The Simplex Method by Example

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The simplex method

Linear Programming (Standard form).

$$
\begin{array}{lll}
\text{(LP)} & \text{maximize} & c^T x \\
 & \text{subject to} & Ax \leq b, \\
 & & x \geq 0,
\end{array}
$$

Example.

$$
\begin{array}{lrcrcl}
\text{maximize} & x_1 + x_2 & & & & \\
\text{subject to} & -x_1 & + & x_2 & \leq & 1, \\
 & x_1 & & & \leq & 3, \\
 & & & x_2 & \leq & 2, \\
 & x_1, x_2 \geq 0. & & & &
\end{array}
$$

After introducing *slack* variables:

$$
\begin{array}{lrclcrcl}
\text{maximize} & x_1 + x_2 & & & & & & \\
\text{subject to} & x_3 & = & 1 & + & x_1 & - & x_2, \\
 & x_4 & = & 3 & - & x_1, & & \\
 & x_5 & = & 2 & & & - & x_2, \\
 & x_1, \ldots, x_5 \geq 0. & & & & & &
\end{array}
$$

Tableaus

Rewrite the LP in *tableau* form.

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}
$$

- Tableau expresses *basic* variables x_B and the objective function value z in terms of *nonbasic* variables x_N
- Any tableau solution with $x \geq 0$ is an LP solution
- $x_N = 0 \Rightarrow$ *basic feasible solution* (BFS)

The pivot step (I)

Given: Tableau

$$
\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}
$$

and associated basic feasible solution

$$x^* = (0, 0, 1, 3, 2).$$

Goal: increase z by making some nonbasic variable positive, keeping the basic ones non-negative

Candidates: the ones with positive coefficient in the z-row.

If x_2 is chosen:

$$
\underbrace{
\begin{array}{rclcl}
x_3 & = & 1 + x_1 - x_2 & \Rightarrow & x_2 \leq 1, \\
x_4 & = & 3 - x_1 & \Rightarrow & \text{no limit for } x_2 \\
x_5 & = & 2 - x_2 & \Rightarrow & x_2 \leq 2.
\end{array}
}_{\Rightarrow \text{ increase } x_2 \text{ by } 1, \; x_3 \text{ becomes zero}}
$$

The pivot step (II)

Rewriting the tableau:

$$\begin{array}{lcl} x_2 & \sim & \textit{entering} \text{ variable} \\ x_3 & \sim & \textit{leaving} \text{ variable} \end{array}$$

Solve the equation

$$\mathbf{x_3} = 1 + x_1 - \mathbf{x_2}$$

for x_2 and subsitute

$$\mathbf{x_2} = 1 + x_1 - \mathbf{x_3}$$

on the right hand side of the tableau.

New tableau:

$$\begin{array}{rcrcrcr} x_2 & = & 1 & + & x_1 & - & x_3 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 1 & - & x_1 & + & x_3 \\ \hline z & = & 1 & + & 2x_1 & - & x_3 \end{array}$$

New BFS:

$$x^* = (0, 1, 0, 3, 1), \quad z = 1$$

The pivot step (III)

$$\begin{array}{rcrcrcr} x_2 & = & 1 & + & x_1 & - & x_3 \\ x_4 & = & 3 & - & x_1 & & \\ x_5 & = & 1 & - & x_1 & + & x_3 \\ \hline z & = & 1 & + & 2x_1 & - & x_3 \end{array}$$

$$x^* = (0, 1, 0, 3, 1), \quad z = 1$$

x_1 enters, x_5 leaves:

$$\begin{array}{rcrcrcr} x_2 & = & 2 & & & - & x_5 \\ x_4 & = & 2 & - & x_3 & + & x_5 \\ x_1 & = & 1 & + & x_3 & - & x_5 \\ \hline z & = & 3 & + & x_3 & - & 2x_5 \end{array}$$

$$x^* = (1, 2, 0, 2, 0), \quad z = 3.$$

x_3 enters, x_4 leaves:

$$\begin{array}{rcrcrcr} x_2 & = & 2 & & & - & x_5 \\ x_3 & = & 2 & - & x_4 & + & x_5 \\ x_1 & = & 3 & - & x_4 & & \\ \hline z & = & 5 & - & x_4 & - & x_5 \end{array}$$

$$x^* = (3, 2, 2, 0, 0), \quad z = 5$$

Termination

$$\begin{array}{rcrcrcr} x_2 & = & 2 & & & - & x_5 \\ x_3 & = & 2 & - & x_4 & + & x_5 \\ x_1 & = & 3 & - & x_4 & & \\ \hline z & = & 5 & - & x_4 & - & x_5 \end{array}$$

$$x^* = (3, 2, 2, 0, 0), \quad z = 5$$

z cannot be increased anymore by increasing nonbasic variables (local optimum).

Observation: Any local optimum is a global optimum.

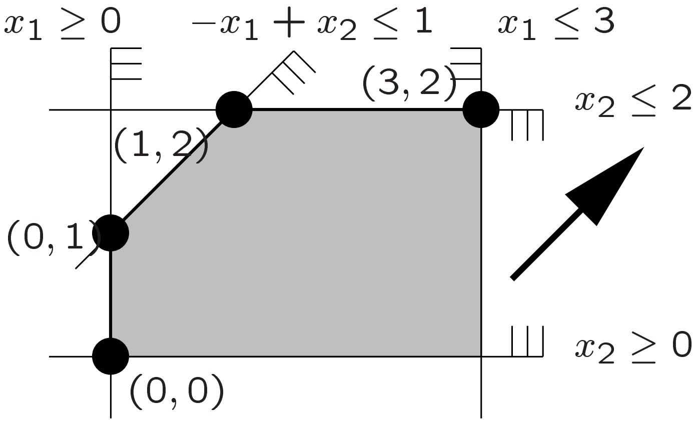
Proof:

$$\underbrace{\begin{array}{rcrcccc} z & = & 5 & - & x_4 & - & x_5 \\ & & & & x_4 \geq 0 & & \\ & & & & & & x_5 \geq 0 \end{array}}_{\text{implies } z \leq 5}$$

Current value $z = 5$ is optimal!

Geometric interpretation

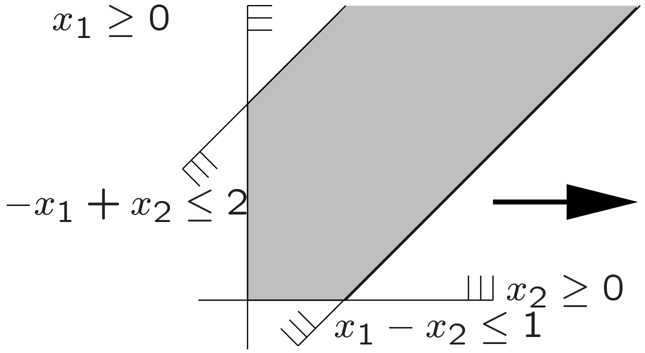
Feasible region	$\rightarrow$	*polyhedron* P
BFS	$\leftrightarrow$	*vertex* of P
Pivot step	$\leftrightarrow$	move along an *edge* of P, monotonically increasing z



BFS	vertex	$z = x_1 + x_2$
$(0,0,1,3,4)$	$(0,0)$	0
$(0,1,0,3,1)$	$(0,1)$	1
$(1,2,0,2,0)$	$(1,2)$	3
$(3,2,2,0,0)$	$(3,2)$	5

Unboundedness (I)

$$
\begin{array}{llrcrcl}
\text{maximize} & & x_1 & & & & \\
\text{subject to} & & x_1 & - & x_2 & \leq & 1, \\
& & -x_1 & + & x_2 & \leq & 2, \\
& x_1, x_2 \geq 0. & & & & &
\end{array}
$$



Initial tableau:

$$
\begin{array}{lcccccc}
x_3 & = & 1 & - & x_1 & + & x_2 \\
x_4 & = & 2 & + & x_1 & - & x_2 \\
\hline
z & = & & & x_1 & &
\end{array}
$$

Unboundedness (II)

$$
\begin{array}{lcccccc}
x_3 & = & 1 & - & x_1 & + & x_2 \\
x_4 & = & 2 & + & x_1 & - & x_2 \\
\hline
z & = & & & x_1 & &
\end{array}
$$

x_1 **enters,** x_3 **leaves:**

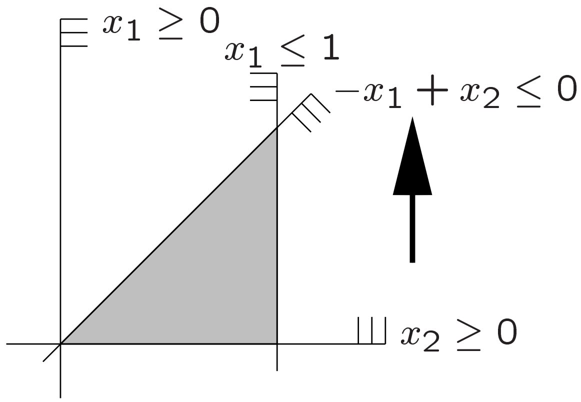
$$
\begin{array}{lcccccc}
x_1 & = & 1 & + & x_2 & - & x_3 \\
x_4 & = & 3 & & & - & x_3 \\
\hline
z & = & 1 & + & x_2 & - & x_3
\end{array}
$$

No bound on the increase of $x_2 \Rightarrow z_{opt} = \infty$

Geometric interpretation: Polyhedron is unbounded in the optimization direction.

Degeneracy (I)

$$\begin{array}{ll} \text{maximize} & x_2 \\ \text{subject to} & -x_1 + x_2 \leq 0, \\ & x_1 \leq 2, \\ & x_1, x_2 \geq 0, \end{array}$$



Initial tableau:

$$\begin{array}{lclclcl} x_3 & = & & & x_1 & - & x_2 \\ x_4 & = & 2 & - & x_1 & & \\ \hline z & = & & & & & x_2 \end{array}$$

Degeneracy (II)

$$\begin{array}{lclclcl} x_3 & = & & & x_1 & - & x_2 \\ x_4 & = & 2 & - & x_1 & & \\ \hline z & = & & & & & x_2 \end{array}$$

x_2 **enters, and . . . the increase in** z **is zero!**

This does *not* imply global optimality!

x_2 **enters,** x_3 **leaves:**

$$\begin{array}{lclclcl} x_2 & = & & & x_1 & - & x_3 \\ x_4 & = & 2 & - & x_1 & & \\ \hline z & = & & & x_1 & - & x_3 \end{array}$$

$$x^* = (0, 0, 0, 2), \quad z = 0$$

x_1 **enters,** x_4 **leaves:**

$$\begin{array}{lclclcl} x_1 & = & 2 & & & - & x_4 \\ x_2 & = & 2 & - & x_3 & - & x_4 \\ \hline z & = & 2 & - & x_3 & - & x_4 \end{array}$$

$$x^* = (2, 2, 0, 0), \quad z = 2 \Rightarrow \text{optimal}$$

Degeneracy (III)

Geometric interpretation: vertex is overdetermined; more than d hyperplanes go through it.

Degeneracy can lead to *cycling*!

Fix: Symbolic perturbation: $Ax \leq b + (\varepsilon, \varepsilon^2, \ldots)^T$

$$\begin{array}{rcrcrcr} x_3 & = & \varepsilon & + & x_1 & - & x_2 \\ x_4 & = & 2+\varepsilon^2 & - & x_1 & & \\ \hline z & = & & & & & x_2 \end{array}$$

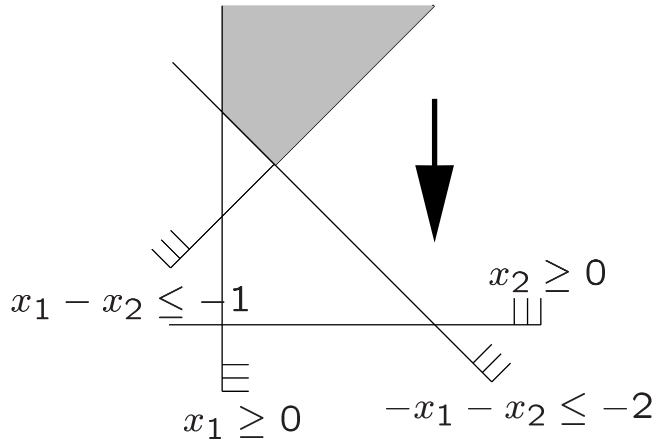
x_2 enters, x_3 leaves:

$$\begin{array}{rcrcrcr} x_2 & = & \varepsilon & + & x_1 & - & x_3 \\ x_4 & = & 2+\varepsilon^2 & - & x_1 & & \\ \hline z & = & \varepsilon & + & x_1 & - & x_3 \end{array}$$

This time, we made ε-progress in z! In the end, ignore the ε's!

Infeasibility (I)

$$\begin{array}{llrcrcr} \text{maximize} & -x_2 & & & & & \\ \text{subject to} & -x_1 & - & x_2 & \leq & -2, \\ & x_1 & - & x_2 & \leq & -1, \\ & x_1, x_2 \geq 0. & & & & \end{array}$$



Initial tableau:

$$\begin{array}{rcrcrcr} x_3 & = & -2 & + & x_1 & + & x_2 \\ x_4 & = & -1 & - & x_1 & + & x_2 \\ \hline z & = & & & & - & x_2 \end{array}$$

does not give a BFS!

Infeasibility (II)

Fix: set up auxiliary problem

$$\begin{array}{llcrcrcrcr}
\text{maximize} & & & -x_0 \\
\text{subject to} & & & -x_1 & - & x_2 & - & x_0 & \leq & -2, \\
 & & & x_1 & - & x_2 & - & x_0 & \leq & -1, \\
 & x_0, x_1, x_2 \geq 0.
\end{array}$$

Observation:

(i) Auxiliary problem is feasible (choose x_0 large)

(ii) Original problem feasible $\Leftrightarrow$ auxiliary problem has optimal value $x_0 = 0$.

Infeasibility (III)

Initial tableau:

$$\begin{array}{rcrcrcrcr}
x_3 & = & -2 & + & x_1 & + & x_2 & + & x_0 \\
x_4 & = & -1 & - & x_1 & + & x_2 & + & x_0 \\
\hline
w & = & & & & & & - & x_0
\end{array}$$

Increase x_0 to obtain a BFS (w gets worse!)

x_0 enters, x_3 leaves:

$$\begin{array}{rcrcrcrcr}
x_0 & = & 2 & - & x_1 & - & x_2 & + & x_3 \\
x_4 & = & 1 & - & 2x_1 & & & + & x_3 \\
\hline
w & = & -2 & + & x_1 & + & x_2 & - & x_3
\end{array}$$

x_2 enters, x_0 leaves:

$$\begin{array}{rcrcrcrcr}
x_2 & = & 2 & - & x_1 & + & x_3 & - & x_0 \\
x_4 & = & 1 & - & 2x_1 & + & x_3 & & \\
\hline
w & = & & & & & & - & x_0
\end{array}$$

In the feasible case, x_0 is nonbasic in the end!

Infeasibility (IV)

To obtain a tableau for the original problem:

- Remove x_0 from the tableau

- Express z in the nonbasic variables

$$\begin{array}{rcrcrcrcr}
x_2 & = & 2 & - & x_1 & + & x_3 & - & x_0 \\
x_4 & = & 1 & - & 2x_1 & + & x_3 & & \\
\hline
w & = & & & & & & - & x_0
\end{array}$$

$$z = -x_2 = -2 + x_1 - x_3$$

Initial tableau for original problem is

$$\begin{array}{rcrcrcr}
x_2 & = & 2 & - & x_1 & + & x_3 \\
x_4 & = & 1 & - & 2x_1 & + & x_3 \\
\hline
z & = & -2 & + & x_1 & - & x_3
\end{array}$$

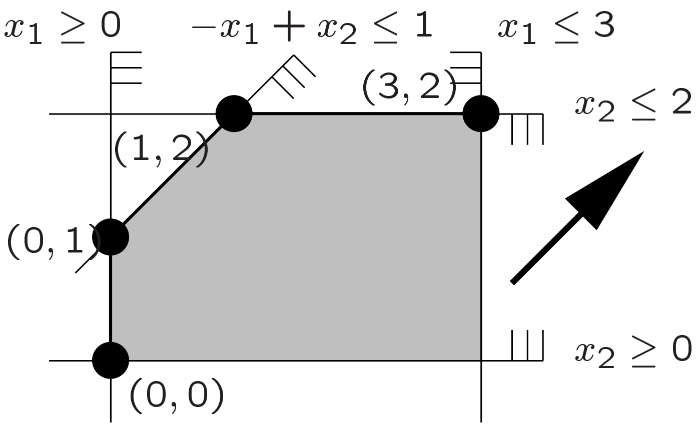
$$x^* = (0, 2, 0, 1)$$

Pivot rules (I)

How to choose the entering variable in case there are several choices?

$$\begin{array}{rcrcrcr}
x_3 & = & 1 & + & x_1 & - & x_2 \\
x_4 & = & 3 & - & x_1 & & \\
x_5 & = & 2 & & & - & x_2 \\
\hline
z & = & & & x_1 & + & x_2
\end{array}$$

Every choice corresponds to one z-increasing edge of the feasible region.



Two increasing edges in the initial vertex $(0, 0)$

Pivot rules (II)

Dantzig's rule: Choose the increasing variable with largest coefficient

Bland's rule: Choose the increasing variable with smallest index (according to some order)

Largest increase: Choose the increasing variable which leads to the largest increase in z

Steepest edge: Choose the increasing edge which is steepest w.r.t. the optimization direction

Random-Edge: Choose an increasing variable uniformly at random