Exercise 2

In the first exercise, we used reliability block diagrams and fault trees for modeling reliability. These methods are based on the assumption that the system is started at time $t = 0$ and only changes by failures. However, we cannot model reconfigurations of a system using reliability block diagrams and fault trees.

In this exercise, we will learn how to do reliability modeling using Markov chains. This technique, as we will see, makes it possible to model system reconfigurations. We will also cover dormancy and coverage factors, and show examples of how they are used.

Problem 3.1

Derive expressions for the reliability and the MTTF for a TMR/Simplex system consisting of three identical modules whose lifetimes are exponentially distributed with the failure rate λ. When a module fails, the failing module and one of the non-faulty modules are taken off-line.

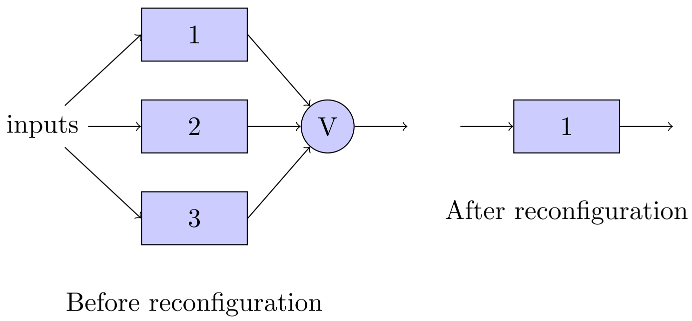

Figure 1: System overview.

Solution

- States in a Markov chain model correspond to system states or system configurations.
- Transitions are events occurring with a certain rate.

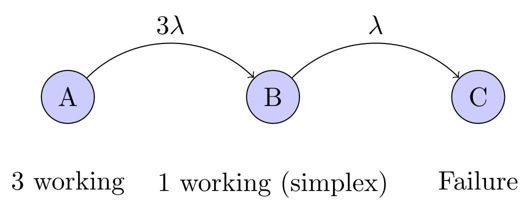

Figure 2: Markov chain

We calculate the reliability of the system by solving the equation system

$$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$$

where

$$\begin{aligned} \mathbf{P}(t) &= \begin{bmatrix} P_A(t) & P_B(t) & P_C(t) \end{bmatrix} \\ \mathbf{Q} &= \text{transition rate matrix} \end{aligned}$$

Initial state:

$$\mathbf{P}(0) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Transition rate matrix:

$$\mathbf{Q} = \begin{bmatrix} -3\lambda & 3\lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

Laplace transform to solve the differential equations (see, e.g., Mathematics Handbook, 5th edition, p. 331, L7).

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= sF(s) - f(0) \Longrightarrow \\ s\mathbf{P}(s) - \mathbf{P}(0) &= \mathbf{P}(s)\mathbf{Q} \Longrightarrow \\ s\begin{bmatrix} P_A & P_B & P_C \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} &= \begin{bmatrix} P_A & P_B & P_C \end{bmatrix} \begin{bmatrix} -3\lambda & 3\lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{cases} sP_A(s) - 1 = -3\lambda P_A(s) \\ sP_B(s) = 3\lambda P_A(s) - \lambda P_B(s) \\ sP_C(s) = \lambda P_B(s) \end{cases}$$

The reliability of the system is

$$R(t) = P_A(t) + P_B(t)$$

Find P_A and P_B:

$$\begin{cases} P_A(s) = \frac{1}{s+3\lambda} \\ P_B(s) = \frac{3\lambda P_A(s)}{s+\lambda} = \frac{3\lambda}{s+\lambda}\frac{1}{s+3\lambda} \end{cases}$$

Decompose the expression for $P_B(s)$ into partial fractions:

$$\begin{aligned} \frac{3\lambda}{s+\lambda}\frac{1}{s+3\lambda} &= \frac{A}{s+\lambda} + \frac{B}{s+3\lambda} \\ &= \frac{A}{s+\lambda}\frac{(s+\lambda)(s+3\lambda)}{(s+\lambda)(s+3\lambda)} + \frac{B}{s+3\lambda}\frac{(s+\lambda)(s+3\lambda)}{(s+\lambda)(s+3\lambda)} \\ &= \frac{A(s+3\lambda)}{(s+\lambda)(s+3\lambda)} + \frac{B(s+\lambda)}{(s+\lambda)(s+3\lambda)} \\ \Longrightarrow 3\lambda &= As + 3\lambda A + Bs + \lambda B \\ & \begin{cases} 3\lambda = 3\lambda A + \lambda B \\ 0 = A + B \end{cases} \Rightarrow \begin{cases} A = -B \\ 3\lambda = 3\lambda A - \lambda A \end{cases} \\ &\Rightarrow \begin{cases} A = \frac{3}{2} \\ B = -\frac{3}{2} \end{cases} \\ \Longrightarrow P_B(s) &= \frac{3}{2}\frac{1}{s+\lambda} - \frac{3}{2}\frac{1}{s+3\lambda} \end{aligned}$$

Inverse Laplace transform (see, e.g., Mathematics Handbook, p. 332, L21):

$$\mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at} \Longrightarrow$$

$$\begin{cases} P_A(t) &= e^{-3\lambda t} \\ P_B(t) &= \frac{3}{2}e^{-\lambda t} - \frac{3}{2}e^{-3\lambda t} \end{cases}$$

Finally,

$$\begin{aligned} R(t) &= P_A(t) + P_B(t) = e^{-3\lambda t} + \frac{3}{2}e^{-\lambda t} - \frac{3}{2}e^{-3\lambda t} \\ &= \frac{3}{2}e^{-\lambda t} - \frac{1}{2}e^{-3\lambda t} \end{aligned}$$

Problem 3.1 b) MTTF

$$\begin{aligned} MTTF &= \int_0^\infty R(t)dt = \int_0^\infty \frac{3}{2}e^{-\lambda t} - \int_0^\infty \frac{1}{2}e^{-3\lambda t} \\ &= \left[-\frac{3}{2}\frac{1}{\lambda}e^{-\lambda t}\right]_0^\infty - \left[-\frac{1}{2}\frac{1}{3\lambda}e^{-3\lambda t}\right]_0^\infty \\ &= \frac{3}{2}\frac{1}{\lambda} - \frac{1}{2}\frac{1}{3\lambda} = \frac{8}{6\lambda} \\ &= \frac{4}{3}\frac{1}{\lambda} \end{aligned}$$

Discussion about the results

$$MTTF = \frac{4}{3}\frac{1}{\lambda} > \frac{1}{\lambda} \quad \text{(simplex)} > \frac{5}{6}\frac{1}{\lambda} \quad \text{(TMR)}$$

The failure intensity in a TMR system after the first failure will be 2λ. The failure intensity for a simplex system is only λ.

Problem 3.2

A computer system consists of three identical modules. At system start-up, two modules are active while one module is acting as a cold stand-by spare. The failure rate is λ for an active module and μ for a cold module. At least two modules must be operational for the whole system to work. Derive an expression for the reliability of the system. Assume that the coverage is ideal.

Coverage and dormancy factors

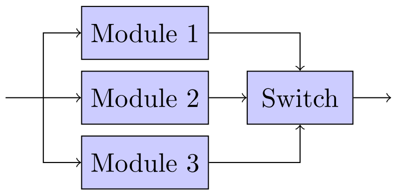

Figure 3: Overview of a standby system.

Coverage: Probability that the fault is handled correctly by the fault tolerance mechanisms.
Dormancy factor: Ratio of active failure rate to inactive failure rate, denoted by k.

$$k = \frac{\lambda_{active}}{\lambda_{standby}} = \frac{\lambda}{\mu}$$

Solution

State	Working	Spares
A	2	1
B	2	0
C	Failure	

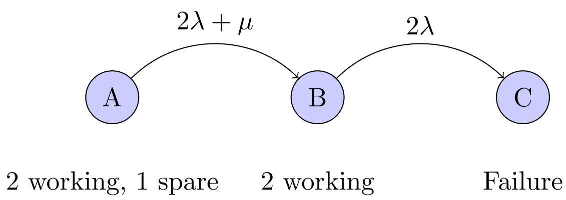

Figure 4: Markov chain

We should assume that the coverage is ideal. *But, how would the Markov chain be affected by a non-ideal coverage factor?*
Coverage: Probability that the fault is handled correctly by the fault tolerance mechanisms. A system with a non-ideal coverage will have a transition going from state A to state C, i.e., the first failure in the system can cause a system failure.

$$\begin{aligned}\mathbf{P}'(t) &= \mathbf{P}(t)\mathbf{Q}\\ \mathbf{Q} &= \begin{bmatrix} -(2\lambda+\mu) & 2\lambda+\mu & 0\\ 0 & -2\lambda & 2\lambda\\ 0 & 0 & 0\end{bmatrix}\\ \mathbf{P}(0) &= \begin{bmatrix} 1 & 0 & 0\end{bmatrix}\end{aligned}$$

Laplace:

$$s\mathbf{P}(s) - \mathbf{P}(0) = \mathbf{P}(s)\mathbf{Q}$$

$$\begin{cases} sP_A - 1 = -(2\lambda+\mu)P_A\\ sP_B = (2\lambda+\mu)P_A - 2\lambda P_B\\ sP_C = 2\lambda P_B\end{cases} \Rightarrow \begin{cases} P_A = \frac{1}{s+(2\lambda+\mu)}\\ P_B = \frac{2\lambda+\mu}{s+2\lambda}P_A\\ P_C = \frac{2\lambda}{s}P_B\end{cases}$$

$$\begin{aligned}\frac{2\lambda+\mu}{s+2\lambda}\frac{1}{s+(2\lambda+\mu)} &= \frac{A}{s+2\lambda} + \frac{B}{s+(2\lambda+\mu)}\\ &= \frac{A(s+(2\lambda+\mu))}{s+2\lambda} + \frac{B(s+2\lambda)}{s+(2\lambda+\mu)}\end{aligned}$$

$$\Rightarrow \begin{cases} A(2\lambda+\mu)+2\lambda B = 2\lambda+\mu\\ A+B = 0\end{cases} \Rightarrow \begin{cases} A = \frac{2\lambda+\mu}{\mu}\\ B = -\frac{2\lambda+\mu}{\mu}\\ \quad = \frac{A(s+(2\lambda+\mu))}{s+2\lambda} + \frac{B(s+2\lambda)}{s+(2\lambda\mu)}\end{cases}$$

Tip for decomposing an expression into partial fractions:

$$\frac{1}{s+a}\frac{1}{s+b} = \frac{1}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right)$$

Inverse Laplace transform:

$$\begin{cases} P_A(t) = e^{-(2\lambda+\mu)t}\\ P_B(t) = \frac{2\lambda+\mu}{\mu}\left(e^{-2\lambda t} - e^{-(2\lambda+\mu)t}\right)\end{cases}$$

Reliability for the system:

$$\begin{aligned}R(t) &= P_A(t)+P_B(t) = e^{-(2\lambda+\mu)t} + \frac{2\lambda+\mu}{\mu}\left(e^{-2\lambda t} - e^{-(2\lambda+\mu)t}\right)\\ &= \left(1 - \frac{2\lambda+\mu}{\mu}\right)e^{-(2\lambda+\mu)t} + \frac{2\lambda+\mu}{\mu}e^{-2\lambda t}\end{aligned}$$

Problem 5.6(variant)

Derive the steady-state availability of a system that from start consists of two active modules and one cold stand-by spare. The system is operational as long as at least one module works. The lifetime is exponentially distributed with a failure rate λ. The cold stand-by spare has a dormancy factor k. The coverage for all reconfigurations are assumed to be ideal.

Solution

State	Working	Spares
A	2	1
B	2	0
C	1	0
D	Failure	

Dormancy factor: ratio of active failure rate to inactive failure rate, denoted by k.

$$k = \frac{\lambda}{\lambda'}$$

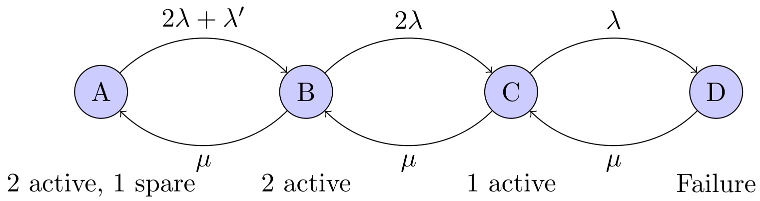

Figure 5: Markov chain

The Markov chain above is a birth-death process, which will be explained in Lecture 6. . Today, we do a variant of 5.6. An expression for the reliability will be derived and repairs will not be considered.

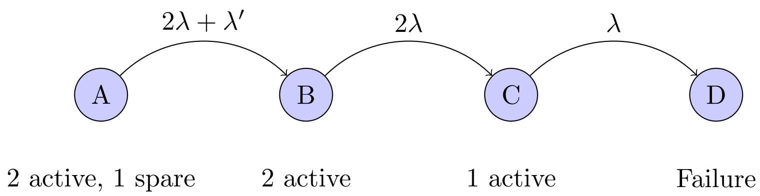

Figure 6: Markov chain

$$\begin{aligned}
\mathbf{P} &= \begin{bmatrix} P_A(t) & P_B(t) & P_C(t) & P_F(t) \end{bmatrix} \\
\mathbf{P}' &= \mathbf{P}(t)\mathbf{Q} \\
\mathbf{P}(0) &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\mathbf{Q} = \begin{bmatrix} -(2\lambda + \lambda/k) & 2\lambda + \lambda/k & 0 & 0 \\ 0 & -2\lambda & 2\lambda & 0 \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Laplace transform:

$$s\mathbf{P}(s) - \mathbf{P}(0) = \mathbf{P}(s)\mathbf{Q}$$

$$\begin{cases} sP_A(s) - 1 &= -\left(2\lambda + \frac{\lambda}{k}\right) P_A(s) \\ sP_B(s) &= \left(2\lambda + \frac{\lambda}{k}\right) P_A(s) - 2\lambda P_B(s) \\ sP_C(s) &= 2\lambda P_B(s) - \lambda P_C(s) \\ sP_F(s) &= \lambda P_C(s) \end{cases}$$

Let:

$$\begin{aligned} a &= 2\lambda + \frac{\lambda}{k} \\ b &= 2\lambda \\ c &= \lambda \end{aligned}$$

$$\Longrightarrow$$

$$\begin{cases} P_A(s) &= \frac{1}{s+a} \\ P_B(s) &= \frac{a}{s+a}\frac{1}{s+b} = \frac{a}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right) \\ P_C(s) &= \frac{b}{s+c}P_B = \frac{b}{s+c}\frac{a}{b-a}\left(\frac{1}{s+a} - \frac{1}{s+b}\right) \\ &= \frac{ab}{(b-a)}\left(\frac{1}{a-c}\left(\frac{1}{s+c} - \frac{1}{s+a}\right) - \frac{1}{b-c}\left(\frac{1}{s+c} - \frac{1}{s+b}\right)\right) \\ &= \frac{ab}{b-a}\frac{1}{c-a}\frac{1}{s+a} + \frac{ab}{b-a}\frac{1}{b-c}\frac{1}{s+b} + \frac{ab}{b-a}\frac{1}{s+c}\left(\frac{1}{a-c} - \frac{1}{b-c}\right) \\ &= \frac{ab}{b-a}\frac{1}{c-a}\frac{1}{s+a} + \frac{ab}{b-a}\frac{1}{b-c}\frac{1}{s+b} + \frac{ab}{a-c}\frac{1}{b-c}\frac{1}{s+c} \end{cases}$$

Inverse Laplace transform:

$$\begin{cases} P_A(t) &= e^{-at} \\ P_B(t) &= \frac{a}{b-a}\left(e^{-at} - e^{-bt}\right) \\ P_C(t) &= \frac{ab}{(b-a)(c-a)}e^{-at} + \frac{ab}{(b-a)(b-c)}e^{-bt} + \frac{ab}{(a-c)(b-c)}e^{-ct} \end{cases}$$

$$R_{sys}(t) = P_A(t) + P_B(t) + P_C(t)$$