In this exercise, we look at four problems (problems 2.2, 2.3, 2.6, and 2.7) from the exercise compendium that involve reliability modeling. You will learn how to transform a textual description of a system into a reliability block diagram. Also, you will learn how to derive expressions for reliability and MTTF from a reliability block diagram.

Exercise 1

Problem 2.2

Derive expressions for the reliability and the MTTF for a hot stand-by system consisting of two identical modules whose liftetimes are exponentially distributed with the failure rate λ.

Solution

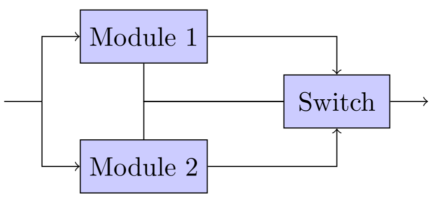


Figure 1: Overview of a hot-standby system.

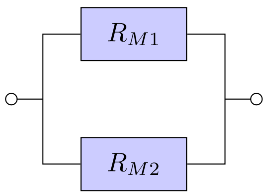


Figure 2: Reliability block diagram

Two identical modules, failure rate exponentially distributed with λ

$$R_{M_1}(t) = R_{M_2}(t) = R_M(t) = e^{-\lambda t}$$

Derive $R_{sys}(t)$ and $MTTF_{sys}$.

$$\begin{aligned}
R_{sys}(t) &= P(\text{System working at t}) \\
&= P(\text{At least one module is working}) \\
&= 1 - P(\text{Both modules are broken}) \\
&= 1 - P(\text{M1 broken})P(\text{M2 broken}) \\
&= 1 - (1 - P\,(\text{M1 working at t}))\,(1 - P\,(\text{M2 working at t})) \\
&= 1 - (1 - R_{m_1}(t))(1 - R_{M_2}(t))
\end{aligned}$$

$$
\begin{aligned}
&= 1-(1-R_M(t))^2 \\
&= 1-(1-2R_M(t)+R_M^2(t)) \\
&= 2R_M(t)-R_M^2(t) \\
&= 2e^{-\lambda t}-e^{-2\lambda t}
\end{aligned}
$$

General expression for parallel system

$$
\begin{aligned}
R_{par} &= 1-F_{par} = 1-\prod_{i=1}^{n} F_i = 1-\prod_{i=1}^{n}(1-R_i) \Longrightarrow \\
R_{sys} &= 1-\prod_{i=1}^{2}\left(1-e^{-\lambda t}\right) \\
&= 1-\left(1-e^{-\lambda t}\right)^2 \\
&= 1-\left(1-2e^{-\lambda t}+e^{-2\lambda t}\right) \\
&= 2e^{-\lambda t}-e^{-2\lambda t}
\end{aligned}
$$

$$
\begin{aligned}
MTTF_{sys} &= \int_0^\infty R_{sys}(t)dt = \int_0^\infty 2e^{-\lambda t}-e^{-2\lambda t}dt \\
&= \int_0^\infty 2e^{-\lambda t}dt - \int_0^\infty e^{-2\lambda t} \\
&= \left[-\frac{2}{\lambda}e^{-\lambda t}\right]_0^\infty - \left[-\frac{1}{2\lambda}e^{-2\lambda t}\right]_0^\infty \\
&= 0-(-\frac{2}{\lambda})-(0-(-\frac{1}{2\lambda})) \\
&= \frac{2}{\lambda}-\frac{1}{2\lambda} = \frac{3}{2\lambda}
\end{aligned}
$$

General expression for parallel system:

$$
\begin{aligned}
MTTF &= \frac{1}{\lambda}\sum_{i=1}^{n}\frac{1}{i} \Longrightarrow \\
MTTF &= \frac{1}{\lambda}\sum_{i=1}^{2}\frac{1}{i} \\
&= \frac{1}{\lambda}\left(1+\frac{1}{2}\right) \\
&= \frac{3}{2\lambda}
\end{aligned}
$$

Problem 2.3

A file server in a local area network uses mirrored disks, as shown in the figure below. This means that each file is stored on two separate disks, which allows the system to remain operational even if one disk fails.

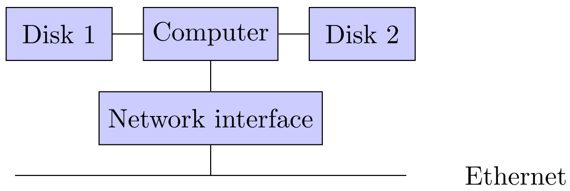


Figure 3: System overview.

Solution

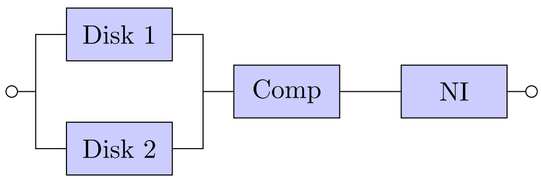


Figure 4: Problem 2.3 a) Reliability block diagram

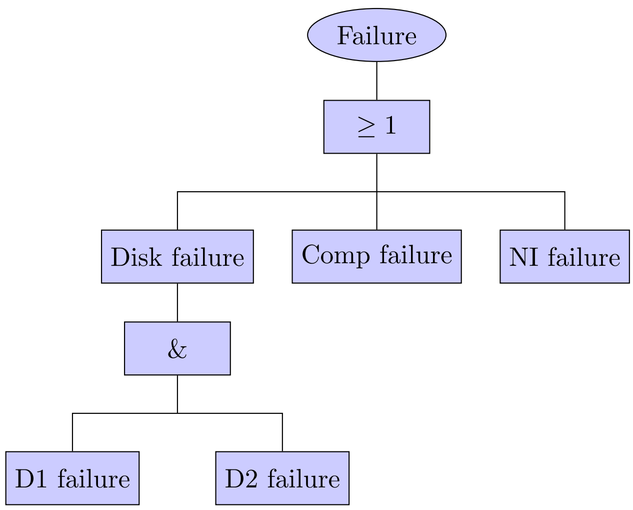


Figure 5: Problem 2.3 b) Fault tree

Problem 2.3 c) Reliability after one year of service

$$\begin{array}{ll} \lambda_{comp} &= 10^-5\text{f/h} \\ \lambda_{disk} &= 10^-4\text{f/h} \\ \lambda_{interface} &= 10^-5\text{f/h} \end{array} \Rightarrow \begin{array}{ll} R_{comp}(t) &= e^{-10^{-5}t} \\ R_{disk}(t) &= e^{-10^{-4}t} \\ R_{interface}(t) &= e^{-10^{-5}t} \end{array}$$

$$\begin{aligned} R_{sys}(t) &= R_{disks}(t) \times R_{comp}(t) \times R_{interface}(t) \\ R_{disks}(t) &= 1 - \prod_{i=1}^{2} (1 - R_{disk}(t)) = 1 - \left(1 - e^{-\lambda_{disk}t}\right)^2 \\ &= 2e^{-\lambda_{disk}t} - e^{-2\lambda_{disk}t} \\ R_{sys}(t) &= \left(2e^{-\lambda_{disk}t} - e^{-2\lambda_{disk}t}\right) \times e^{-\lambda_{comp}t} \times e^{-\lambda_{interface}t} \\ &= 2e^{-(\lambda_{disk}+\lambda_{comp}+\lambda_{interface})t} - e^{-(2\lambda_{disk}+\lambda_{comp}+\lambda_{interface})t} \end{aligned}$$

$$R_{sys}(t = 1 \text{ year} = 8760h) = 0.5535$$

Problem 2.3 d) MTTF

$$\begin{aligned} MTTF &= \int_0^\infty R(t)dt \\ &= \int_0^\infty 2e^{-(\lambda_{disk}+\lambda_{comp}+\lambda_{interface})t} - e^{-(2\lambda_{disk}+\lambda_{comp}+\lambda_{interface})t} \\ &= \left[-\frac{2}{\lambda_{disk}+\lambda_{comp}+\lambda_{interface}} e^{-(\lambda_{disk}+\lambda_{comp}+\lambda_{interface})t}\right]_0^\infty \\ &\quad - \left[-\frac{1}{2\lambda_{disk}+\lambda_{comp}+\lambda_{interface}} e^{-(2\lambda_{disk}+\lambda_{comp}+\lambda_{interface})t}\right]_0^\infty \\ &= \frac{2}{\lambda_{disk}+\lambda_{comp}+\lambda_{interface}} - \frac{1}{2\lambda_{disk}+\lambda_{comp}+\lambda_{interface}} \\ &= \cdots = 1.21 \times 10^4 h \approx 1.4 \text{ years} \end{aligned}$$

Problem 2.6

Derive an expression for the reliability of the system shown below.

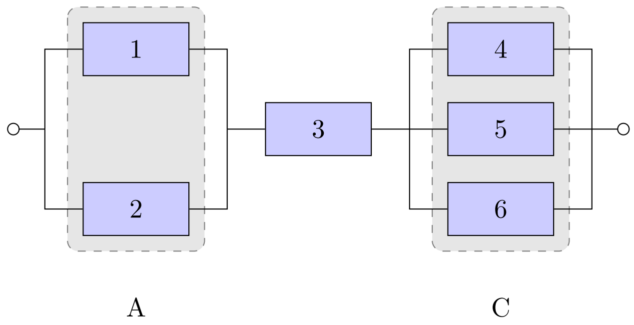
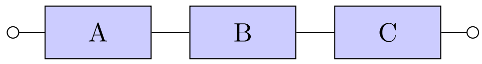


Figure 6: Problem 2.6

Simplified system:



Solution

$$
\begin{aligned}
R_A(t) &= 1 - (1 - R_1(t))(1 - R_2(t)) \\
R_B(t) &= R_3(t) \\
R_C &= 1 - (1 - R_4(t))(1 - R_5(t))(1 - R_6(t)) \\
R_{sys} &= (1 - (1 - R_1(t))(1 - R_2(t)))\, R_3(t)\, (1 - (1 - R_4(t))(1 - R_5(t))(1 - R_6(t)))
\end{aligned}
$$

Problem 2.7

A fault-tolerant computer system consists of four identical computers. The system is a hot stand-by system. The system delivers service as long as two comptuters are operating correctly. The system is shut down when only one correctly working computer system remains.

This is a 2-of-4 system and we are asked to calculate the reliability and MTTF for the system.

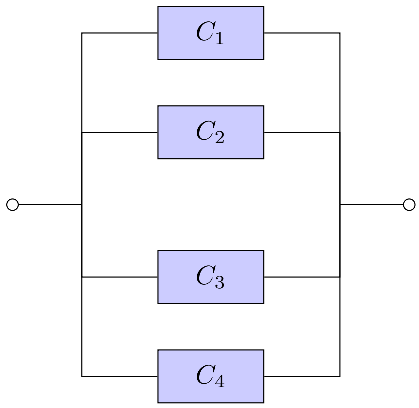


Figure 7: 2-of-4 system.

Problem 2.7 a) Reliablity after one year of service

The lifetime of one computer is exponentially distributed with a failure rate of 10^{-6} f/h.

$$
\begin{aligned}
R_{m-of-n} &= \sum_{i=m}^{n} \binom{n}{i} R^i \left(1-R\right)^{n-i} \\
\binom{n}{i} &= \frac{n!}{i!(n-i)!} \Longrightarrow \\
R_{2-of-4} &= \sum_{i=2}^{4} \binom{4}{i} R^i \left(1-R\right)^{4-i} \\
&= \binom{4}{2} R^2(1-R)^2 + \binom{4}{3} R^3(1-R) + \binom{4}{4} R^4(1-R)^0 \\
&= \frac{4!}{2!2!} R^2(1-R)^2 + \frac{4!}{3!1!} R^3(1-R)^1 + \frac{4!}{4!} R^4(1-R)^0 \\
&= \frac{24}{4} R^2(1-R)^2 + \frac{24}{6} R^3(1-R) + R^4 \\
&= 6\left(R^2 - 2R^3 + R^4\right) + 4\left(R^3 - R^4\right) + R^4 \\
&= 3R^4 - 8R^3 + 6R^2 \\
&= \left\{R(t) = e^{-\lambda t}\right\} = 3e^{-4\lambda t} - 8e^{-3\lambda t} + 6e^{-2\lambda t}
\end{aligned}
$$

$$R_{sys}(1 \text{ year}) = 0.999997$$

Problem 2.7 b) MTTF

$$\begin{aligned}
MTTF &= \int_0^\infty R_{sys}(t)dt \\
&= \int_0^\infty \left(3e^{-4\lambda t} - 8e^{-3\lambda t} + 6e^{-2\lambda t}\right) dt \\
&= \left[-\frac{3}{4\lambda}e^{-4\lambda t} + \frac{8}{3\lambda}e^{-3\lambda t} - \frac{3}{\lambda}e^{-2\lambda t}\right]_0^\infty \\
&= 0 - \left(-\frac{3}{4\lambda} + \frac{8}{3\lambda} - \frac{3}{\lambda}\right) \\
&= \frac{13}{12\lambda} \approx 1.08 \times 10^6 h
\end{aligned}$$