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Computers and Intractability

A Guide to the Theory of NP-Completeness

The "Bible" of complexity theory

M. R. Garey and D. S. Johnson

W. H. Freeman and Company, 1979

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The "Bandersnatch" problem

Background:

Find a good method for determining whether or not any given set of specifications for a new bandersnatch component can be met and, if so, for constructing a design that meets them.

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The "Bandersnatch" problem

Initial attempt:

Pull down your reference books and plunge into the task with great enthusiasm.

Some weeks later ...

Your office is filled with crumpled-up scratch paper, and your enthusiasm has lessened considerable because …

… the solution seems to be to examine all possible designs!

New problem:

How do you convey the bad information to your boss?

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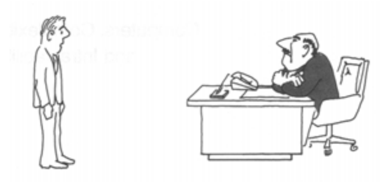
The "Bandersnatch" problem

Approach #1: Take the loser's way out



"I can't find an efficient algorithm, I guess I'm just too dumb."

Drawback: Could seriously damage your position within the company

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The "Bandersnatch" problem

Approach #2: Prove that the problem is inherently intractable

"I can't find an efficient algorithm, because no such algorithm is possible!"

Drawback: Proving inherent intractability can be as hard as finding efficient algorithms. Even the best theoreticians have failed!

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The "Bandersnatch" problem

Approach #3: Prove that the problem is NP-complete

"I can't find an efficient algorithm, but neither can all these famous people."

Advantage: This would inform your boss that it is no good to fire you and hire another expert on algorithms.

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NP-complete problems

NP-complete problems:
Problems that are "just as hard" as a large number of other problems that are widely recognized as being difficult by algorithmic experts.

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NP-complete problems

Problem:

- A general question to be answered

  Example: The "traveling salesman optimization problem"

Parameters:

- Free problem variables, whose values are left unspecified

  Example: A set of "cities" $C = \{c_1, \ldots, c_n\}$ and a "distance" $d(c_i, c_j)$ between each pair of cities c_i and c_j

Instance:

- An instance of a problem is obtained by specifying particular values for all the problem parameters

  Example: $C = \{c_1, c_2, c_3, c_4\}, d(c_1, c_2) = 10, d(c_1, c_3) = 5, d(c_1, c_4) = 9,$
  $d(c_2, c_3) = 6, d(c_2, c_4) = 9, d(c_3, c_4) = 3$

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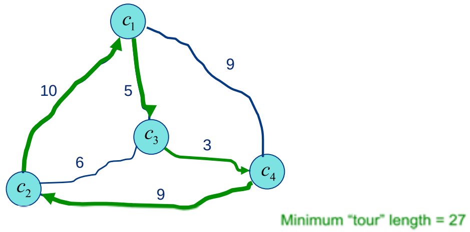
NP-complete problems

The Traveling Salesman Optimization Problem:



Minimize the length of the "tour" that visits each city in sequence, and then returns to the first city.

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Intractability

Input length:

- The number of information symbols needed for describing a problem instance using a "reasonable" encoding scheme

Largest number:

- The magnitude of the largest number in a problem instance

Time-complexity function:

- Expresses an algorithm's time requirements by giving, for each possible input length, the largest amount of time needed by the algorithm to solve a problem instance of that size

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Intractability

Polynomial-time algorithm:

- An algorithm whose time-complexity function is $O(p(n))$ for some polynomial function p, where n is the input length.

Exponential-time algorithm:

- Any algorithm whose time-complexity function cannot be so bounded.

A problem is said to be <u>intractable</u> if it is so hard that no polynomial-time algorithm can possibly solve it

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Intractability

Reasonable encoding scheme:

- Conciseness:
  - The encoding of an instance I should be concise and not "padded" with unnecessary information or symbols
  - Numbers occurring in I should be represented in binary (or decimal, or octal, or in any fixed base other than 1)
- Decodability:
  - It should be possible to specify a polynomial-time algorithm that can extract a description of any component of I.

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Decision problems

The theory of NP-completeness applies only to <u>decision problems</u>, where the solution is either a “Yes” or a “No”.

↓

If an optimization problem asks for a structure of a certain type that has minimum “cost” among such structures, we can associate with that problem a decision problem that includes a numerical bound B as an additional parameter and that asks whether there exists a structure of the required type having cost <u>no more than</u> B.

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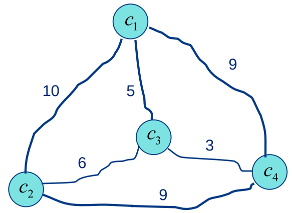
Decision problems

The Traveling Salesman <u>Decision</u> Problem:



Is there a “tour” of all the cities in C having a total length of no more than B?

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Class P

Deterministic algorithm: (Deterministic Turing Machine)

- Finite-state control:
  - The algorithm can pursue only one computation at a time
  - Given a problem instance I, some structure (= solution) S is derived by the algorithm
  - The correctness of S is inherent in the algorithm

The <u>class P</u> is the class of all decision problems Π that, under reasonable encoding schemes, can be solved by polynomial-time deterministic algorithms.

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Class NP

Non-deterministic algorithm: (Non-Deterministic Turing Machine)

1. Guessing stage:
   - Given a problem instance I, some structure S is “guessed”.
   - The algorithm can pursue an <u>unbounded</u> number of independent computational sequences in parallel.
2. Checking stage:
   - The correctness of S is verified in a normal deterministic manner

The <u>class NP</u> is the class of all decision problems Π that, under reasonable encoding schemes, can be solved by polynomial-time non-deterministic algorithms.

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Relationship between P and NP

Observations:

1. P ⊆ NP
   – Proof: use a polynomial-time deterministic algorithm as the checking stage and ignore the guess ....
2. P ≠ NP
   – This is a wide-spread belief, but …
   – … no proof of this conjecture exists!

The question of whether or not the NP-complete problems are intractable is now considered to be one of the foremost open questions of contemporary mathematics and computer science!

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NP-complete problems

Reducibility:

- A problem Π' is reducible to problem Π if, for any instance of Π', an instance of Π can be constructed in polynomial time such that solving the instance of Π will solve the instance of Π' as well.

When Π' is reducible to Π, we write $\Pi' \propto \Pi$

A decision problem Π is said to be NP-complete if $\Pi \in$ NP and, for all other decision problems $\Pi' \in$ NP, Π' polynomially reduces to Π.

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NP-hard problems

Turing reducibility:

- A problem Π' is Turing reducible to problem Π if there exists an algorithm A that solves Π' by using a hypothetical subroutine S for solving Π such that, if S were a polynomial time algorithm for Π, then A would be a polynomial time algorithm for Π' as well.

When Π' is Turing reducible to Π, we write $\Pi' \propto_T \Pi$

A search problem Π is said to be NP-hard if there exists some decision problem $\Pi' \in$ NP that Turing-reduces to Π.

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NP-hard problems

Observations:

- All NP-complete problems are NP-hard
- Given an NP-complete decision problem, the corresponding optimization problem is NP-hard
  To see this, imagine that the optimization problem (that is, finding the optimal cost) could be solved in polynomial time.
  The corresponding decision problem (that is, determining whether there exists a solution with a cost no more than B) could then be solved by simply comparing the found optimal cost to the bound B. This comparison is a constant-time operation.
- While an NP-complete problem is solvable in polynomial time if and only if P = NP, an NP-hard problem cannot be solved in polynomial time unless P = NP.

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Strong NP-completeness

Pseudo-polynomial-time algorithm:

- An algorithm whose time-complexity function is $O(p(n,m))$ for some polynomial function p, where n is the input length and m is the largest number.

Number problem:

- A decision problem for which there exists no polynomial function p such that $m \leq p(n)$ for all instances of the problem.
- Examples:
  - PARTITION, KNAPSACK, TRAVELING SALESMAN
  - MULTIPROCESSOR SCHEDULING

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Strong NP-completeness

If a decision problem Π is NP-complete and is not a number problem, then it cannot be solved by a pseudo-polynomial-time algorithm unless P = NP.

↓

Assuming P ≠ NP, the only NP-complete problems that are potential candidates for being solved by pseudo-polynomial-time algorithms are those that are number problems.

↓

A decision problem Π which cannot be solved by a pseudo-polynomial-time algorithm, unless P = NP, is said to be NP-complete in the strong sense.

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Avoiding NP-completeness

Tricks for circumventing the intractability:

1. Limiting the largest number m
2. Redefining the problem (e.g. edge vs vertex cover)
3. Exploiting problem structure (e.g. limits on vertex degrees, "intree" vs "outtree" task graphs)
4. Fixing problem parameters (e.g. fixed # of processors in multiprocessor scheduling)

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History of NP-completeness

S. Cook: (1971)
"The Complexity of Theorem Proving Procedures"
Every problem in the class NP of decision problems polynomially reduces to the SATISFIABILITY problem:

Given a set U of Boolean variables and a collection C of clauses over U, is there a satisfying truth assignment for C ?

R. Karp: (1972)
"Reducibility among Combinatorial Problems"
Decision problem versions of many well-known combinatorial optimization problems are "just as hard" as SATISFIABILITY.

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History of NP-completeness

D. Knuth: (1974)
"A Terminological Proposal"
Initiated a researcher's poll in search of a better term for "at least as hard as the polynomial complete problems".

One suggestion by S. Lin was PET problems:

- "Probably Exponential Time" (if P = NP remain open question)
- "Provably Exponential Time" (if P $\neq$ NP)
- "Previously Exponential Time" (if P = NP)

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Proving NP-completeness

Proving NP-completeness for a decision problem Π:

1. Show that Π is in NP
2. Select a known NP-complete problem Π'
3. Construct a transformation $\propto$ from Π' to Π
4. Prove that $\propto$ is a (polynomial) transformation

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Basic NP-complete problems

SATISFIABILITY

3-DIMENSIONAL MATCHING
VERTEX COVER
HAMILTONIAN CIRCUIT

3-SATISFIABILITY
CLIQUE
PARTITION

KNAPSACK
3-PARTITION
ANNIHILATION
CLUSTERING

MINIMUM COVER
LONGEST PATH
BIN PACKING
MAX CUT

INTEGER PROGRAMMING
DEADLOCK AVOIDANCE
JOB-SHOP SCHEDULING
REGISTER SUFFICIENCY

MULTIPROCESSOR SCHEDULING
TRAVELING SALESMAN
GRAPH COLORABILITY
PREEMPTIVE SCHEDULING

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NP-complete scheduling problems

Uniprocessor scheduling with offsets and deadlines:

Independent tasks with individual offsets and deadlines.
Transformation from 3-PARTITION (Garey and Johnson, 1977)

NP-complete in the strong sense.
Solvable in pseudo-polynomial time if number of allowed values for offsets and deadlines is bounded by a constant.
Solvable in polynomial time if execution times are identical, preemptions are allowed, or all offsets are 0.

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NP-complete scheduling problems

Multiprocessor scheduling:

Independent tasks with an overall deadline.
Transformation from PARTITION (Garey and Johnson, 1979)

- NP-complete in the strong sense for arbitrary number of processors.
- NP-complete in the normal sense for two processors.
- Solvable in pseudo-polynomial time for any fixed number of processors.
- Solvable in polynomial time if execution times are identical.

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NP-complete scheduling problems

Precedence-constrained multiprocessor scheduling:

Precedence-constrained tasks with identical execution times and an overall deadline.
Transformation from 3-SATISFIABILITY (Ullman, 1975)

- NP-complete in the normal sense for arbitrary number of processors.
- Solvable in polynomial time for two processors, or for arbitrary number of processors and "forest-like" precedence constraints.
- Remains an open problem for fixed number of processors (≥ 3).

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NP-complete scheduling problems

Multiprocessor scheduling with individual deadlines:

Precedence-constrained tasks with identical execution times and individual deadlines.
Transformation from VERTEX COVER (Brucker, Garey and Johnson, 1977)

- NP-complete in the normal sense for arbitrary number of processors.
- Solvable in polynomial time for two processors or "in-tree" precedence constraints.

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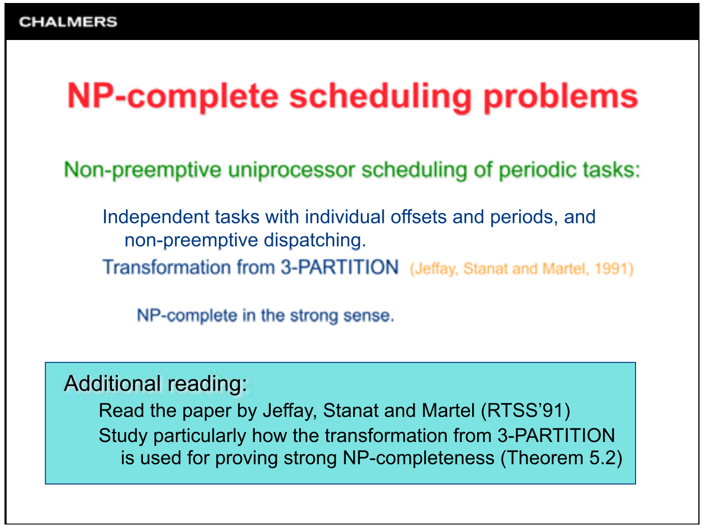
NP-complete scheduling problems

Preemptive uniprocessor scheduling of periodic tasks:

Independent tasks with individual offsets and periods, and preemptive dispatching.
Transformation from CLIQUE (Leung and Merrill, 1980)

- NP-complete in the normal sense.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NP-complete scheduling problems
Non-preemptive uniprocessor scheduling of periodic tasks:
Independent tasks with individual offsets and periods, and non-preemptive dispatching.
Transformation from 3-PARTITION (Jeffay, Stanat and Martel, 1991)
NP-complete in the strong sense.
Additional reading:
Read the paper by Jeffay, Stanat and Martel (RTSS'91)
Study particularly how the transformation from 3-PARTITION is used for proving strong NP-completeness (Theorem 5.2)