Exercise 3

This exercise covers availability modeling. Availability is the probability that a system is working at a given time t. When we calculate availability, we consider both failures and repairs. In comparison to calculating reliability, the system cannot suffer permanent failures. We will solve Problems 3.12, 3.11, and 5.2.

Problem 3.12

Derive an expression for the steady-state availability of a cold stand-by system with one primary module and one spare. Assume that the coverage is c. One module at a time can be repaired.

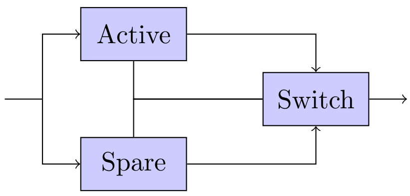

Figure 1: System overview.

Solution

State	Working	Spares
A	1	1
B	1	0
C	System failure	

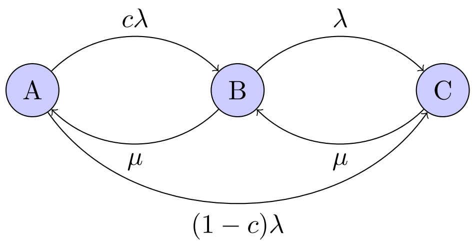

Figure 2: TMR – Markov chain for $A(t)$

$$\begin{aligned} A(t) &= P_A(t) + P_B(t) \\ \mathbf{P}'(t) &= \mathbf{P}(t)\mathbf{Q} \qquad \mathbf{Q} \text{ transition rate matrix} \\ \mathbf{P}(t) &= \begin{bmatrix} P_A(t) & P_B(t) & P_C(t) \end{bmatrix} \end{aligned}$$

Transition rate matrix:

$$\mathbf{Q} = \begin{bmatrix} -\lambda & c\lambda & (1-c)\lambda \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$$

Steady-state solution $\Rightarrow \mathbf{P'} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.
Let

$$\Pi_i = \lim_{t\to\infty} P_i(t)$$

We now have

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \Pi_A & \Pi_B & \Pi_C \end{bmatrix} \begin{bmatrix} -\lambda & c\lambda & (1-c)\lambda \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{bmatrix}$$

We also now that

$$\sum_i \Pi_i = 1$$

We get the following equations:

$$\begin{cases} 0 & = & -\lambda\Pi_A + \mu\Pi_B \\ 0 & = & c\lambda\Pi_A - (\lambda+\mu)\Pi_B + \mu\Pi_C \\ 0 & = & (1-c)\lambda\Pi_A + \lambda\Pi_B - \mu\Pi_C \\ \Pi_A + \Pi_B + \Pi_C & = & 1 \end{cases}$$

$$\begin{aligned}
\Pi_A &= \frac{\mu}{\lambda}\Pi_B \\
\Pi_C &= \frac{(1-c)\lambda}{\mu}\Pi_A + \frac{\lambda}{\mu}\Pi_B \\
&= \frac{(1-c)\lambda}{\mu}\frac{\mu}{\lambda}\Pi_B + \frac{\lambda}{\mu}\Pi_B \\
&= \frac{(1-c)\mu + \lambda}{\mu}\Pi_B \\
1 &= \Pi_A + \Pi_B + \Pi_C \\
&= \frac{\mu}{\lambda}\Pi_B + \Pi_B + \frac{(1-c)\mu+\lambda}{\mu}\Pi_B \\
&= \Pi_B\left(\frac{\mu^2 + \lambda\mu + (1-c)\lambda\mu + \lambda^2}{\lambda\mu}\right) \\
\Rightarrow \Pi_B &= \frac{\lambda\mu}{\mu^2 + (2-c)\lambda\mu + \lambda^2} \\
\Pi_A &= \frac{\mu}{\lambda}\Pi_B = \frac{\mu^2}{\mu^2 + (2-c)\mu\lambda + \lambda^2}
\end{aligned}$$

Π_C is not needed.

$$\begin{aligned}
\lim_{t\to\infty} A(t) &= \lim_{t\to\infty} P_A(t) + P_B(t) = \Pi_A + \Pi_B \\
&= \frac{\mu^2 + \lambda\mu}{\mu^2 + (2-c)\lambda\mu + \lambda^2}
\end{aligned}$$

3.12 – Ideal coverage

If the coverage is ideal, we use the following Markov chain to describe the system.

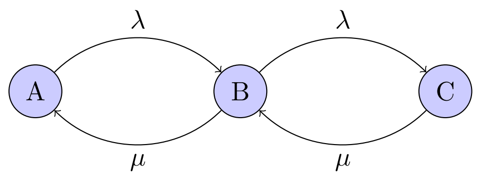

Figure 3: TMR – Markov chain for $A(t)$

This is a birth and death process (see, e.g., Mathematics Handbook, pp. 440-441).

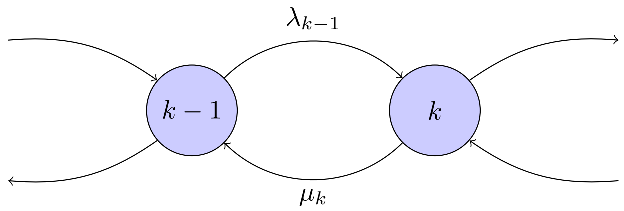

Figure 4: General birth and death process

$$\begin{aligned} Pi_k &= \frac{\lambda_{k-1}}{\mu_k}\Pi_{k-1} \\ \sum_k \Pi_k &= 1 \end{aligned}$$

Using the formulas for a birth and death process, we get:

$$\begin{cases} \Pi_B &= \frac{\lambda}{\mu}\Pi_A \\ \Pi_C &= \frac{\lambda}{\mu}\Pi_B = \frac{\lambda^2}{\mu^2}\Pi_A \\ \Pi_A + \Pi_B + \Pi_C &= 1 \end{cases}$$

$$\begin{aligned} \Pi_A + \Pi_B + \Pi_C &= 1 \\ &\Longleftrightarrow \\ \Pi_A\left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}\right) &= 1 \\ &\Longleftrightarrow \\ \Pi_A\left(\frac{\mu^2 + \lambda\mu + \lambda^2}{\mu^2}\right) &= 1 \\ \Rightarrow \Pi_A &= \frac{\mu^2}{\mu^2 + \lambda\mu + \lambda^2} \end{aligned}$$

$$\begin{aligned}\Pi_B &= \frac{\lambda}{\mu}\Pi_A = \frac{\lambda\mu}{\mu^2 + \lambda\mu + \lambda^2} \\ \lim_{t\to\infty} A(t) &= \Pi_A + \Pi_B = \frac{\mu^2 + \lambda\mu}{\mu^2 + \lambda\mu + \lambda^2}\end{aligned}$$

Compare with expression for the system with coverage factor:

$$A_\infty = \frac{\mu^2 + \lambda\mu}{\mu^2 + (2-c)\lambda\mu + \lambda^2}$$

Problem 3.11

Derive an expression for the steady-state availability of a TMR system where failed modules are repaired.

Solution

The TMR-system is described by the following Markov chain, which is a birth and death process:

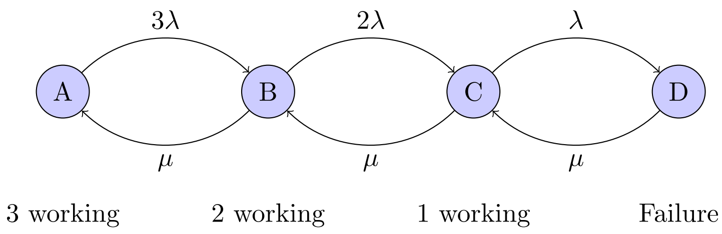

Figure 5: Markov chain

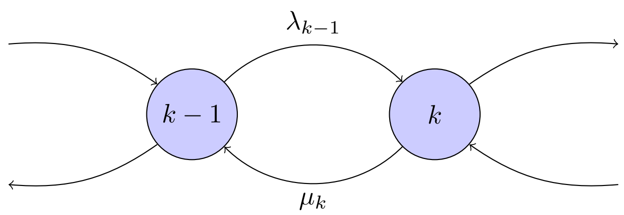

Figure 6: General birth and death process

$$
\begin{aligned}
\Pi_k &= \frac{\lambda_{k-1}}{\mu_k}\Pi_{k-1} \\
\sum_k \Pi_k &= 1
\end{aligned}
$$

The TMR system is a 2-of-3 system $\Rightarrow A(t) = P_A(t) + P_B(t)$

$$
\begin{cases}
\Pi_B & = \frac{3\lambda}{\mu}\Pi_A \\
\Pi_C & = \frac{2\lambda}{\mu}\Pi_B = \frac{6\lambda^2}{\mu^2}\Pi_A \\
\Pi_D & = \frac{\lambda}{\mu}\Pi_C = \frac{6\lambda^3}{\mu^3}\Pi_A \\
\Pi_A + \Pi_B + \Pi_C + \Pi_D & = 1
\end{cases}
$$

$$
\begin{aligned}
\Pi_A\left(1 + \frac{3\lambda}{\mu} + \frac{2\lambda}{\mu}\frac{3\lambda}{\mu} + \frac{\lambda}{\mu}\frac{2\lambda}{\mu}\frac{3\lambda}{\mu}\right) &= 1 \\
&\Longleftrightarrow \\
\Pi_A\left(\frac{\mu^3 + 3\lambda\mu^2 + 6\lambda^2\mu + 6\lambda^3}{\mu^3}\right) &= 1
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \Pi_A &= \frac{\mu^3}{\mu^3 + 3\lambda\mu^2 + 6\lambda^2\mu + 6\lambda^3} \\
\Pi_B &= \frac{3\lambda}{\mu}\Pi_A \\
A(t) &= P_A(t) + P_B(t) \\
\lim_{t\to\infty} A(t) &= \Pi_A + \Pi_B \\
&= \frac{\mu^3 + 3\lambda\mu^2}{\mu^3 + 3\lambda\mu^2 + 6\lambda^2\mu + 6\lambda^3}
\end{aligned}
$$

Problem 5.2

A fault tolerant computer system consists of four identical computers. From system start, one is the primary computer, one is a hot stand-by spare, and the other two are cold stand-by spares. The cold stand-by spares are powered up one after the other as the active computers fail, acting first as the hot stand-by spare and then as the primary computer. The system is shut-down when only one computer remains operational.

5.2 a

Derive an expression for the system reliability. A primary module and a hot stand-by spare have failure rates that are exponentially distributed with λ. The failure rate for a cold stand-by spare is zero. Covergage is assumed to be 100%.

State	Primary	Hot stand-by	Cold stand-by
A	1	1	2
B	1	1	1
C	1	1	0
D	1	0	0

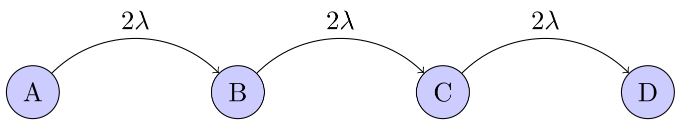

Figure 7: Markov chain

$$
\begin{aligned}
P'(t) &= P(t)Q \\
P(0) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\
Q &= \begin{bmatrix} -2\lambda & 2\lambda & 0 & 0 \\ 0 & -2\lambda & 2\lambda & 0 \\ 0 & 0 & -2\lambda & 2\lambda \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Laplace transform:

$$\mathcal{L}\left\{P'(t) = P(t)Q\right\} \Rightarrow sP(s) - P(0) = P(s)Q$$

$$
\begin{cases}
sP_A - 1 = -2\lambda P_A \\
sP_B = 2\lambda P_A - 2\lambda P_B \\
sP_C = 2\lambda P_B - 2\lambda P_C \\
sP_D = 2\lambda P_C
\end{cases}
$$

$$P_A(s) = \frac{1}{s + 2\lambda}$$

$$\begin{aligned}
P_B(s) &= \frac{2\lambda}{s+2\lambda}P_A = \frac{2\lambda}{s+2\lambda}\frac{1}{s+2\lambda} = \frac{2\lambda}{(s+2\lambda)^2} \\
P_C(s) &= \frac{2\lambda}{s+2\lambda}P_B = \frac{2\lambda}{s+2\lambda}\frac{2\lambda}{(s+2\lambda)^2} = \frac{4\lambda^2}{(s+2\lambda)^3} \\
P_D(s) &= \frac{2\lambda}{s}P_C = \dots
\end{aligned}$$

Inverse Laplace transform (see, e.g., Mathematics Handbook, pp. 332, L22.):

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s+a)^{n+1}}\right\} = t^n e^{-at}$$

$$\begin{aligned}
P_A(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s+2\lambda}\right\} = e^{-2\lambda t} \\
P_B(t) &= \mathcal{L}^{-1}\left\{\frac{2\lambda}{(s+2\lambda)^2}\right\} = 2\lambda\mathcal{L}^{-1}\left\{\frac{1}{(s+2\lambda)^2}\right\} = 2\lambda t e^{-2\lambda t} \\
P_C(t) &= 2\lambda^2\mathcal{L}^{-1}\left\{\frac{2}{(s+2\lambda)^3}\right\} = 2\lambda^2 t^2 e^{-2\lambda t} \\
\Rightarrow R(t) &= P_A(t) + P_B(t) + P_C(t) = \left(1 + 2\lambda t + 2\lambda^2 t^2\right) e^{-2\lambda t}
\end{aligned}$$

5.2 b

The repair of a failed computer is started immediately after the failure occurs. Only one repair person is available, the repair rate is μ. Derive an expression for the steady-state availability.

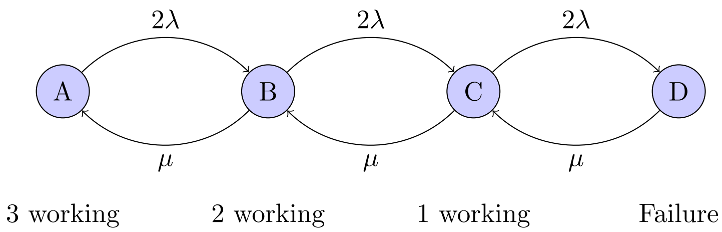

Figure 8: Markov chain

General birth-and-death process:

$$\begin{aligned}
\Pi_i &= \lim_{t\to\infty} P_i(t) \\
\Pi_k &= \frac{\lambda_{k-1}}{\mu_k}\Pi_{k-1} \\
\sum_k \Pi_k &= 1
\end{aligned}$$

Using the formulas above, we get:

$$\Pi_B = \frac{2\lambda}{\mu}\Pi_A$$

$$
\begin{aligned}
\Pi_C &= \frac{2\lambda}{\mu}\Pi_B = \frac{4\lambda^2}{\mu^2}\Pi_A \\
\Pi_D &= \frac{2\lambda}{\mu}\Pi_C = \frac{8\lambda^3}{\mu^3}\Pi_A \\
1 &= \Pi_A + \Pi_B + \Pi_C + \Pi_D \Rightarrow \\
1 &= \Pi_A\left(1 + \frac{2\lambda}{\mu} + \frac{4\lambda^2}{\mu^2} + \frac{8\lambda^3}{\mu^3}\right) \Rightarrow \\
\Pi_A &= \frac{\mu^3}{\mu^3 + 2\lambda\mu^2 + 4\lambda^2\mu + 8\lambda^3} \\
A(\infty) &= \Pi_A + \Pi_B + \Pi_C \\
&= \Pi_A\left(1 + \frac{2\lambda}{\mu} + \frac{4\lambda^2}{\mu^2}\right) \\
&= \Pi_A\left(\frac{\mu^2 + 2\lambda\mu + 4\lambda^2}{\mu^2}\right) \\
&= \frac{\mu^3}{\mu^3 + 2\lambda\mu^2 + 4\lambda^2\mu + 8\lambda^3}\,\frac{\mu^2 + 2\lambda\mu + 4\lambda^2}{\mu^2} \\
&= \frac{\mu^3 + 2\lambda\mu^2 + 4\lambda^2\mu}{\mu^3 + 2\lambda\mu^2 + 4\lambda^2\mu + 8\lambda^3}
\end{aligned}
$$