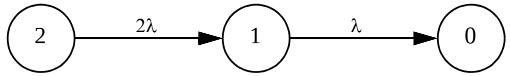
Blackboard example, lecture 4

We use a Markov chain model to calculate the reliability for a hot stand-by system with one spare module. We obtain the following state diagram:



The state labelling shows the number of working modules. Thus, in state 2 both modules are working, while one module is working in state 1. The reliability of the system is $R(t) = P_2(t) + P_1(t)$

The state probabilities $P_2(t)$ och $P_1(t)$ are determined by the following system of differential equations:

$$\mathbf{P'(t)} = \mathbf{P(t)} \cdot \mathbf{Q}$$

where

$$\mathbf{P(t)} = \begin{bmatrix} P_2(t) & P_1(t) & P_0(t) \end{bmatrix}$$

$$\mathbf{P'(t)} = \begin{bmatrix} P'_2(t) & P'_1(t) & P'_0(t) \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & 0 \end{bmatrix}$$

The equation system thus consists of the following equations:

$$P'_2(t) = -2\lambda \cdot P_2(t)$$
$$P'_1(t) = 2\lambda \cdot P_2(t) - \lambda \cdot P_1(t)$$
$$P'_0(t) = \lambda \cdot P_1(t)$$

We can solve the equation system using the following Laplace transform: $f'(t) \propto s\tilde{f}(s) - f(0)$. The system of differential equations is then transformed into an algebraic equation system, which can be written in matrix form as follows: .

$$s \cdot \tilde{\mathbf{P}}(\mathbf{s}) - \mathbf{P}(\mathbf{0}) = \tilde{\mathbf{P}}(\mathbf{s}) \cdot \mathbf{Q}$$

where

$$\mathbf{P}(\mathbf{0}) = \begin{bmatrix} P_2(0) & P_1(0) & P_0(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

is a row vector of the initial probabilities for the different states. We assume that the system is in state 2 at the time $t = 0$, hence $P_2(0) = 1$ and $P_1(0) = P_0(0) = 0$

We then obtain the following equation system:

$$\begin{aligned} s \cdot \tilde{P}_2(s) - 1 &= -2\lambda \cdot \tilde{P}_2(s) \\ s \cdot \tilde{P}_1(s) - 0 &= 2\lambda \cdot \tilde{P}_2(s) - \lambda \cdot \tilde{P}_1(s) \\ s \cdot \tilde{P}_0(s) - 0 &= \lambda \cdot \tilde{P}_1(s) \end{aligned}$$

For $\tilde{P}_2(s)$ we obtain:

$$\tilde{P}_2(s) = \frac{1}{s + 2\lambda}$$

which give us the following time function

$$P_2(t) = e^{-2\lambda \cdot t}$$

We then calculate $\tilde{P}_1(s)$:

$$\begin{aligned} \tilde{P}_1(s) &= \frac{2\lambda \cdot \tilde{P}_2(s)}{(s + \lambda)} = \frac{2\lambda}{(s + \lambda)(s + 2\lambda)} \\ &= 2\lambda\left[\frac{A}{s + \lambda} + \frac{B}{s + 2\lambda}\right] \\ &= \left\{A = \frac{1}{\lambda}, B = -\frac{1}{\lambda}\right\} \\ &= 2\lambda\left[\frac{1}{\lambda} \cdot \frac{1}{s + \lambda} - \frac{1}{\lambda} \cdot \frac{1}{s + 2\lambda}\right] \\ &= \frac{2}{s + \lambda} - \frac{2}{s + 2\lambda} \end{aligned}$$

$$P_1(t) = 2e^{-\lambda \cdot t} - 2e^{-2\lambda \cdot t}$$

The reliability of the system is

$$R(t) = P_2(t) + P_1(t) = e^{-2\lambda \cdot t} + 2e^{-\lambda \cdot t} - 2e^{-2\lambda \cdot t} = 2e^{-\lambda \cdot t} - e^{-2\lambda \cdot t}$$

Note that this is the same result we obtain using a reliability block diagram

$$R(t) = 1-(1 - R)^2 = 2R - R^2 = 2e^{-\lambda \cdot t} - e^{-2\lambda \cdot t}$$

where $R = e^{-\lambda \cdot t}$ is the reliability for one module.

