for Σ_0.
- Similarly, T_1 for Σ_1 with leading 1 removed.
- Create T from T_0 and T_1 by adding root r and making T_0 the left sub-tree and T_1 the right sub-tree.

□

$\gamma(a) = 0101$

$\gamma(b) = 010$

Optimal Codes and Full Trees

Definition

A binary tree is full if every internal node has two children.

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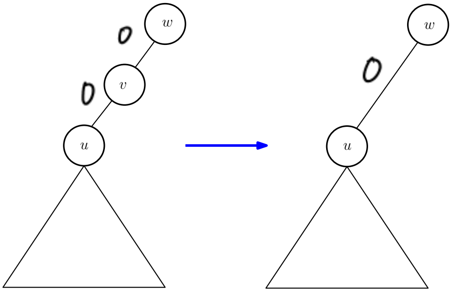


Optimal Codes and Full Trees

Definition

A binary tree is full if every internal node has two children.

Proposition

The binary tree corresponding to the optimal code is full.

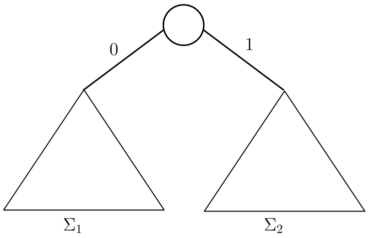
Proof.

- Suppose (for contradiction) T is optimal code, where u has only one child v
- Consider T' where u is removed; if u is the root make v root, otherwise, attach v to parent of u
- T' has a smaller average code, as the code of leaves below u has been shortened by 1 bit. □

Top-Down Approach

Algorithm [Shannon-Fano]

1. Divide Σ into Σ_1 and Σ_2 such that total frequency of Σ_1 and Σ_2 is (if possible) $\frac{1}{2}$
2. Recursively find code for γ_1 for Σ_1 and γ_2 for Σ_2.
3. Code for Σ: $\gamma(x) = 0\gamma_1(x)$, $x \in \Sigma_1$ & $\gamma(x) = 1\gamma_2(x)$, $x \in \Sigma_2$



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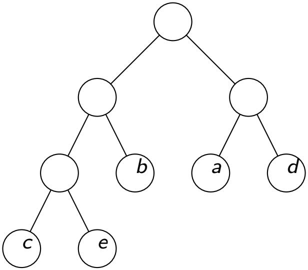
Consider $\Sigma = \{a, b, c, d, e\}$ and $p_a = 0.32,\ p_b = 0.25,\ p_c = 0.2,\ p_d = 0.18,\ p_e = 0.05$. First split results in $\{b, c, e\}$ and $\{a, d\}$ and recursively find codes. Resulting code is $\gamma(a) = 11,\ \gamma(b) = 01,\ \gamma(c) = 001,\ \gamma(d) = 10,$ $\gamma(e) = 000$.



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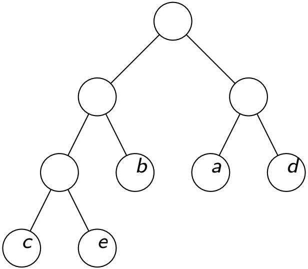

Understanding an Optimal Solution

- Given Σ and p_x for each $x \in \Sigma$
- Suppose we knew the (optimum) tree T but not a labeling of the leaves by Σ. Can we label the leaves?

$$\Sigma = \{a, b, c, d, e\}$$

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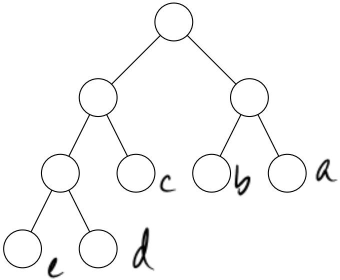

Depth and Probability

Proposition

Let T^* be an optimal prefix code. For leaves u and v with labels x and y, respectively, if $\mathrm{depth}(u) < \mathrm{depth}(v)$ then $p_x \geq p_y$.

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- Suppose (for contradiction) $p_x < p_y$
- Consider tree T_1^* where the labels of leaves u and v have been exchanged.

$$\mathrm{ABL}(T^*) - \mathrm{ABL}(T_1^*) \quad = \quad \sum_{z \in \Sigma} p_z \mathrm{depth}_{T^*}(z) - \sum_{z \in \Sigma} p_z \mathrm{depth}_{T_1^*}(z)$$

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&= (\mathrm{depth}(u) p_x + \mathrm{depth}(v) p_y) \\
&\qquad -(\mathrm{depth}(u) p_y + \mathrm{depth}(v) p_x)
\end{aligned}$$

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\end{aligned}$$

- T_1^* is better, which contradicts optimality of T^* □

Maximum Depth

Corollary

Least frequent symbol labels the leaf of maximum depth.

Observation

If u and v are leaves of T of same depth d, labeled with x and y then T' has the same ABL as T if labels of u and v are swapped.

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Observation

Any full binary tree with more than two leaves has leaves u and v at maximum depth and which are siblings (share a parent).

Proof.

Let u be a leaf at maximum depth and let w be its parent.
w has another child other than u — this has to be a leaf v since u is at maximum depth. □

Technical Observation

Lemma

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Proof.

- Let u, v be two sibling leaves of T^* at maximum depth — they exist by previous observation.
- x, y are at maximum depth since they are least frequent.
- If x, y do not label u, v, by observation, can swap them to label u, v without increasing ABL. □

Huffman's Algorithm

Algorithm

1. Find x, y with the two lowest probabilities
2. If $|\Sigma| = 2$ return two-leaf tree with x, y as labels.
3. Let $\Delta = (\Sigma \setminus \{x, y\}) \cup \{\omega\}$ with $p_\omega = p_x + p_y$
4. Recursively find optimal code T' for Σ'
5. Code T for Σ is: Add two leaves to leaf labeled ω in T' and label the leaves x and y

Example

$\Sigma = \{a, b, c, d, e\}$ and $p_a = 0.32$, $p_b = 0.25$, $p_c = 0.2$, $p_d = 0.18$, $p_e = 0.05$

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$\Sigma = \{a, b, c, \omega_1\}$ and
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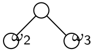

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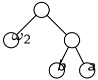

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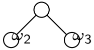

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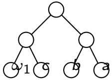

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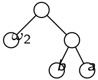

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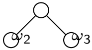

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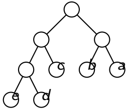

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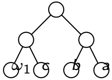

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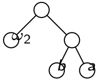

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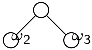

Property about Recursive Step

Proposition

Let $\Sigma' = (\Sigma \setminus \{x, y\}) \cup \{\omega\}$, T' *be the Huffman code for* Σ' *and* T *the huffman code for* Σ. *Then,*

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$$\begin{aligned}
\mathrm{ABL}(T) &= \sum_{z \in S} p_z \mathrm{depth}_T(z) \\
&= p_x \mathrm{depth}_T(x) + p_y \mathrm{depth}_T(y) + \sum_{z \neq x,y} p_z \mathrm{depth}_T(z) \\
&= (p_x + p_y)(1 + \mathrm{depth}_{T'}(\omega)) + \sum_{z \neq x,y} p_z \mathrm{depth}_{T'}(z) \\
&= p_\omega + p_\omega \mathrm{depth}_{T'}(\omega) + \sum_{z \neq x,y} p_z \mathrm{depth}_{T'}(z) \\
&= p_\omega + \mathrm{ABL}(T') \quad \square
\end{aligned}$$

Property about Optimal Encoding

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Let Z be an optimal tree for Σ and let Z' be an optimal tree for $(\Sigma \cup \{\omega\}) \setminus \{x, y\}$ where x, y are the two least probable letters. Then $\mathrm{ABL}(Z') \leq \mathrm{ABL}(Z) - (p_x + p_y)$.

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Proof.

- From Lemma on optimal trees, assume x, y are siblings in Z.
- Obtain a tree Y for $(\Sigma \cup \{\omega\}) \setminus \{x, y\}$ from Z by removing x, y from Z and labeling parent of x, y with ω.
- Y is a valid tree for $(\Sigma \cup \{\omega\}) \setminus \{x, y\}$
- $\mathrm{ABL}(Y) = \mathrm{ABL}(Z) - (p_x + p_y)$.
- An optimal tree Z' cannot be worse than Y. □

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① $ABL(T') \leq ABL(Z')$

② $ABL(T) = ABL(T') + p_w$

③ $ABL(Z') \leq ABL(Z) - p_w$

$$\begin{aligned} ABL(T) &= ABL(T') + p_w && ② \\ &\leq ABL(Z') + p_w && ① \\ &\leq ABL(Z) - p_w + p_w && ③ \\ &\leq ABL(Z) && ✓ \end{aligned}$$

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- By Proposition, $\mathrm{ABL}(Z') \leq \mathrm{ABL}(Z) - p_\omega$.
- Implies $\mathrm{ABL}(T) \leq \mathrm{ABL}(Z)$ and hence T is optimal. □

Implementation and Analysis

```
if Σ has two letters then
    encode one as 0 and the other as 1
else
    let x,y be the lowest probability letters
    remove x,y and add ω to get Σ'
    recursively find code T' for Σ'
    code T for Σ is as follows
        for z ≠ x,y T(z) = T'(z)
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- Store Σ in a priority queue with the probability as key
- Each iteration takes $O(\log n)$
- Total time is $O(n \log n)$ for the $n-2$ iterations