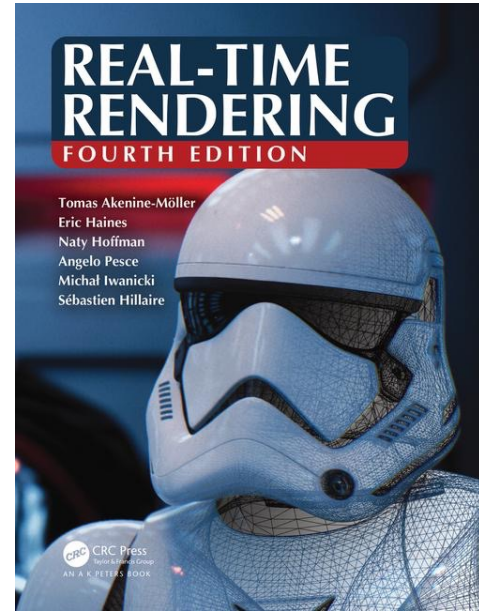# Vectors and Transforms

In

3D Graphics

Course Structure

- 14 lectures
  - Book is the verbal format / more meticulous explanations
  - Lecture slides are only short summary
    - Perhaps not enough to fully understand
  - Exam (salstentamen):
    - I will only assume that you have studied the topics covered by the slides.
    - Reading instructions are pointers to more verbal descriptions in the book
    - May come a few "harder" questions, intended to force you to think beyond what's in the slides (and that could of course accidentally be covered by the book).

- Tutorials – the practical experience
  - 1-6 "holds your hand". Very fast. Intentionally lots of copy/paste. Do them in 2-3 weeks. No need to wait for their deadlines.
  - Project – Here, you apply the knowledge from tutorial 1-6, so you must have understood them.
    - You will need the 3-4 weeks for the project.

The Bonus Material

- Bonus material on home page
  - http://www.cse.chalmers.se/edu/course/TDA362/schedule.html
  - Purpose: **only** to be of help in case lectures and course book is not enough for you to understand. Sometimes, it helps having same topics explained in a second way.

  - Skip the bonus material if you are not very interested.
  - No exam questions on bonus material!

Quick Repetition of Vector Algebra

QUICK STUDY ACADEMIC

PHYSICS

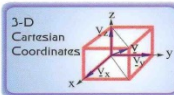## VECTORS AND COORDINATE SYSTEMS

DEFINITIONS

• **Scalar and Vector Quantities.** Physical quantities such as temperature, T, distance, s, density, ρ, work, W, etc., that can be fully described by a single number are called *scalars*. Scalars are not associated with any direction. Physical quantities that have both *magnitude* and *direction* are called *vectors*, e.g. force, F, velocity, V, acceleration, a, momentum, P, etc.

• **Coordinate Systems.** A vector, V, can be described in reference to a *coordinate system*. Two-dimensional coordinate systems can be *cartesian* or *polar*. Three-dimensional coordinate systems can be *cartesian*, *cylindrical* or *spherical*.
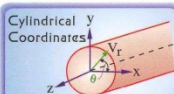
TWO-DIMENSIONAL (2-D) COORDINATE SYSTEMS

• **Cartesian Coordinates (x,y).** A vector, V, in a 2-D cartesian coordinate system can be written as:

$$V = V(V_x, V_y) = V_x\underline{i} + V_y\underline{j}$$

where V_x, V_y are the *vector components*, and $\underline{i}$, $\underline{j}$ are the *unit vectors* along the x and y axis respectively. The *magnitude* of the vector, $|V|$, is:

$$|V| = \sqrt{V_x^2 + V_y^2}$$

• **Polar Coordinates (r,θ).** A vector, V, in polar coordinates can be written as:

$$V = V(V_r, \theta), \text{ where:}$$

$$V_r = \sqrt{V_x^2 + V_y^2} \text{ and}$$

$$\theta = tan^{-1}\left(\frac{V_y}{V_x}\right)$$

• **Relation Between Cartesian and Polar Coordinates:**

$$V_x = |V|\cos\theta \qquad V_y = |V|\sin\theta$$

$$r = \sqrt{x^2 + y^2} \qquad \vartheta = tan^{-1}\left(\frac{y}{x}\right)$$

2-D Cartesian Coordinate System   **2-D Polar Coordinate System**

THREE-DIMENSIONAL (3-D) COORDINATE SYSTEMS:

Cartesian Coordinates (x,y,z). A vector, V, in a 3-D cartesian coordinate system can be written as:

$$|V| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

$$V = V(V_x, V_y, V_z) = V_x\underline{i} + V_y\underline{j} + V_z\underline{k}$$

3-D Cartesian Coordinates

• **Cylindrical Coordinates (r, θ, z).** A vector, V, in a cylindrical coordinate system can be written as:

$$V = V(V_r, \theta, z), \text{ where}$$

$$V_r = \sqrt{V_x^2 + V_y^2} \text{ and } z = z$$

Cylindrical Coordinates

• **Spherical Coordinates (r,θ,φ).** A vector, V, in a spherical coordinate system is written as: $V = V(V_r, \theta, \phi)$, where:

$$V_r = \sqrt{V_x^2 + V_y^2 + V_z^2},$$

$$\theta = tan^{-1}\left(\frac{V_y}{V_x}\right),$$

$$\phi = cos^{-1}\frac{V_z}{\sqrt{V_x^2 + V_y^2 + V_z^2}}$$

Spherical Coordinates

• **Relation Between Cartesian and Spherical Coordinates**

$$V_x = |V|\sin\phi\cos\theta,$$

$$V_y = |V|\sin\phi\sin\theta,$$
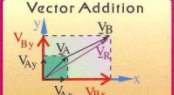
$$V_z = |V|\cos\phi$$

$$x = r\sin\phi\cos\vartheta$$

$$y = r\sin\phi\sin\vartheta$$

$$z = r\cos\phi$$

VECTOR ALGEBRA

• **Vector Addition.** The sum of two vectors, V_A and V_B, in a 2-D cartesian coordinate system is a vector, V_R, defined as:

$$V_R = V_A + V_B$$

In component notation, the summation is given as:

$$V_{RX} = V_{AX} + V_{BX}, \quad V_{RY} = V_{AY} + V_{BY}$$

Vector Addition

• **Commutative Law of Vector Addition:**

$$V_A + V_B = V_B + V_A$$

• **Associative Law of Vector Addition:**

$$(V_A + V_B) + V_C = V_A + (V_B + V_C)$$

• **Distributive Law for Multiplication by a Scalar (ε):**

$$\varepsilon(V_A + V_B) = \varepsilon V_A + \varepsilon V_B$$

• **Scalar or Dot Product:**

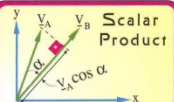$$V_A \cdot V_B = V_B \cdot V_A = |V_A||V_B|\cos\alpha$$

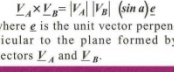where α is the angle between the two vectors. If the two vectors are perpendicular to each other then:

$$|V_A||V_B| = 0, \quad V_A \perp V_B$$

If the vectors are given in terms of their components, then in a 3-D cartesian coordinate system:

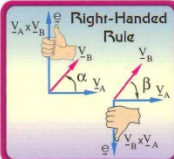$$V_A \cdot V_B = V_{AX}V_{BX} + V_{AY}V_{BY} + V_{AZ}V_{BZ}$$

since:

$$\underline{i} \cdot \underline{i} = 1 \quad \underline{j} \cdot \underline{j} = 1 \quad \underline{k} \cdot \underline{k} = 1$$
$$\underline{i} \cdot \underline{j} = 0 \quad \underline{i} \cdot \underline{k} = 0 \quad \underline{j} \cdot \underline{k} = 0$$
$$\underline{j} \cdot \underline{i} = 0 \quad \underline{k} \cdot \underline{i} = 0 \quad \underline{k} \cdot \underline{j} = 0$$

Scalar Product

• **Vector or Cross Product**

$$V_A \times V_B = |V_A||V_B|(\sin\alpha)\underline{e}$$

where $\underline{e}$ is the unit vector perpendicular to the plane formed by vectors V_A and V_B.

• **Right-hand rule:** the direction of the vector $\underline{e}$ can be found by curling the fingers of the right hand around a hypothetical axis perpendicular to plane V_A - V_B so that the vector V_A rotates along the angle α until is aligned with vector V_B. The thumb then gives the direction of $\underline{e}$.
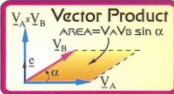
Right-Handed Rule

$$V_A \times V_B = -V_B \times V_A$$

A cartesian coordinate system is called a *right-handed system* if $\underline{i} \times \underline{j} = \underline{k}$.

If two vectors are parallel to each then:
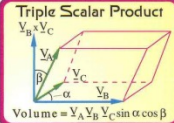
$$V_A \times V_B = 0, \quad V_A \parallel V_B$$

If the vectors are given in terms of their components, then in a 3-D cartesian coordinate system:

$$V_A \times V_B = \begin{Bmatrix} \underline{i} & \underline{j} & \underline{k} \\ V_{AX} & V_{AY} & V_{AZ} \\ V_{BX} & V_{BY} & V_{BZ} \end{Bmatrix}$$

Vector Product   AREA = $V_A V_B \sin\alpha$

• **Triple Scalar Product**
The magnitude of the triple scalar product is equal to the volume of the parallelepiped formed by the three vectors V_A, V_B, V_C: $V_A \cdot (V_B \times V_C)$.

Triple Scalar Product

$$\text{Volume} = V_A V_B V_C \sin\alpha \cos\beta$$

• **Differentiation Formulas of Vectors**

$$\frac{d}{dt}[\underline{u}(t) + \underline{v}(t)] = \frac{du}{dt} + \frac{dv}{dt}$$

$$\frac{d}{dt}[c\underline{u}(t)] = c\frac{du}{dt}$$

$$\frac{d}{dt}[f(t)\underline{u}(t)] = \frac{df}{dt}\underline{u} + f\frac{du}{dt}$$

$$\frac{d}{dt}[\underline{u}(t) \cdot \underline{v}(t)] = \frac{du}{dt} \cdot \underline{v}(t) + \underline{u}(t) \cdot \frac{dv}{dt}$$

• **Integration of a Vector**

$$\int_a^b \underline{r}(t)dt = [\underline{R}(t)]_a^b = \underline{R}(b) - \underline{R}(a)$$

Excellent interactive online linear algebra repetition:
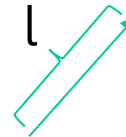
- http://immersivemath.com/ila/index.html
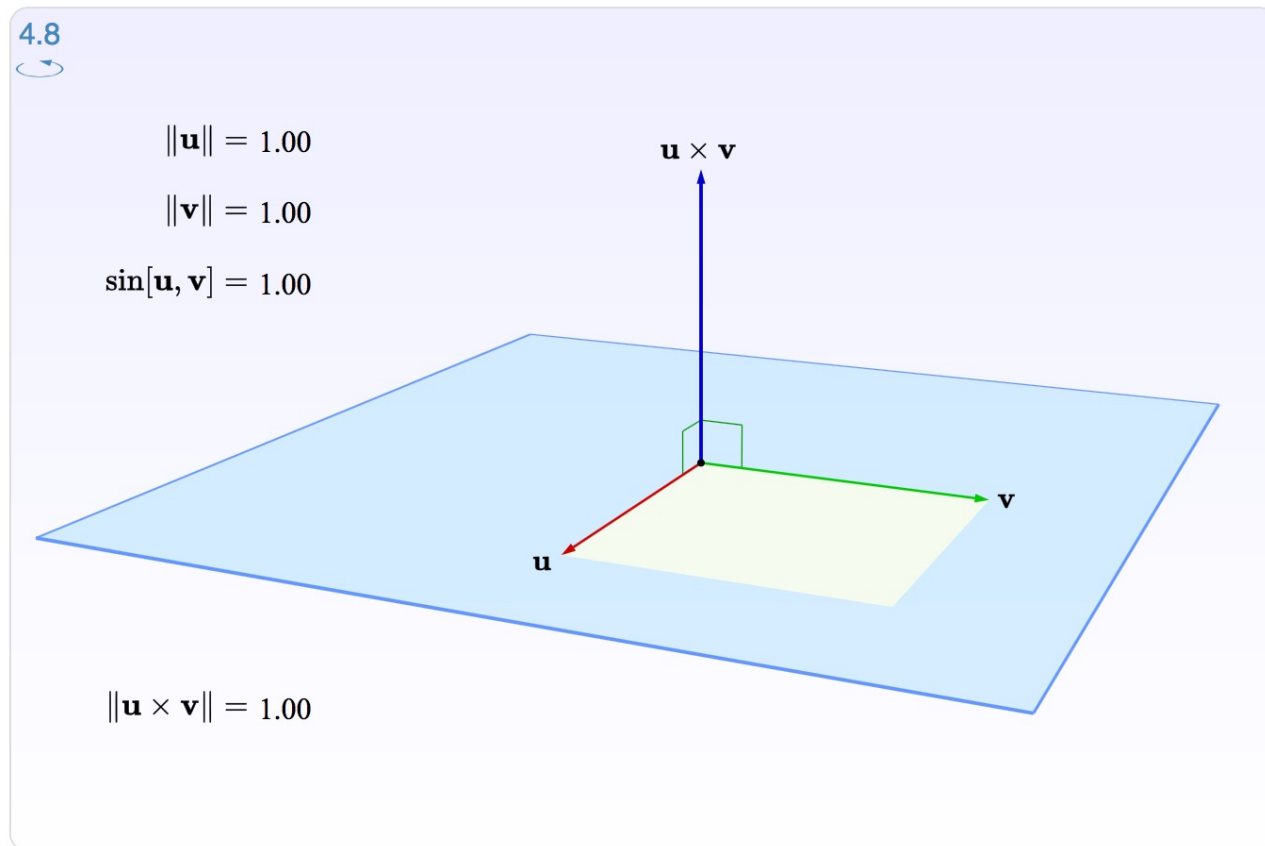
Quick Repetition of Vector Algebra for 3D graphics

A 3D vector, **a**, contains 3 elements: (a_x, a_y, a_z), which are coordinates (or lengths) along the 3 coordinage axes.
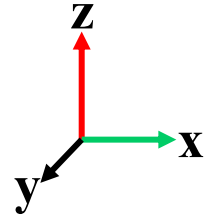
The length of a vector is: $\|\mathbf{x}\| = \sqrt{\left(x^2 + y^2 + z^2\right)}$

Normalizing a vector, **n**, means to scale the vector such that it becomes a unit vector, $\hat{\boldsymbol{n}}$, i.e., its length = 1.

E.g.,: $\hat{\boldsymbol{n}} = \frac{\boldsymbol{n}}{\|\boldsymbol{n}\|} = \frac{\boldsymbol{n}}{\sqrt{n_x^2 + n_y^2 + n_z^2}} = \left(\frac{n_x}{c}, \frac{n_y}{c}, \frac{n_z}{c}\right)$, where $c = \sqrt{n_x^2 + n_y^2 + n_z^2}$

The dot product is typically used to find the angle, a, between two vectors. If **a** and **b** are of unit length (normalized), then $\cos a = \boldsymbol{a} \cdot \boldsymbol{b}$

where $\mathbf{a} \bullet \mathbf{b} = (a_x b_x + a_y b_y + a_z b_z)$

The definition of the dot product is: $\cos \alpha = \dfrac{\mathbf{v}_a \bullet \mathbf{v}_b}{\|\mathbf{v}_a\|\|\mathbf{v}_b\|}$

*(so for non-normalized **a** and **b**, we must divide with their lengths)*
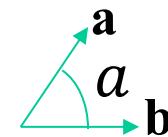
Quick Repetition of Vector Algebra for 3D graphics

Cross product

Definition:

Geometric meaning:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin \theta$$
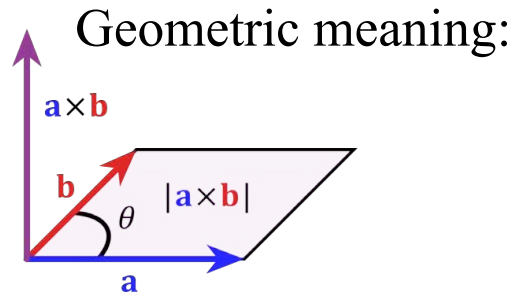
The cross product is typically used to find a vector, **a**, that is perpendicular to two others (**b** and **c**): $\boldsymbol{a} = \boldsymbol{b} \times \boldsymbol{c}$
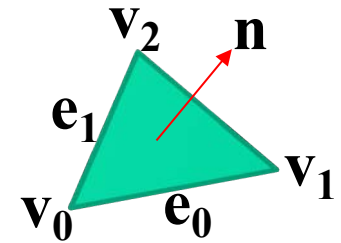
Example to find a triangle normal: $n = e_0 \times e_1$, where $e_0 = (v_1 - v_0)$ and $e_1 = (v_2 - v_0)$

In code: `n = cross(e0, e1);`

In maths:
$$\boldsymbol{n} = ((e0_y e1_z - e0_z e1_y), (e0_z e1_x - e0_x e1_z), (e0_x e1_y - e0_y e1_x))$$

Note that the length of **n** then is two times the size of the triangle area. (So the cross product can be used to find the area between two vectors). We typically want normals to be of unit length (=1), and therefore we normalize n:

In code: `n = normalize(n);`

Structure of today's lecture

- Matrices
  - Matrix mult.
  - Transformation Pipeline
  - Practical usage of matrices
  - Rotations
  - Translations
  - Homogeneous coordinates
  - Shear / scale / normal matrix
  - Euler matrices
  - Quaternions
  - Projections

- Bresenham's line drawing algorithm

Why transforms?

- We want to be able to **animate** objects and the camera
  - Translations
  - Rotations
  - Shears
  - …
- We want to be able to use **projection** transforms

How implement transforms?

- Matrices!
- Can you really do everything with a matrix?
- Not everything, but a lot!
- We use 3x3 and 4x4 matrices

$$\mathbf{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix}$$

Matrix multiplication

Matrix-vector multiplication:

$$\begin{pmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \\ m_{20} & m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} m_{00}p_x + m_{01}p_y + m_{02}p_z \\ m_{10}p_x + m_{11}p_y + m_{12}p_z \\ m_{20}p_x + m_{21}p_y + m_{22}p_z \end{pmatrix}$$

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Matrix multiplication

Matrix-matrix multiplication:

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}$$

where $c_{ij} = a_{row_i} \bullet b_{col_j}$

Matrix multiplication

Matrix-matrix multiplication:

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Matrix multiplication

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where $c_{ij} = a_{row_i} \cdot b_{col_j}$

Matrix multiplication

Matrix-matrix multiplication:

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{00} & b_{01} & b_{02} \\ b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{00} & c_{01} & c_{02} \\ c_{10} & c_{11} & c_{12} \\ c_{20} & c_{21} & c_{22} \end{pmatrix}$$

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where $c_{ij} = a_{row_i} \bullet b_{col_j}$

Model to World Matrix

Model space

World to View Matrix

World space

View space

ModelViewMtx = "Model to View Matrix"

ModelViewMtx * v = $(M_{V \leftarrow W} * M_{W \leftarrow M}) * v$
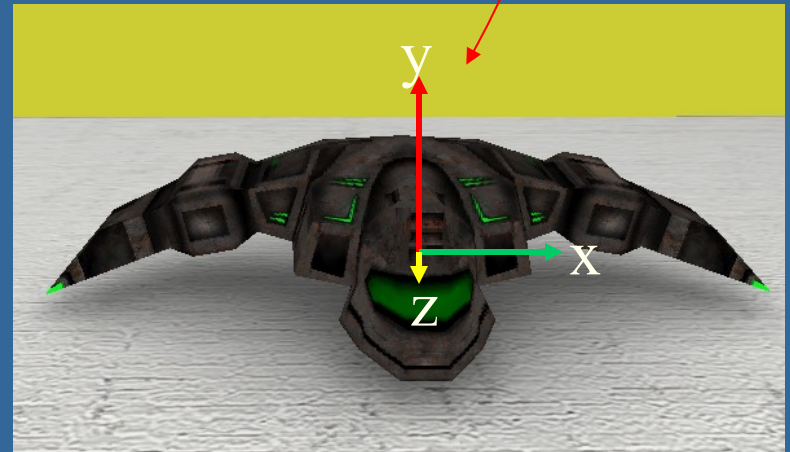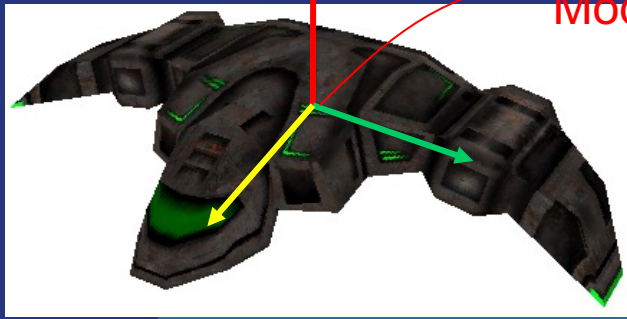
Model space

World space

Model to World Matrix

World to View Matrix

camera
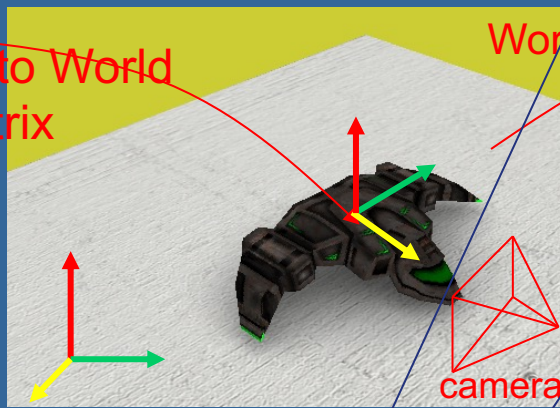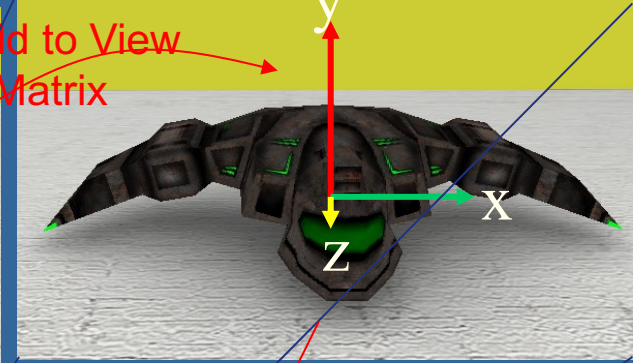
View space

Projection Matrix

ModelViewMtx = "Model to View Matrix"

ModelViewMtx * v = $(M_{V \leftarrow W} * M_{W \leftarrow M}) * v$

Full projection:

V_{clip_space} = projectionMatrix * ModelViewMatrix * v_{model_space}

Or simply: $v_{clip\_space} = M_{MVP} * v$

Transformation Pipeline

Clip space: clipping is nowadays typically done in homogeneous space. However, it used to be done in unit-cube space. Both terminologies are still used.

object space

View/Eye/ Camera space

Homogeneous coord. space

Unit-cube space / Normalized device coords

Window coords. Screen space

| v e r t e x | → | *Modelview Matrix* | → | *Projection Matrix* | → | *Perspective Division* | → | *Viewport Transform* | → |
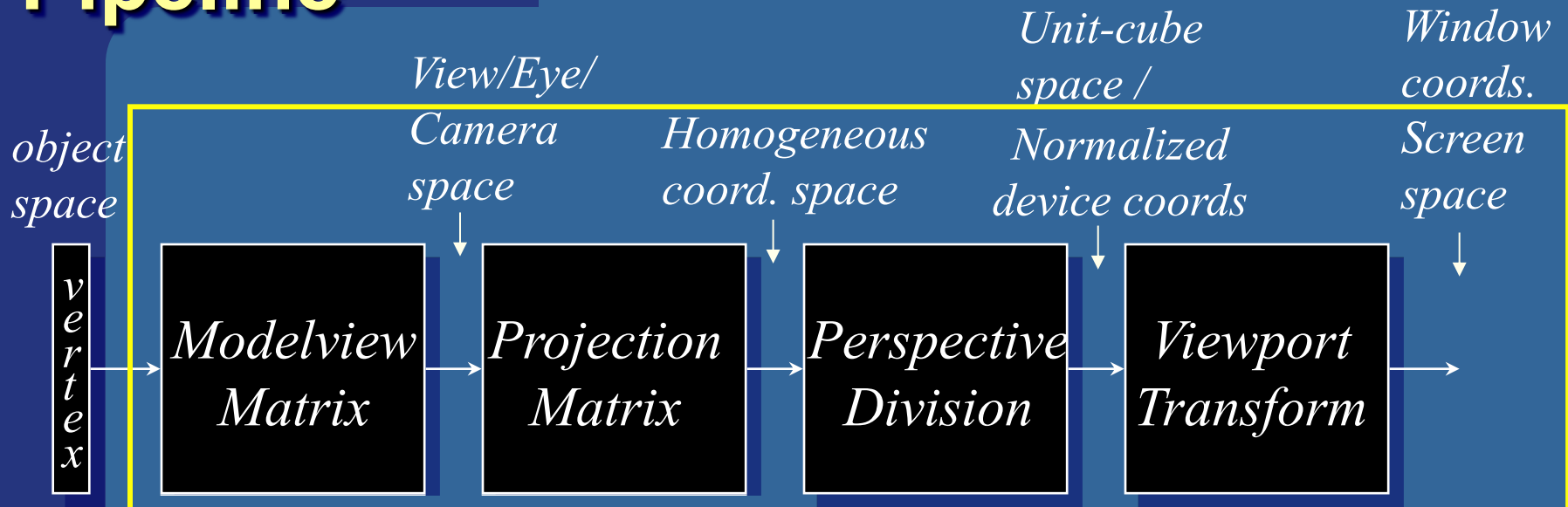
Done by the vertex shader:

gl_Position = modelViewProjectionMatrix*vec4(vertex,1);

OpenGL | Geometry stage | done on GPU

The OpenGL Pipeline

From http://deltronslair.com/glpipe.html

How do I use transforms practically?

- Say you have a circle with origin at (0,0,0) and with radius 1, i.e., a unit circle

```
mat4 m = translate({8,0,0});   // create translation matrix
RenderCircle(m);                // Draw circle using m as
                                // model-to-world matrix
```

```
mat4 s = scale({2,2,2});       // create scaling matrix
mat4 t = translate({3,2,0});   // create translation matrix
RenderCircle(t*s);             // use matrix (t*s)
```

What happens?
See next slide...

Cont'd from previous slide
A simple 2D example

● A circle in **model space**

y

```
mat4 s = scale({2,2,2});
mat4 t = translate({3,2,0});
RenderCircle(t*s); // Effect= first scaling, then translation
```
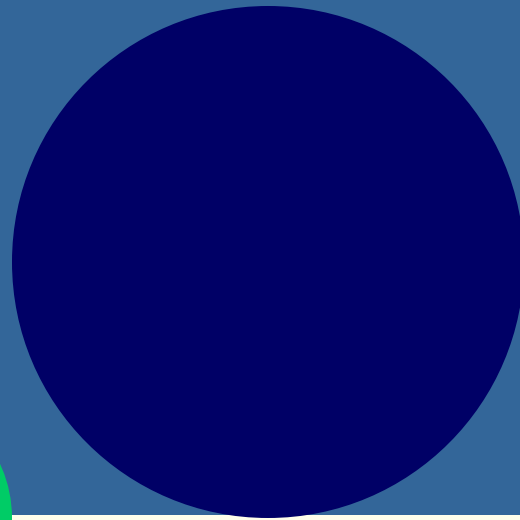
```
mat4 m = translate({8,0,0});
RenderCircle(m);
```
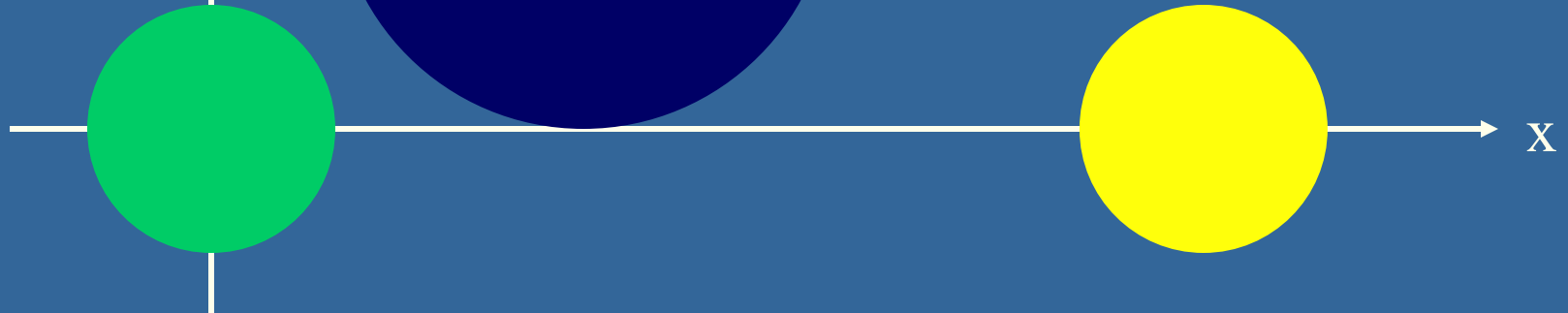
x

Cont'd from previous slide
A simple 2D example

● A circle in **model space**

y

```
mat4 s = scale({2,2,2});
mat4 t = translate({3,2,0});
RenderCircle(s*t); // Effect= first translation, then scaling.
                   // Each vertex in the sphere will first
                   // be translated (3,2,0) and then have its
                   // coordinate doubled in x,y,z
```

// This is less intuitive so humans
// prefer to do scaling first and then
// translation.

x

Example of a simple GfxObject class

```cpp
class GfxObject {
public:
    load("filename"); // Creates m_shaderProgram + m_vertexArrayObject
    render(mat4 projectionMatrix, mat4 viewMatrix)
    {
        …
        mat4 modelViewProjectionMatrix = projectionMatrix * viewMatrix *
                                         m_modelMatrix;

        int loc = glGetUniformLocation(shaderProgram, "modelViewProjectionMatrix");
        glUniformMatrix4fv(loc, 1, false, &modelViewProjectionMatrix[0].x);

        glEnableVertexAttribArray(0);
        glEnableVertexAttribArray(1);
        glUseProgram(m_shaderProgram);

        glBindVertexArray(m_vertexArrayObject);

        glDrawArrays( GL_TRIANGLES, 0,
    };
private:
    mat4   m_modelMatrix;
    uint   numVertices;
    Gluint m_shaderProgram;
    GLuint m_vertexArrayObject;
};
```

```glsl
#version 420    VERTEX SHADER
layout(location = 0) in vec3 position;
layout(location = 1) in vec3 color;

out vec4 outColor;
uniform mat4 modelViewProjectionMatrix;

void main()
{
    gl_Position = modelViewProjectionMatrix *
                  vec4(position, 1.0);
    outColor = vec4(color, 1.0);
}
```

Rotation (2D)

Consider rotation about the origin by θ degrees
–radius stays the same, angle increases by θ

$$x' = r \cos (\phi + \theta)$$
$$y' = r \sin (\phi + \theta)$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x = r \cos \phi$$
$$y = r \sin \phi$$

Answer: $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$
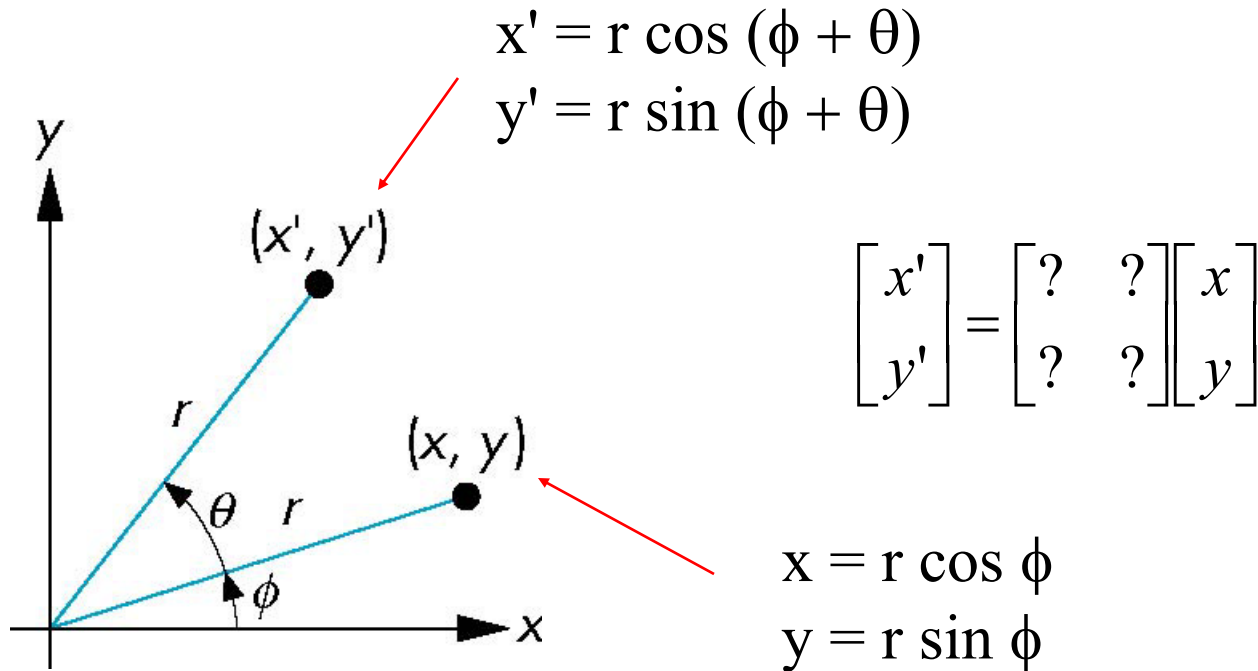
Derivation of rotation matrix in 2D
$\mathbf{n} = \mathbf{R}_z \mathbf{p}$?

$\mathbf{p} = re^{i\varphi} = r(\cos\varphi + i\sin\varphi)$ [rotation is mult by $e^{i\alpha}$]

$\mathbf{n} = e^{i\alpha}\mathbf{p} = re^{i\alpha}e^{i\varphi} =$

$= r[(\cos\alpha + i\sin\alpha)(\cos\varphi + i\sin\varphi)] =$

$= r(\cos\alpha\cos\varphi - \sin\alpha\sin\varphi) +$

$ir(\cos\alpha\sin\varphi + \sin\alpha\cos\varphi)$
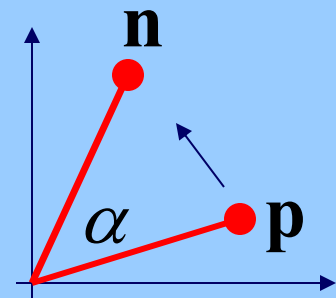
In vector form:

$\mathbf{p} = (p_x, p_y)^T = (r\cos\varphi, r\sin\varphi)^T$

$\mathbf{n} = (n_x, n_y)^T = (r(\cos\alpha\cos\varphi - \sin\alpha\sin\varphi),$

$r(\cos\alpha\sin\varphi + \sin\alpha\cos\varphi))^T$

Derivation 2D rotation, cont'd

In vector form:

$$\mathbf{p} = (p_x, p_y)^T = (r\cos\phi, r\sin\phi)^T$$

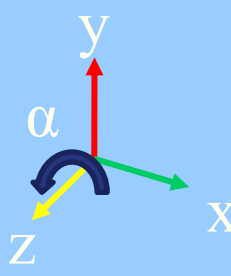$$\mathbf{n} = (n_x, n_y)^T = (r(\cos\alpha\cos\phi - \sin\alpha\sin\phi),$$

$$r(\sin\alpha\cos\phi + \cos\alpha\sin\phi))^T$$

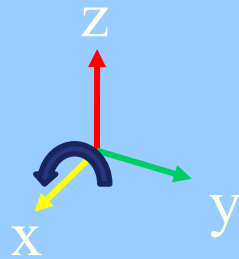$$\mathbf{n} = \mathbf{R}_z\mathbf{p} \quad \text{what is } \mathbf{R}_z?$$

$$\begin{pmatrix} n_x \\ n_y \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}_{\mathbf{R}_z} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

Rotations in 3D

- Same as in 2D for Z-rotations, but with a 3x3 matrix

$$\mathbf{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \Rightarrow \mathbf{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- For X

$$\mathbf{R}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

- For Y
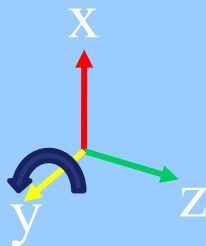
$$\mathbf{R}_y(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

Translations must be simple?

Translation   Rotation

$$\begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{pmatrix} \mathbf{p} = \mathbf{p} + \mathbf{t} \qquad \mathbf{n} = \mathbf{R}\mathbf{p}$$

- Rotation is matrix mult, translation is add
- Would be nice if we could only use matrix multiplications…
- Turn to **homogeneous coordinates**
- Add a new component to each vector

Homogeneous notation

- A point: $\mathbf{p} = \begin{pmatrix} p_x & p_y & p_z & 1 \end{pmatrix}^T$
- Translation becomes:

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\mathbf{T(t)}} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x + t_x \\ p_y + t_y \\ p_z + t_z \\ 1 \end{pmatrix}$$

- A vector (direction): $\mathbf{d} = \begin{pmatrix} d_x & d_y & d_z & 0 \end{pmatrix}^T$
- Translation of vector: $\mathbf{Td} = \mathbf{d}$
- Also allows for projections (later)

Rotations in 4x4 form

- Just add a row at the bottom, and a column at the right:

$$\mathbf{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Vector Product

$\underline{V}_A \times \underline{V}_B$

AREA=$V_A V_B \sin\alpha$

- **Triple Scalar Product**
The magnitude of the triple scalar product is equal to the volume of the parallelepiped formed by the three vectors $\underline{V}_A, \underline{V}_B, \underline{V}_C$: $\underline{V}_A \cdot (\underline{V}_B \times \underline{V}_C)$.

Triple Scalar Product

$\underline{V}_B \times \underline{V}_C$

Volume = $\underline{V}_A \underline{V}_B \underline{V}_C \sin\alpha \cos\beta$

- Similarly for X and Y
- Determinant = volume change when the transform is applied to a unit cube
  - det(R) = 1 for all rot. matrices (=tripple scal. prod for 3x3 mtx)
- Trace(R) = 1+2cos(alpha) (for 3x3 rot-matrices)
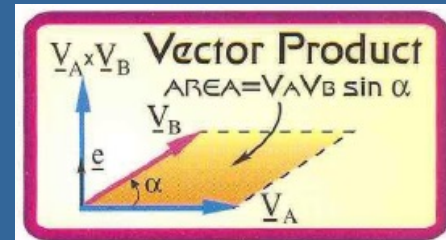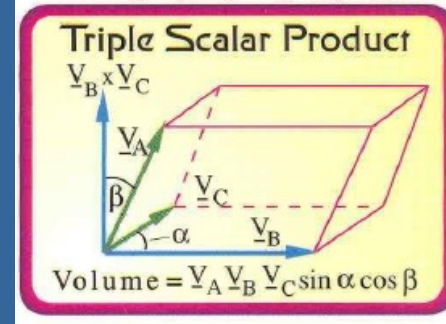
Change of Frames

- How to get the $M_{\text{model-to-world}}$ matrix:

$$P = (0,5,0,1)$$

$$M_{\text{model-to-world}} = \begin{bmatrix} a_x & b_x & c_x & o_x \\ a_y & b_y & c_y & o_y \\ a_z & b_z & c_z & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$
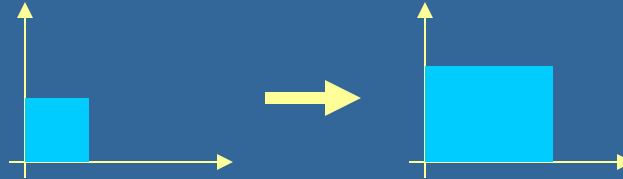

model space

world space

(Both coordinate systems are right-handed)

The basis vectors **a**, **b**, **c** are expressed in the world coordinate system

E.g.: $\mathbf{p}_{\text{world}} = M_{m \to w} \mathbf{p}_{\text{model}} = M_{m \to w} (0,5,0,1)^T = 5\,\mathbf{b} + \mathbf{o}$

More basic transforms

● Scaling

● Shear
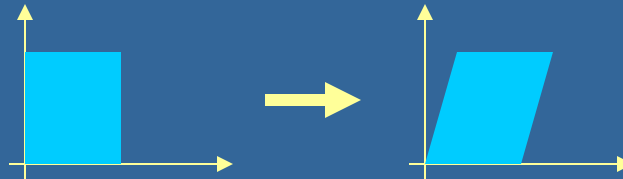
● Rigid-body: rotation and/or (then) translation

$$\mathbf{X = TR}$$

● Concatenation of matrices
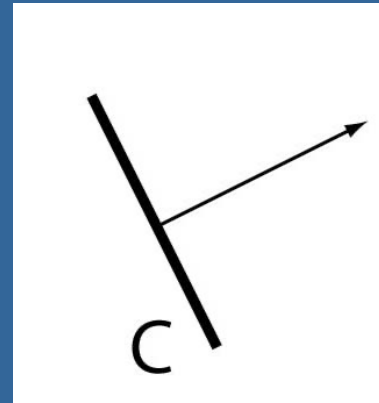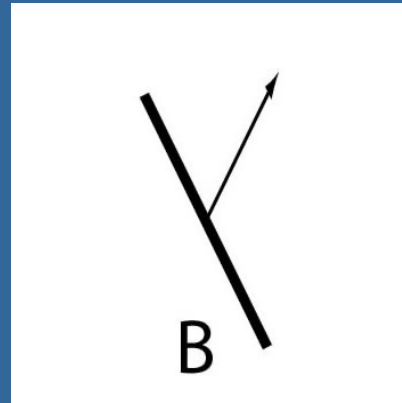 - Not commutative, i.e., $\mathbf{RT \neq TR}$
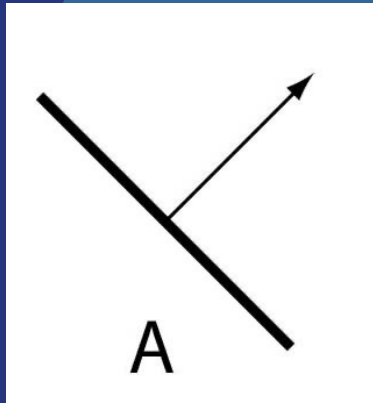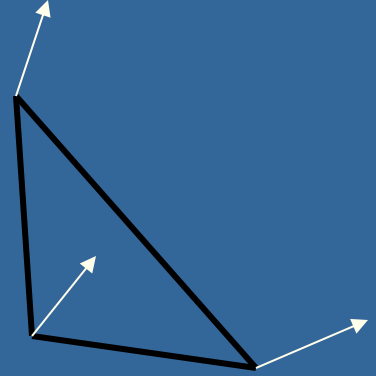 - In $\mathbf{X = TR}$, the rotation is done first

● Inverses and rotation about arbitrary axis:
  - Rigid body: $X^{-1} = X^T$ (for 3x3 matrices)

Normal transforms Not so normal…

- Cannot use same matrix to transform normals

$$\text{Use}: \mathbf{N} = \left(\mathbf{M}^{-1}\right)^{T} \quad \text{instead of } \mathbf{M}$$

- **M** works for rotations and translations, though

The Euler Transform

- Assume the camera or object looks down the negative z-axis, with up in the y-direction, x to the right

- *h*=head

- *p*=pitch

- *r*=roll

- Optional
  - You may read about Gimbal lock in book, p: 67
  - See also
    - http://mathworld.wolfram.com/EulerAngles.html
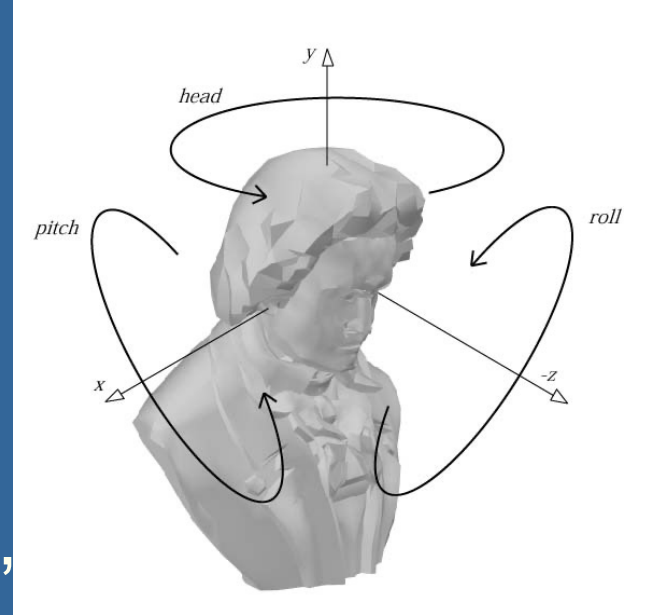
Using Euler transforms

Head:

- Rotate around y-axis
- Recompute x- and z-axes
    - By rotating them as vectors

Pitch:

- Rotate around x'-axis
- Recompute y- and z'-axes

Roll:

- Rotate around z''-axis

How do we rotate vectors (axes) and points around an **arbitrary** axis?
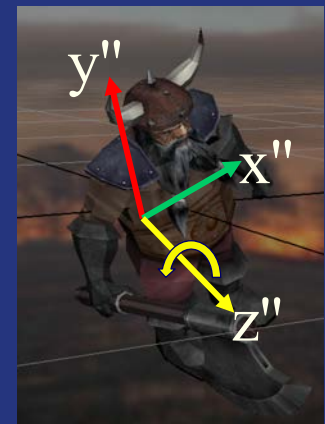
Quaternions

$$\hat{\mathbf{q}} = (\mathbf{q}_v, q_w) = (q_x, q_y, q_z, q_w)$$
$$= iq_x + jq_y + kq_z + q_w$$

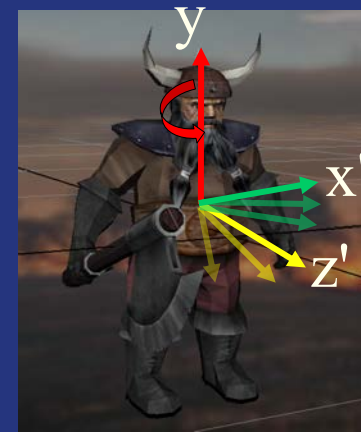- Extension of imaginary numbers
- Compact+fast representation of rotations
- Focus on unit quaternions:
  - Norm (or length): $n(\hat{\mathbf{q}}) = \sqrt{q_x^2 + q_y^2 + q_z^2 + q_w^2} = 1$

- A unit quaternion can be written as:

$$\hat{\mathbf{q}} = (\sin\phi\,\mathbf{u}_q, \cos\phi) \quad \text{where } \|\mathbf{u}_q\| = 1$$

Unit quaternions are perfect for rotations!

$$\hat{\mathbf{q}} = (\sin \phi \mathbf{u}_q, \cos \phi)$$

- Compact (4 components)
- Can show that $\hat{\mathbf{q}}\hat{\mathbf{p}}\hat{\mathbf{q}}^{-1}$
- …represents a rotation of 2ϕ radians around $\mathbf{u}_q$ of $\mathbf{p}$

- That is: a unit quaternion represents a rotation as a **rotation axis** and an **angle**
- `rotate(ux,uy,uz,angle);`
  - `See p:76 how to convert q to matrix.`
- Interpolation from one quaternion to another is much simpler, and gives optimal results
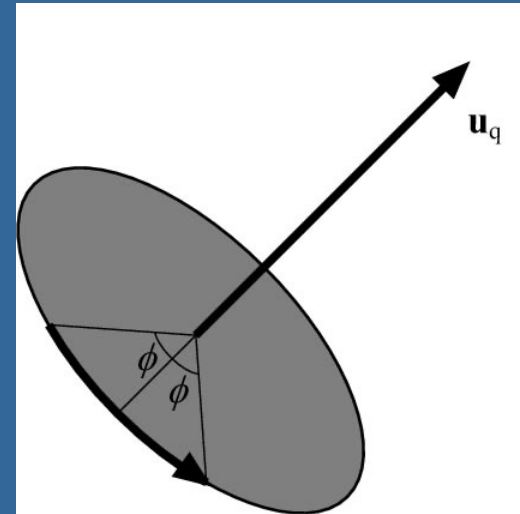
Projections

- Orthogonal (parallel) and Perspective
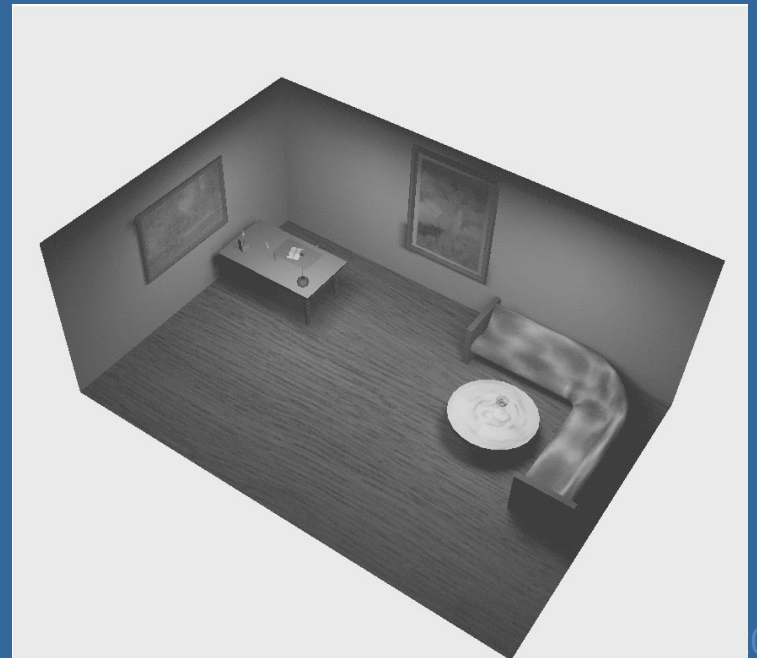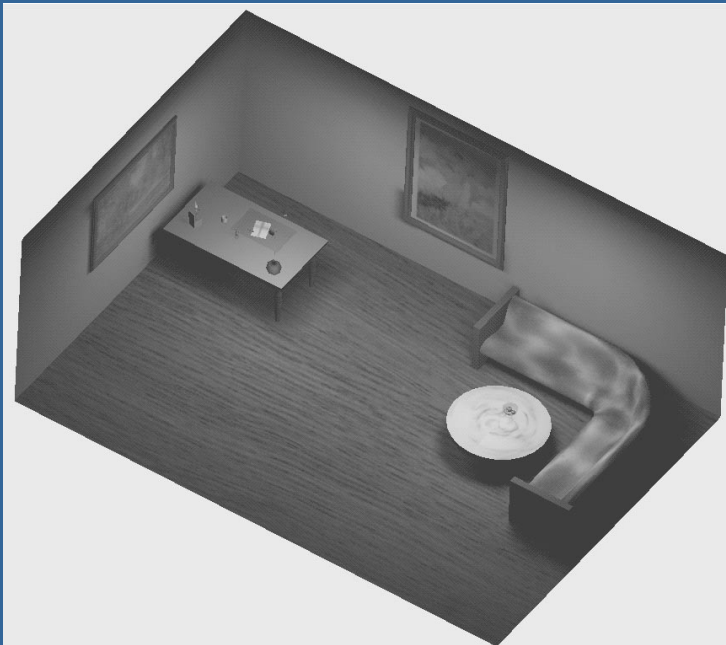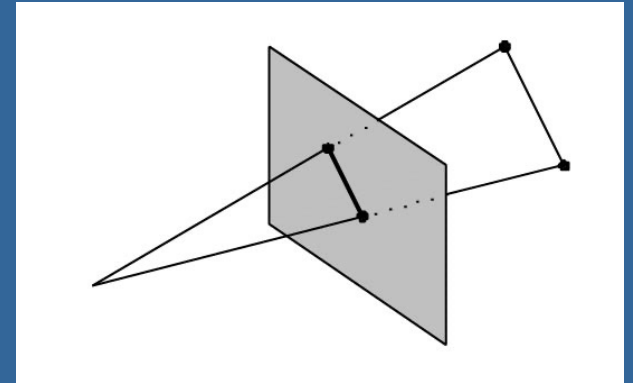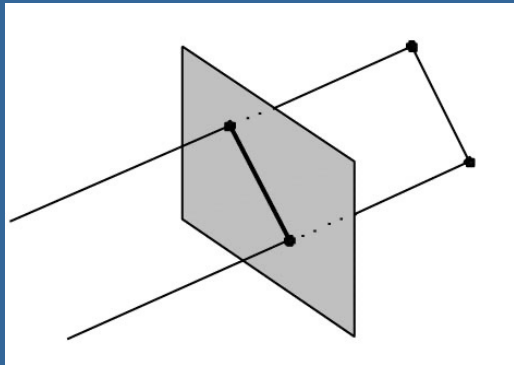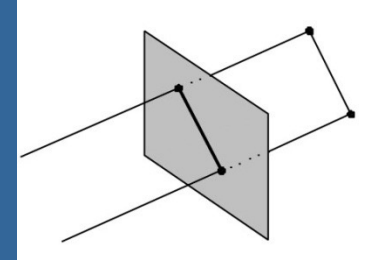
Orthogonal projection

- Simple, just skip one coordinate
  - Say, we're looking along the z-axis
  - Then drop z, and render

$$\mathbf{M}_{ortho} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \mathbf{M}_{ortho} \begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ 0 \\ 1 \end{pmatrix}$$

z   →   z
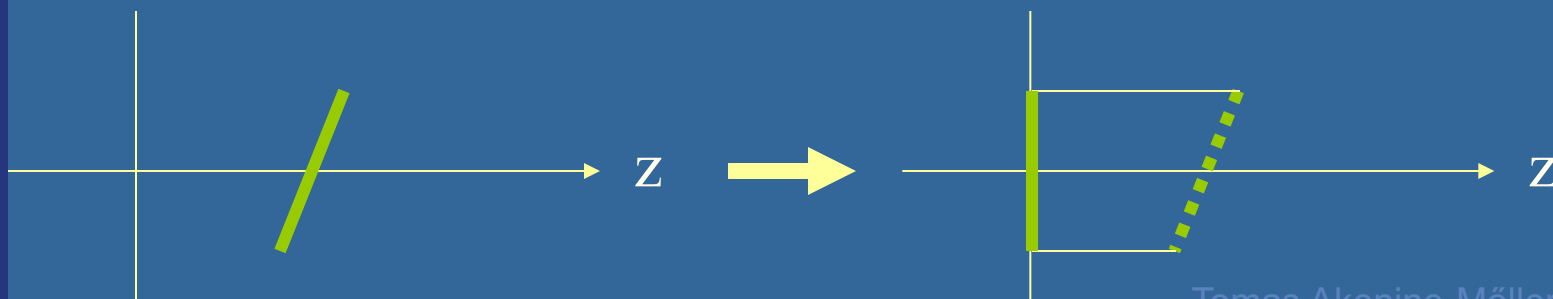
Orthogonal projection
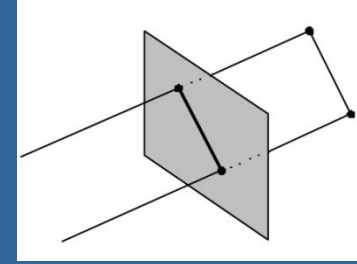
- Not invertible! (determinant is zero)
  - i.e., depth information is lost
- For Z-buffering
  - It is not sufficient to project to a plane
  - Rather, we need to "project" to a box

far

near

image plane

Unit cube: [-1,-1,-1] to [1,1,1]

eye
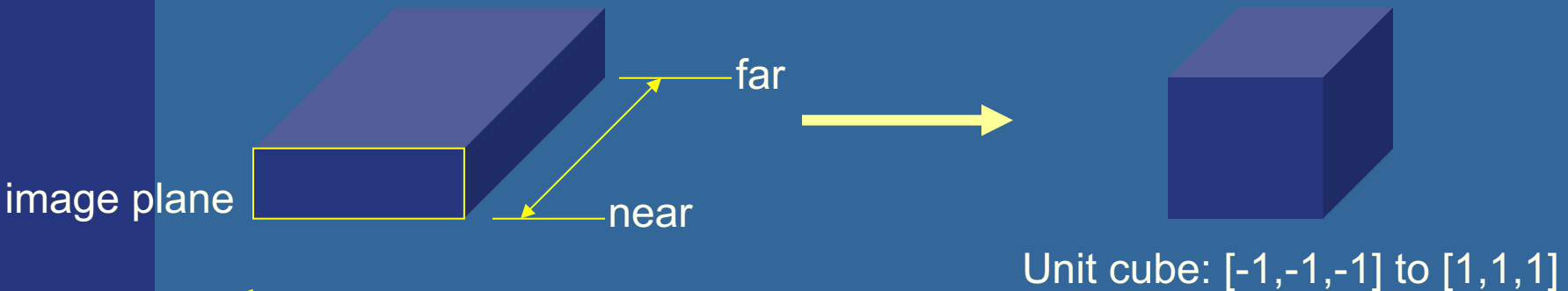
- Unit cube is also used for perspective proj.
- Simplifies clipping

Orthogonal projection
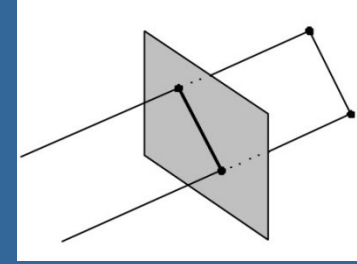
- The "unitcube projection" is invertible
- Simple to derive
  - Just a translation and scale
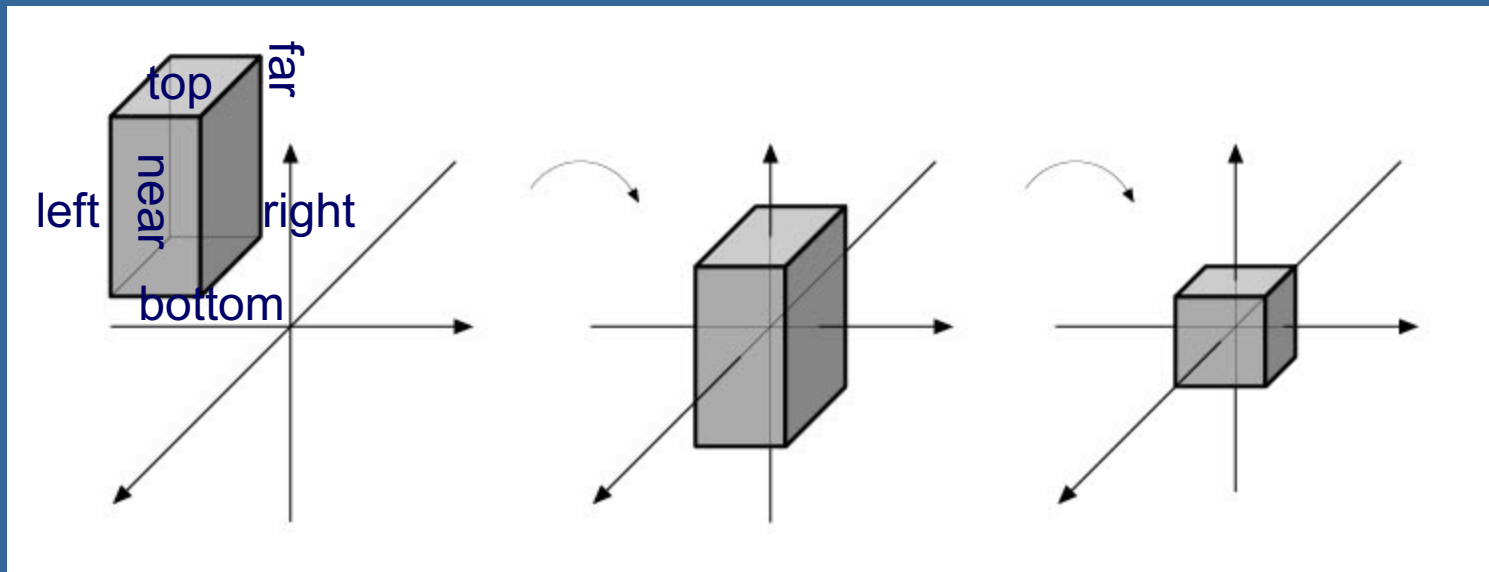
What about those homogenenous coordinates?

$$\mathbf{p} = \begin{pmatrix} p_x & p_y & p_z & p_w \end{pmatrix}^T$$

- $p_w=0$ for vectors, and $p_w=1$ for points
- What if p_w is **not** 1 or 0?
- Solution is to divide all components by p_w

$$\mathbf{p} = \begin{pmatrix} p_x / p_w & p_y / p_w & p_z / p_w & 1 \end{pmatrix}^T$$

- Gives a point again!

- Can be used for projections, as we will see

Perspective projection

projection plane, $z = -d$
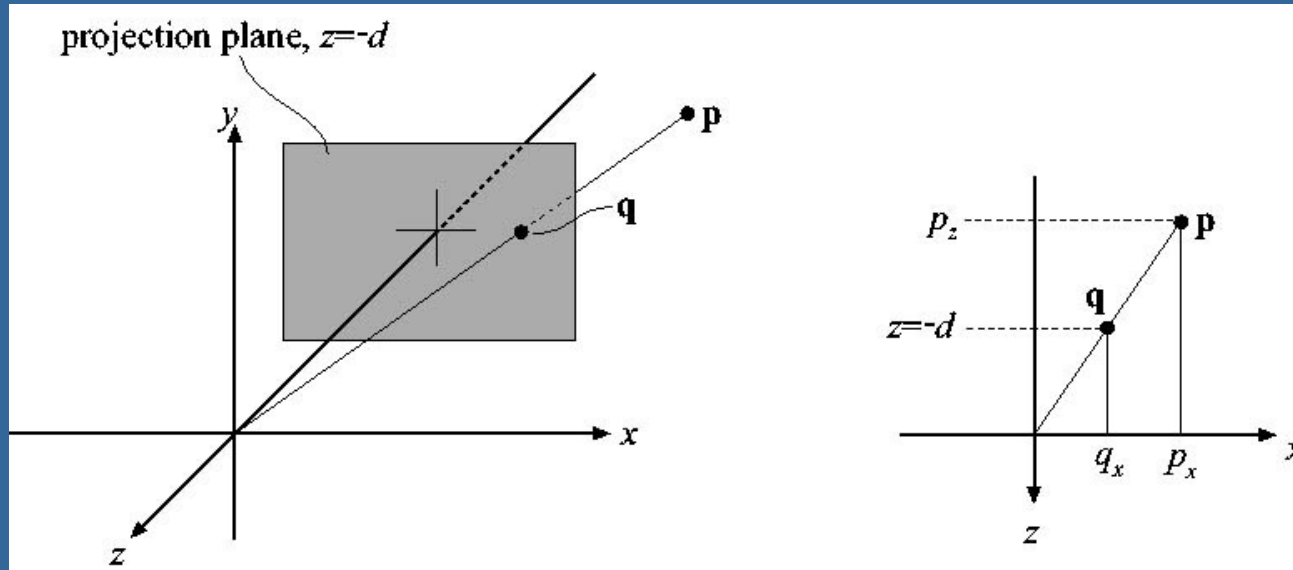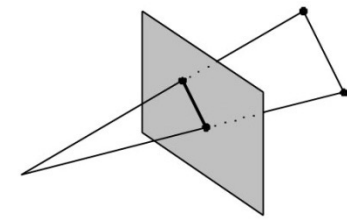
$$\frac{q_x}{p_x} = \frac{-d}{p_z} \quad \Rightarrow \quad q_x = -d\,\frac{p_x}{p_z}$$

For y: $\quad q_y = -d\,\frac{p_y}{p_z}$

$$\mathbf{P}_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{pmatrix}$$

Perspective projection
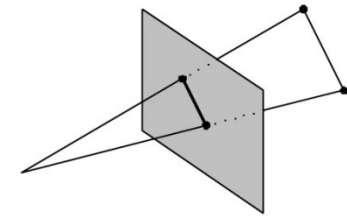
$$\mathbf{P}_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{pmatrix}$$

$$\mathbf{P}_p \mathbf{p} = \mathbf{q}$$

projection plane, z=-d
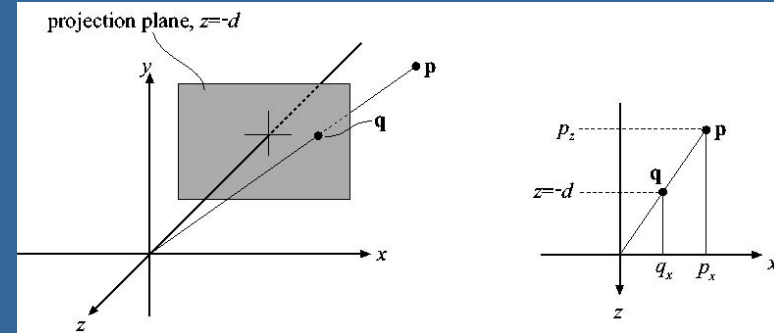
$$q_x = -d\,\frac{p_x}{p_z} \qquad q_y = -d\,\frac{p_y}{p_z} \qquad q_z = -d$$

$$\mathbf{P}_p\mathbf{p} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1/d & 0 \end{pmatrix}\begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \\ p_z \\ -p_z/d \end{pmatrix} \Rightarrow \mathbf{q} = \begin{pmatrix} -dp_x/p_z \\ -dp_y/p_z \\ -dp_z/p_z \\ 1 \end{pmatrix} = \begin{pmatrix} -dp_x/p_z \\ -dp_y/p_z \\ -d \\ 1 \end{pmatrix}$$

● The "arrow" is the homogenization process

Perspective projection
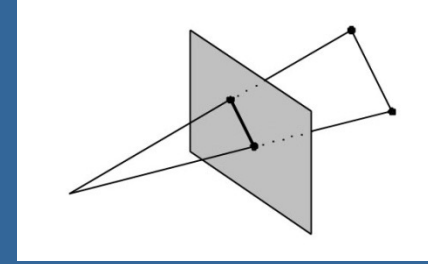
- Again, the determinant is 0 (not invertible)
- To make the rest of the pipeline the same as for orhogonal projection:
  - project into unit-cube
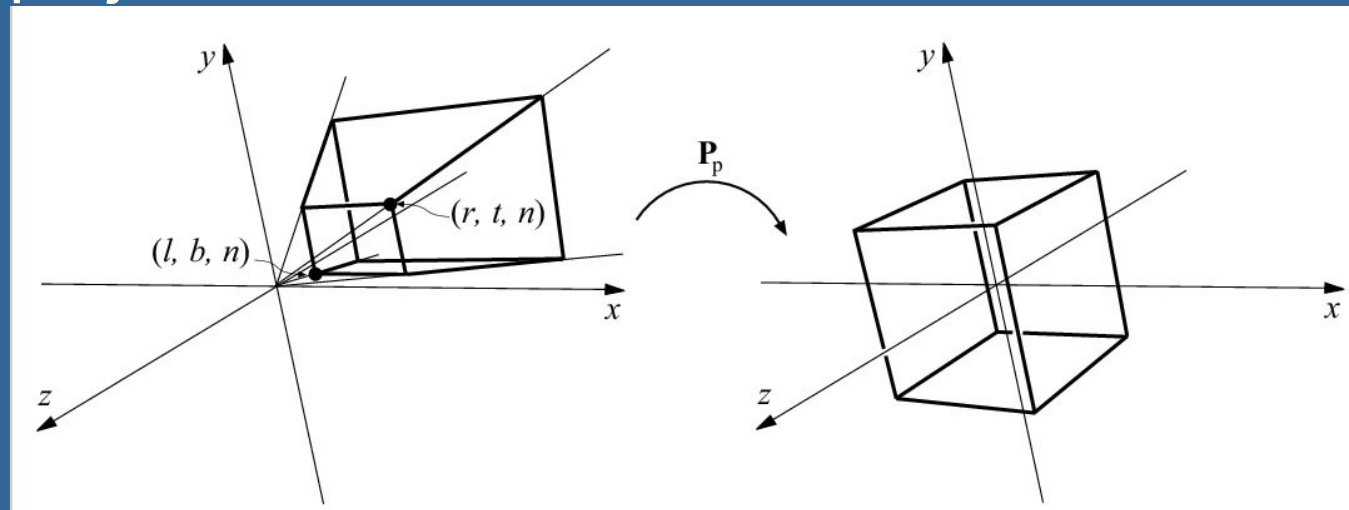
- Not much different from P_p
- Do not collapse z-coord to a plane

Understanding the projection matrix
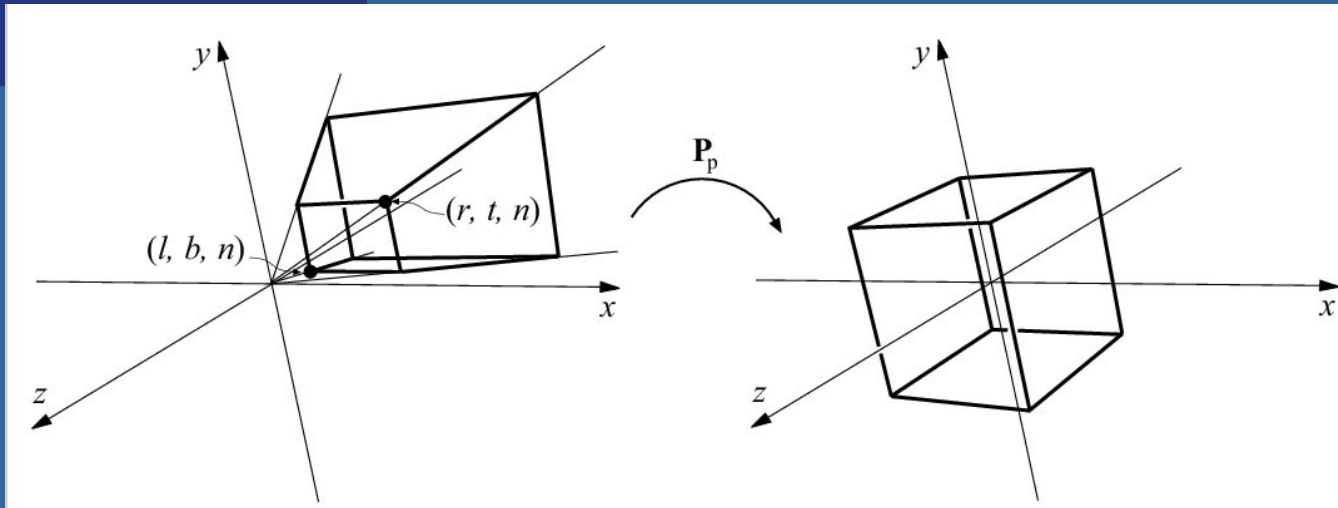
$$\mathbf{P}_p\mathbf{p} = \begin{pmatrix} s_x & 0 & a & 0 \\ 0 & s_y & b & 0 \\ 0 & 0 & s_z & c \\ 0 & 0 & -1/d & 0 \end{pmatrix}\begin{pmatrix} p_x \\ p_y \\ p_z \\ 1 \end{pmatrix} = \begin{pmatrix} s_x p_x + a p_z \\ s_y p_y + b p_z \\ s_z p_z + c \\ -p_z/d \end{pmatrix} \Rightarrow \mathbf{q} = \begin{pmatrix} -d(s_x p_x / p_z + a) \\ -d(s_y p_y / p_z + b) \\ -d(s_z p_z + c)/p_z \\ 1 \end{pmatrix}$$

- s_x, s_y, s_z –Scaling
- a, b – Due to homogenization, this controls asymmetry of the frustum
- c – Keep z-info
- -1/d – Perspective division based on p_z

OpenGL projection matrix
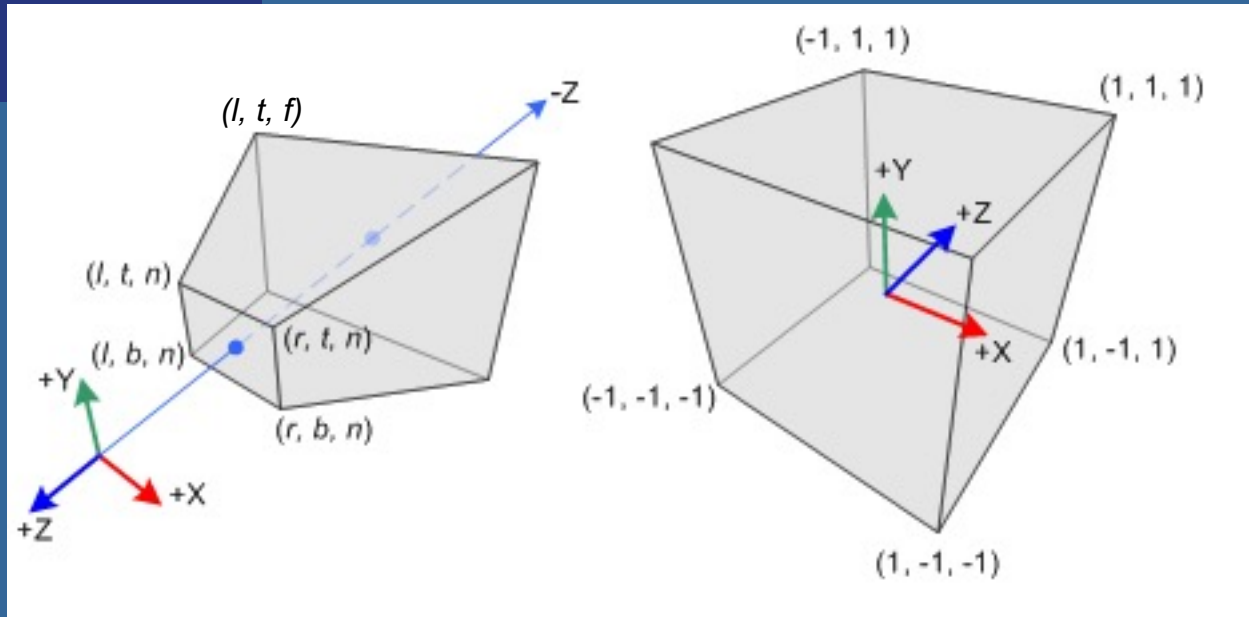
$$P_{OpenGL} = \begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{-(f+n)}{f-n} & \frac{-2fn}{f-n} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

l = left
r = right
t = top
b = bottom
n = near
f = far

mat4 projectionMtx = perspective(fov, width / height, near, far);

Ray/Plane Intersections

- Ray: $r(t) = \mathbf{o} + t\mathbf{d}$

- Plane: $\mathbf{n} \cdot \mathbf{x} + d = 0$; $(d = -\mathbf{n} \cdot \mathbf{p_0})$

- Set $\mathbf{x} = r(t)$:

  $\mathbf{n} \cdot (\mathbf{o} + t\mathbf{d}) + d = 0$

  $\mathbf{n} \cdot \mathbf{o} + t(\mathbf{n} \cdot \mathbf{d}) + d = 0$

  $t = (-d - \mathbf{n} \cdot \mathbf{o}) / (\mathbf{n} \cdot \mathbf{d})$
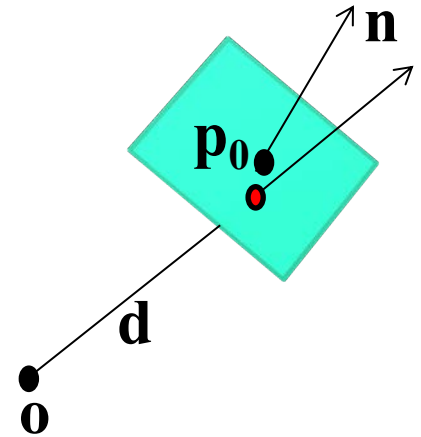
```
Vec3f rayPlaneIntersect(vec3f o,dir, n, d)
{
        float t=(-d-n.dot(o)) / (n.dot(dir));
        return o + dir*t;
}
```

Line/Line intersection in 2D

- $r_1(s) = o_1 + sd_1$
- $r_2(t) = o_2 + td_2$

- $r_1(s) = r_2(t)$ (1)
- $o_1 + sd_1 = o_2 + td_2$ (2)

noting that $d \cdot d^\perp = 0$, $[d=(a,b) \rightarrow d^\perp=(b,-a)]$

$sd_1 \cdot d_2{}^\perp = (o_2-o_1) \cdot d_2{}^\perp$

$td_2 \cdot d_1{}^\perp = (o_1-o_2) \cdot d_1{}^\perp$

$$s = \frac{(o_2 - o_1) \cdot d_2^\perp}{(d_1 \cdot d_2^\perp)}$$

$$t = \frac{(o_1 - o_2) \cdot d_1^\perp}{(d_2 \cdot d_1^\perp)}$$

From book, p: 988

Line/Line intersection in 3D

- $r_1(s) = o_1 + s d_1$
- $r_2(t) = o_2 + t d_2$

s, t correspond to closest points
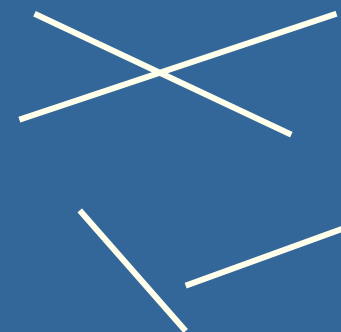
- $r_1(s) = r_2(t)$ (1)
- $o_1 + s d_1 = o_2 + t d_2$ (2)

noting that d x d=0 $\left\| (\mathbf{d}_1 \times \mathbf{d}_2) \right\|^2 = 0$ means parallel lines

$s\mathbf{d}_1 \times \mathbf{d}_2 = (\mathbf{o}_2 - \mathbf{o}_1) \times \mathbf{d}_2$ (i.e., cross mult. both sides with $\mathbf{d}_2$ to drop t)

$t\mathbf{d}_2 \times \mathbf{d}_1 = (\mathbf{o}_1 - \mathbf{o}_2) \times \mathbf{d}_1$ (i.e., cross mult. both sides with $\mathbf{d}_1$ to drop s)

=>

$s\, (\mathbf{d}_1 \times \mathbf{d}_2) \bullet (\mathbf{d}_1 \times \mathbf{d}_2) = ((\mathbf{o}_2 - \mathbf{o}_1) \times \mathbf{d}_2) \bullet (\mathbf{d}_1 \times \mathbf{d}_2)$

$t\, (\mathbf{d}_2 \times \mathbf{d}_1) \bullet (\mathbf{d}_2 \times \mathbf{d}_1) = ((\mathbf{o}_1 - \mathbf{o}_2) \times \mathbf{d}_1) \bullet (\mathbf{d}_2 \times \mathbf{d}_1)$

$$s = \frac{\det(\mathbf{o}_2 - \mathbf{o}_1, \mathbf{d}_2, \mathbf{d}_1 \times \mathbf{d}_2)}{\left\| (\mathbf{d}_1 \times \mathbf{d}_2) \right\|^2}$$

$$t = \frac{\det(\mathbf{o}_2 - \mathbf{o}_1, \mathbf{d}_1, \mathbf{d}_1 \times \mathbf{d}_2)}{\left\| (\mathbf{d}_1 \times \mathbf{d}_2) \right\|^2}$$

Area and Perimeter
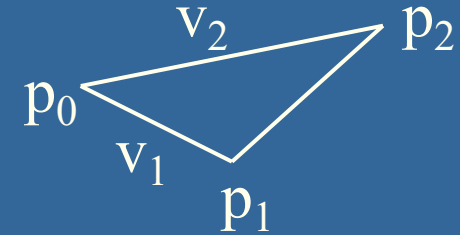
For polygon p_0, p_1…p_n

Perimeter = omkrets = sum of length of each edge in 2D and 3D:

$$O = \sum_{i=0}^{n-1} \left\| p_{i+1} - p_i \right\| = \sum_{i=0}^{n-1} \sqrt{\left(x_{i+1} - x_i\right)^2 + \left(y_{i+1} - y_i\right)^2 + \left(z_{i+1} - z_i\right)^2}$$

Area in 2D:

$$A = \frac{1}{2} \left| \sum_{i=1}^{n-1} \left\langle x_i y_{i+1} - x_{i+1} y_i \right\rangle \right|$$
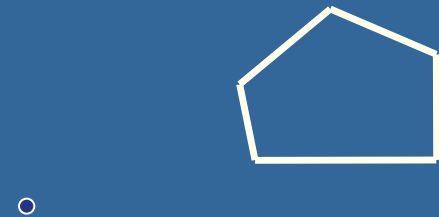
We can understand the formula from using Greens theorem: integrating over border to get area

Choose arbitrary point to integrate from, e.g. Origin (0,0,0)

$$A_{triangle} = \frac{1}{2}\left(v_1 \times v_2\right)$$

Works for non-convex polygons as well

Volume in 3D

The same trick for computing area in 2D can be used to easily compute the volume in 3D for triangulated objects
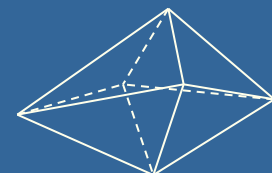
Again, choose arbitrary point-of-integration, e.g. Origin (0,0,0)

With respect to point-of-integration

- For all backfacing triangles, add volume

- For all frontfacing triangles, subtract volume

Works for non-convex objects as well

$$V_{tetrahedron} = \frac{1}{3!}\left|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})\right| = \frac{1}{3!}\left|\det(\mathbf{a}, \mathbf{b}, \mathbf{c})\right|$$

where
$a = p_1 - origin$
$b = p_2 - origin$
$c = p_3 - origin$

$$V_{object} = \frac{1}{3!}\sum_{i=1}^{n}\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$$

The sign of the determinant will automatically handle positive and negative contribution

Scan Conversion of Line Segments

- Start with line segment in window coordinates with integer values for endpoints

- Assume implementation has a `write_pixel` function

$$y = kx + m$$

$$k = \frac{\Delta y}{\Delta x}$$
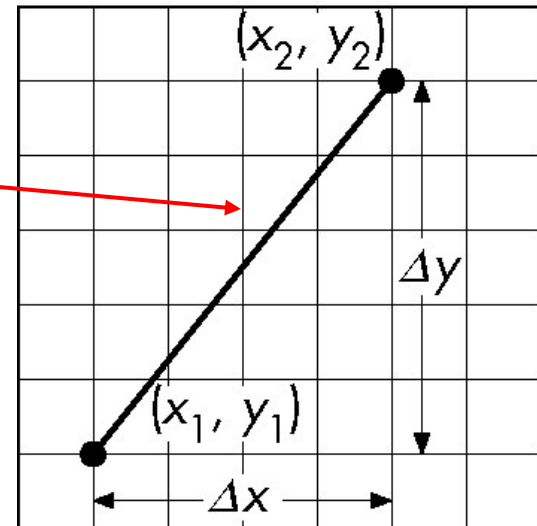
DDA Algorithm
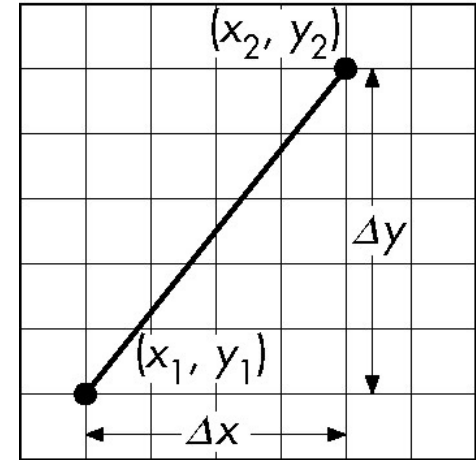
- Digital Differential Analyzer
  - DDA was a mechanical device for numerical solution of differential equations
  - Line $y=kx+m$ satisfies differential equation

    dy/dx = k = Δy/Δx = y$_2$-y$_1$/x$_2$-x$_1$

- Along scan line Δx = 1

```
y=y1;
For(x=x1; x<=x2,ix++) {
  write_pixel(x, round(y), line_color)
  y+=k;
}
```

Problem

- DDA = for each x plot pixel at closest y
  - Problems for steep lines
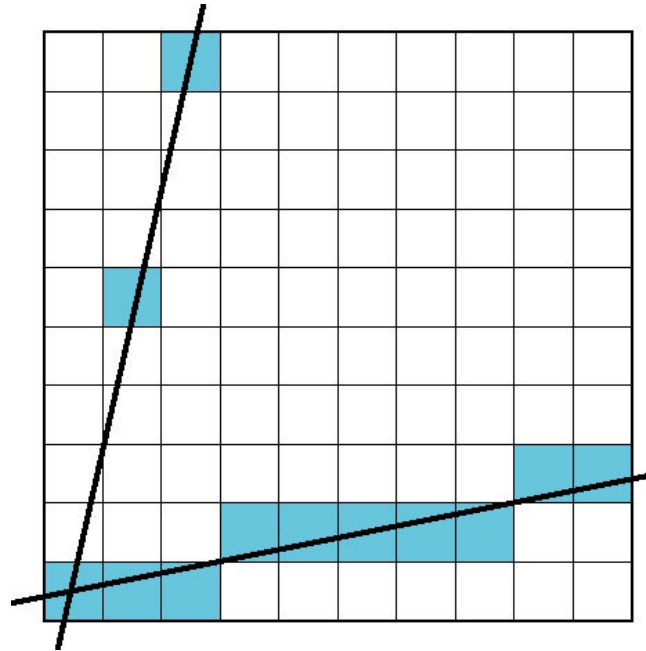
Using Symmetry

- Use for $1 \geq k \geq 0$

- For $k > 1$, swap role of x and y
  - For each y, plot closest x
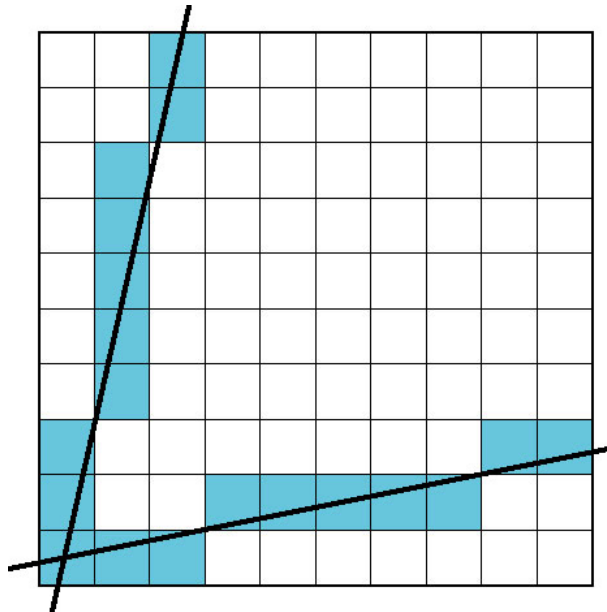
- The problem with DDA is that it uses floats which was slow in the old days
- Bresenhams algorithm only uses integers

Bresenham's line drawing algorithm
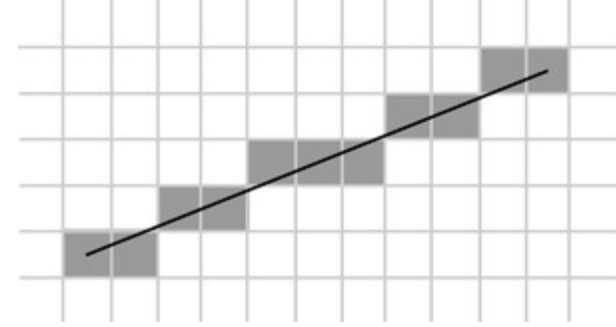
- The line is drawn between two points (x_0, y_0) and (x_1, y_1)

- Slope $\quad k = \dfrac{(y_1 - y_0)}{(x_1 - x_0)}$ $\qquad$ $(y = kx + m)$

- Each time we step 1 in x-direction, we should increment y with k. Otherwise the error in y increases with k.

- If the error surpasses 0.5, the line has become closer to the next y-value, so we add 1 to y, simultaneously decreasing the error by 1

```
function line(x0, x1, y0, y1)
    int deltax := abs(x1 - x0)
    int deltay := abs(y1 - y0)
    real error := 0
    real deltaerr := deltay / deltax
    int y := y0
    for x from x0 to x1
        plot(x,y)
        error := error + deltaerr
        if error ≥ 0.5
            y := y + 1
            error := error - 1.0
```

See also
http://en.wikipedia.org/wiki/Bresenham's_line_algorithm

Bresenham's line drawing algorithm
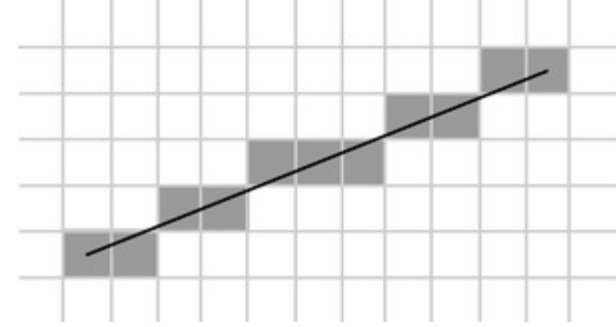
- Now, convert algorithm to only using integer computations
- Trick: multiply the fractional number, *deltaerr*, by *deltax*
  - enables us to express *deltaerr* as an integer.
  - The comparison *if error>=0.5* is multiplied on both sides by *2*deltax*

Old float version:

```
function line(x0, x1, y0, y1)
    int deltax := abs(x1 - x0)
    int deltay := abs(y1 - y0)
    real error := 0
    real deltaerr := deltay / deltax
    int y := y0
    for x from x0 to x1
        plot(x,y)
        error := error + deltaerr
        if error ≥ 0.5
            y := y + 1
            error := error - 1.0
```

New integer version:

```
function line(x0, x1, y0, y1)
    int deltax := abs(x1 - x0)
    int deltay := abs(y1 - y0)
    real error := 0
    real deltaerr := deltay              ← Multiply by deltax
    int y := y0
    for x from x0 to x1
        plot(x,y)
        error := error + deltaerr
        if 2*error ≥ deltax              ← Multiply by 2 deltax
            y := y + 1
            error := error - deltax      ← Multiply by deltax
```

Ulf Assarsson © 2006

Complete Bresenham's line drawing algorithm

function line(x0, x1, y0, y1)

    boolean steep := abs(y1 - y0) > abs(x1 - x0)

    if steep then

        swap(x0, y0)

        swap(x1, y1)   Swap loop axis

    if x0 > x1 then

        swap(x0, x1)   Swap start and end

        swap(y0, y1)   points

    int deltax := x1 - x0

    int deltay := abs(y1 - y0)

    int error := 0

    int ystep

    int y := y0

    if y0 < y1 then ystep := 1 else ystep := -1

    for x from x0 to x1

        if steep then plot(y,x) else plot(x,y)

        error := error + deltay

        if 2×error ≥ deltax

            y := y + ystep

            error := error - deltax

The first case is allowing us to draw lines that still slope downwards, but head in the opposite direction. I.e., swapping the initial points if x0 > x1.

To draw lines that go up, we check if y0 >= y1; if so, we step y by -1 instead of 1.

To be able to draw lines with a slope less than one, we take advantage of the fact that a steep line can be reflected across the line y=x to obtain a line with a small slope. The effect is to switch the x and y variables.
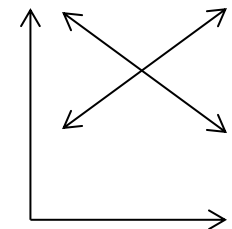
You need to know

- How to create a simple Scaling matrix, rotation matrix, translation matrix and orthogonal projection matrix
- Change of frames (creating model-to-view matrix)
- Understand how quaternions are used
- Understanding of Euler transforms
- DDA line drawing algorithm
- Understand what is good with Bresenhams line drawing algorithm, i.e., uses only integers.

Most of the following slides are from

Ed Angel

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University of New Mexico

Scalars

- Need three basic elements in geometry
  - Scalars, Vectors, Points
- Scalars can be defined as members of sets which can be combined by two operations (addition and multiplication) obeying some fundamental axioms (associativity, commutivity, inverses)
- Examples include the real and complex number systems under the ordinary rules with which we are familiar
- Scalars alone have no geometric properties

Vector Operations

- Physical definition: a vector is a quantity with two attributes
  - Direction
  - Magnitude

- Examples include
  - Force
  - Velocity
  - Directed line segments
    - Most important example for graphics
    - Can map to other types. Every vector can be multiplied by a scalar.

- There is a zero vector
  - Zero magnitude, undefined orientation

- The sum of any two vectors is a vector

v $-v$ αv v w u
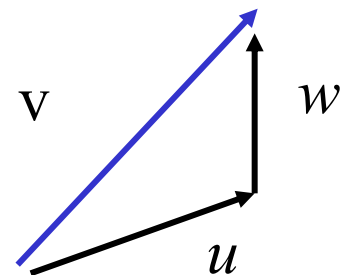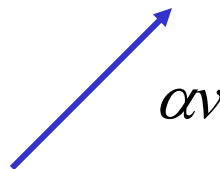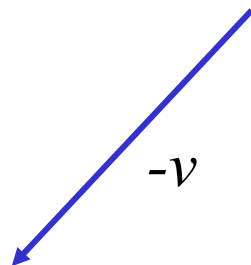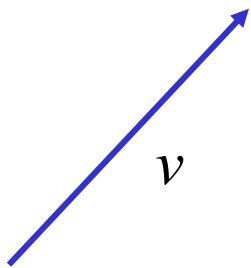
Vectors Lack Position

- These vectors are identical
  - Same length and magnitude
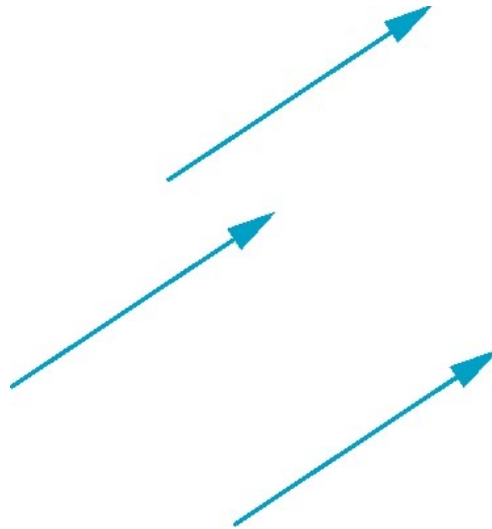
- Vectors insufficient for geometry
  - Need points

Points

- Location in space

- Operations allowed between points and vectors

  - Point-point subtraction yields a vector
  - Equivalent to point-vector addition
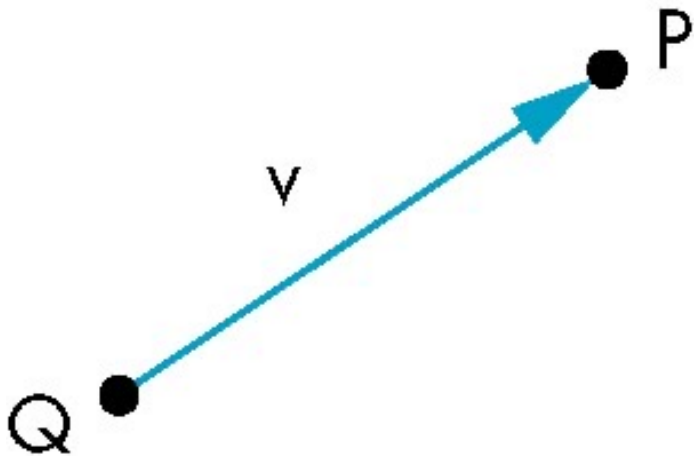
v=P-Q

P=v+Q

Affine Spaces

- Point + a vector space
- Operations
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Scalar-scalar operations
- For any point define
  - $1 \cdot P = P$
  - $0 \cdot P = \mathbf{0}$ (zero vector)

Lines

- Consider all points of the form
  - $P(\alpha) = P_0 + \alpha\ \mathbf{d}$
  - Set of all points that pass through P_0 in the direction of the vector $\mathbf{d}$
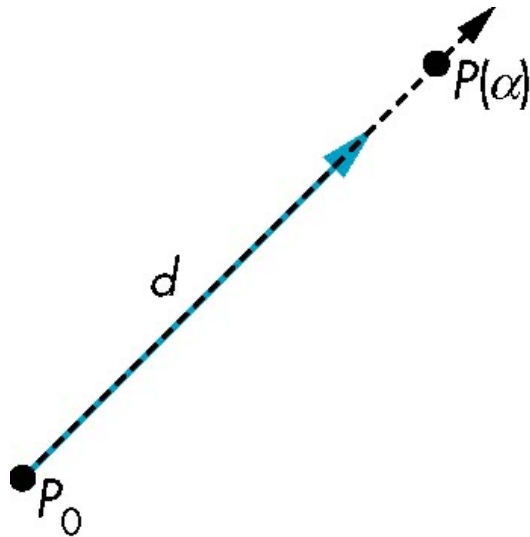
Parametric Form

- This form is known as the parametric form of the line
  - More robust and general than other forms
  - Extends to curves and surfaces
- Two-dimensional forms
  - Explicit: $y = kx + m$
  - Implicit: $ax + by + c = 0$
  - Parametric:

$$x(\alpha) = \alpha x_0 + (1-\alpha)x_1$$
$$y(\alpha) = \alpha y_0 + (1-\alpha)y_1$$

Rays and Line Segments

- If $\alpha >= 0$, then $P(\alpha)$ is the *ray* leaving P_0 in the direction **d**

If we use two points to define v, then

$P(\alpha) = Q + \alpha (R-Q) = Q + \alpha v$

$= \alpha R + (1-\alpha)Q$

For $0 <= \alpha <= 1$ we get all the

points on the *line segment*

joining R and Q
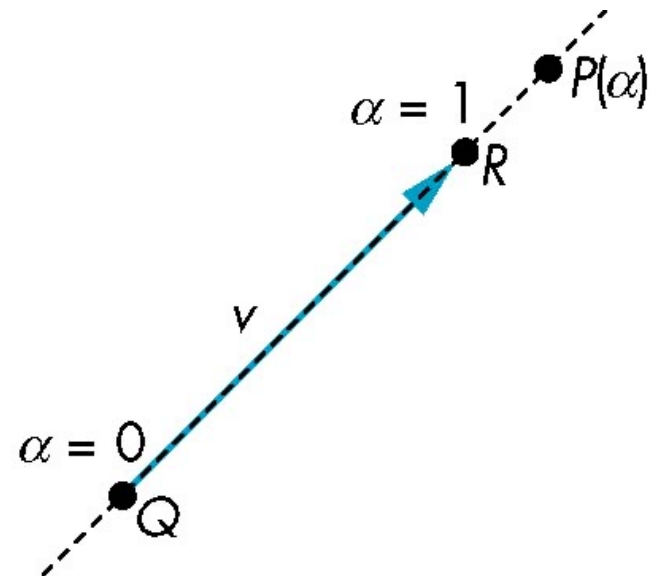
Planes

- A plane can be defined by a point and two vectors or by three points

$$P(\alpha,\beta)=R+\alpha u+\beta v$$

$$P(\alpha,\beta)=R+\alpha(Q-R)+\beta(P-Q)$$

Triangles

convex sum of S(α) and R

convex sum of P and Q

$T(\alpha,\beta)$

R

P

$S(\alpha)$

Q
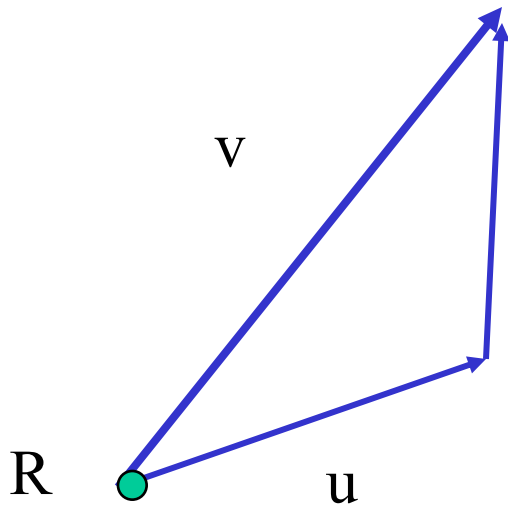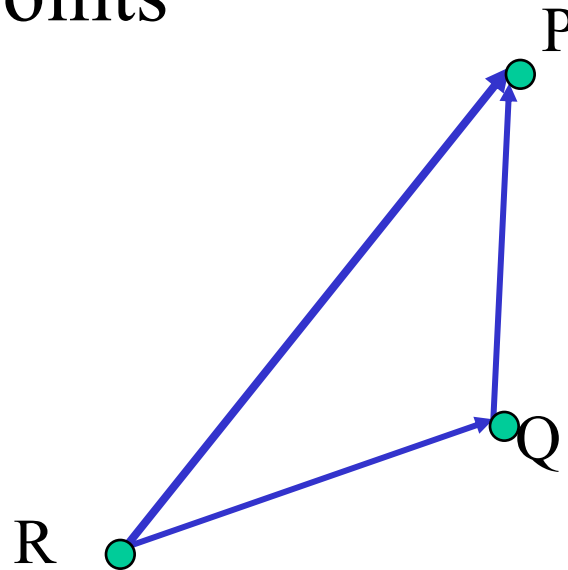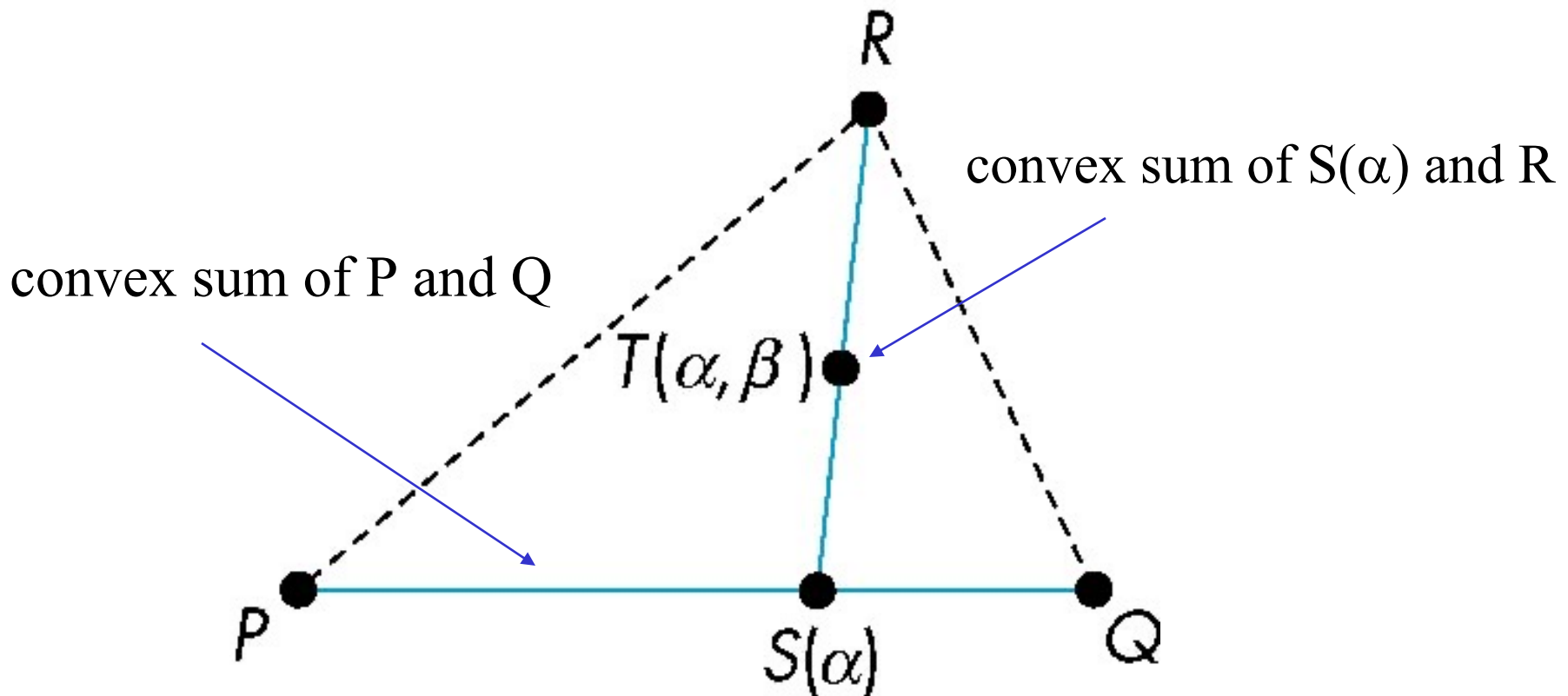
for $0 <= \alpha, \beta <= 1$, we get all points in triangle

Normals

- Every plane has a vector n normal (perpendicular, orthogonal) to it

- From point/vector form
  - $P(\alpha,\beta)=R+\alpha u+\beta v$

  we know we can use the cross product to find
  - $n = u \times v$

- Plane equation:
  - $n \cdot \mathbf{x} - d = 0$,
  - where $d = -n \cdot p$ and p is any point in the plane
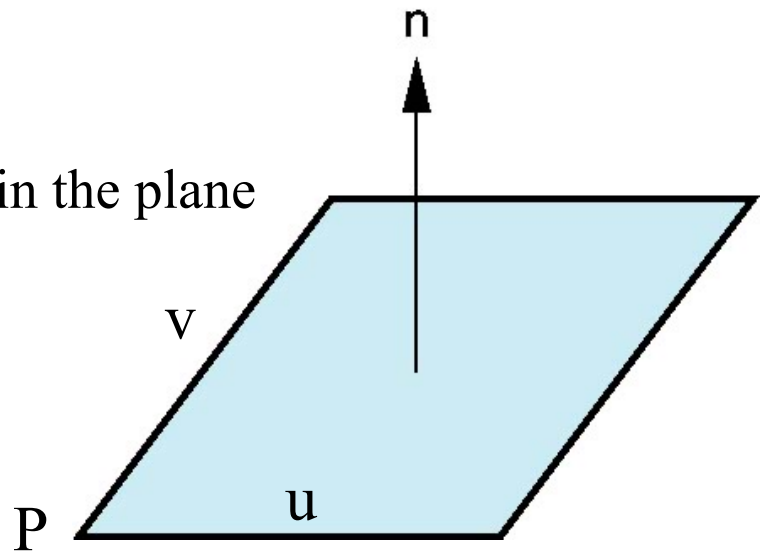
Normal for Triangle

plane $\quad \mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$

$\mathbf{n} = (\mathbf{p}_2 - \mathbf{p}_0) \times (\mathbf{p}_1 - \mathbf{p}_0)$

normalize $\mathbf{n} \quad \leftarrow \quad \mathbf{n}/|\mathbf{n}|$
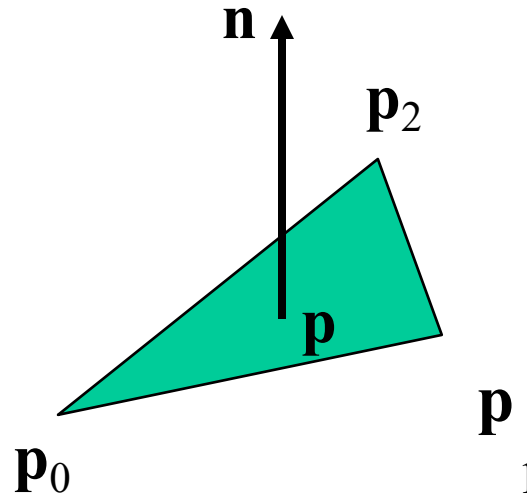
Note that right-hand rule determines outward face

Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object
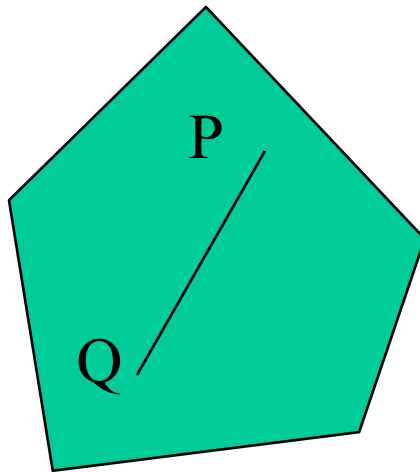
convex
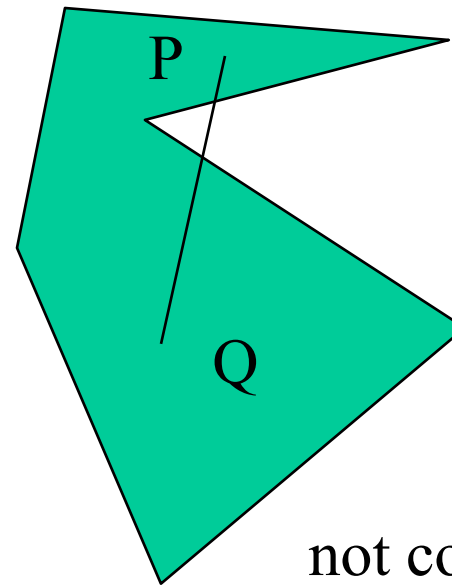
not convex

Affine Sums

- Consider the "sum"

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \dots + \alpha_n P_n$$

Can show by induction that this sum makes sense iff

$$\alpha_1 + \alpha_2 + \dots \alpha_n = 1$$

in which case we have the *affine sum* of the points $P_1, P_2, \dots P_n$

- If, in addition, $\alpha_i >= 0$, we have the *convex hull* of $P_1, P_2, \dots P_n$

Convex Hull

Consider the linear combination

$$P=\alpha_1 P_1 + \alpha_2 P_2 + \ldots + \alpha_n P_n$$

- If $\alpha_1 + \alpha_2 + \ldots \alpha_n = 1$

  – (in which case we have the *affine sum* of the points $P_1, P_2, \ldots P_n$)

  and if $\alpha_i >= 0$, we have the *convex hull* of $P_1, P_2, \ldots P_n$

- Smallest convex object containing $P_1, P_2, \ldots P_n$
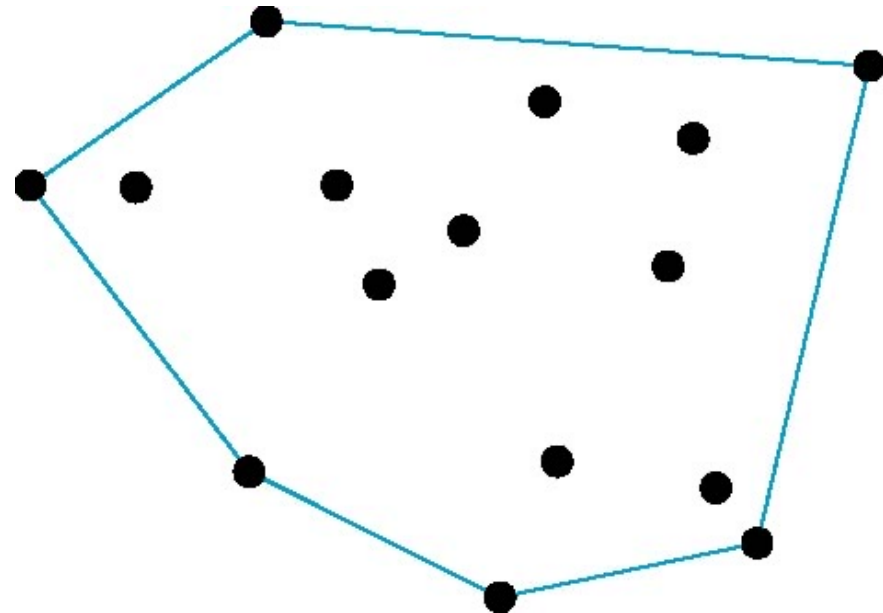
Frames

- A coordinate system is insufficient to represent points

- If we work in an affine space we can add a single point, the *origin*, to the basis vectors to form a *frame*
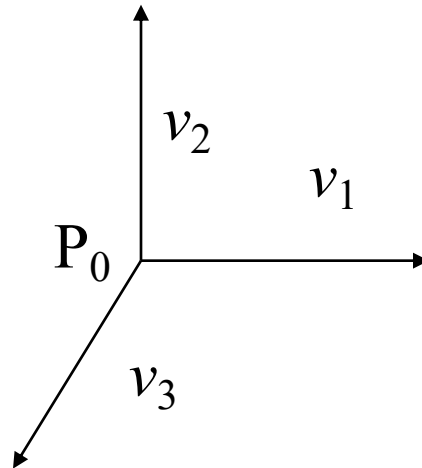
Representing one basis in terms of another

Each of the basis vectors, u1, u2, u3, are vectors that can be represented in terms

$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$
$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$
$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$
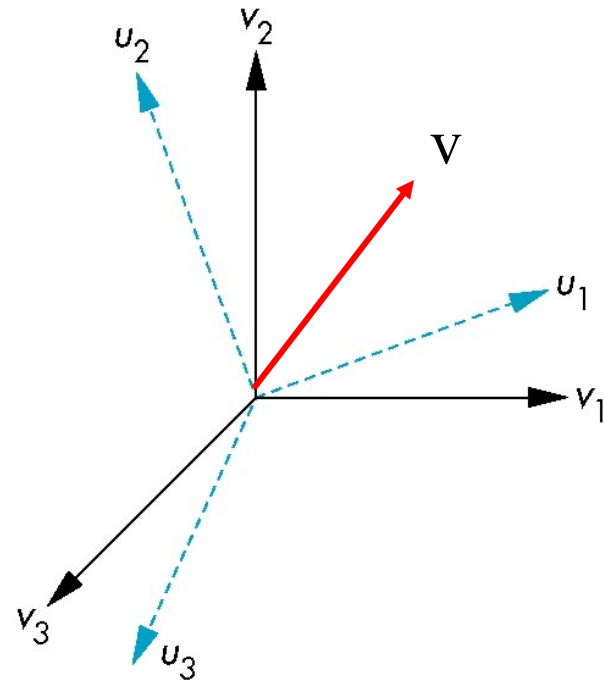
Matrix Form

The coefficients define a 3 x 3 matrix

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

and the bases can be related by

$$\mathbf{a} = \mathbf{M}^T \mathbf{b}$$

Translation

- Move (translate, displace) a point to a new location

P'

d

P
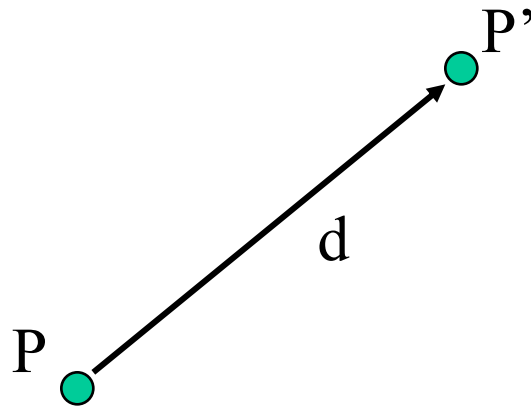
- Displacement determined by a vector d
  - Three degrees of freedom
  - P'=P+d

How many ways?

Although we can move a point to a new location in infinite ways, when we move many points there is usually only one way
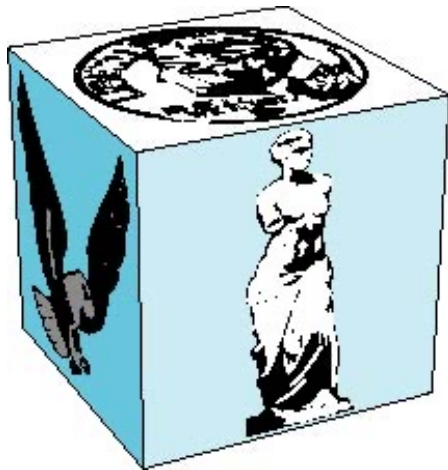
object
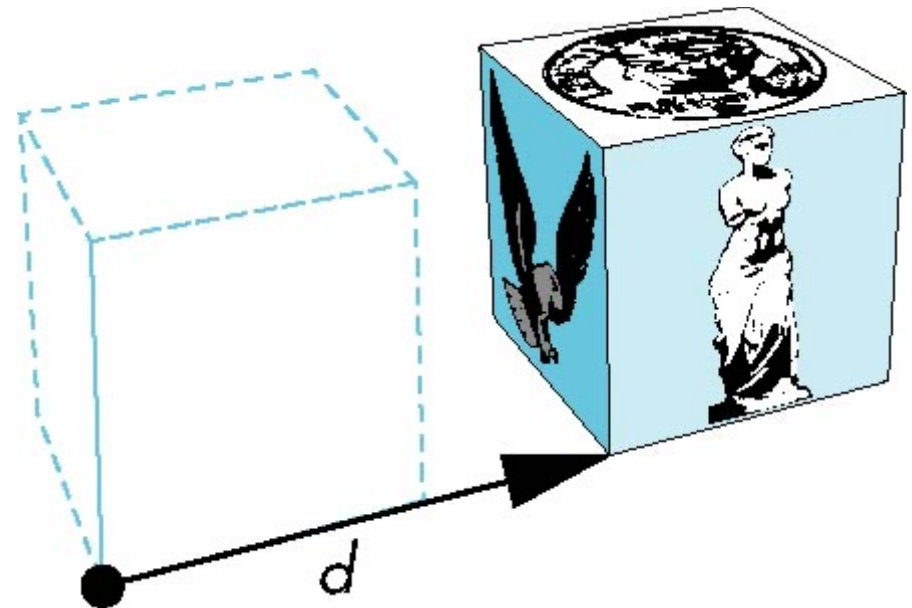
translation: every point displaced by same vector

Translation Using Representations

Using the homogeneous coordinate representation in some frame

$\mathbf{p}=[\ x\ y\ z\ 1]^T$

$\mathbf{p}'=[x'\ y'\ z'\ 1]^T$

$\mathbf{d}=[dx\ dy\ dz\ 0]^T$

Hence $\mathbf{p}' = \mathbf{p} + \mathbf{d}$ or

$x'=x+d_x$

$y'=y+d_y$

$z'=z+d_z$

note that this expression is in four dimensions and expresses point = vector + point

Translation Matrix

We can also express translation using a
4 x 4 matrix $\mathbf{T}$ in homogeneous coordinates
$\mathbf{p'}=\mathbf{Tp}$ where

$$\mathbf{T} = \mathbf{T}(d_x, d_y, d_z) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This form is better for implementation because all affine
transformations can be expressed this way and multiple
transformations can be concatenated together

Homogeneous Coordinates

The homogeneous coordinates form for a three dimensional point [x y z] is given as

$\mathbf{p} = [x'\ y'\ z'\ w]^T = [wx\ wy\ wz\ w]^T$

We return to a three dimensional point (for w≠0) by

x←x'/w

y←y'/w

z←z'/w

If w=0, the representation is that of a vector

Note that homogeneous coordinates replaces points in three dimensions by lines through the origin in four dimensions

For w=1, the representation of a point is [x y z 1]

Homogeneous Coordinates and Computer Graphics

- Homogeneous coordinates are key to all computer graphics systems
  - All standard transformations (rotation, translation, scaling) can be implemented with matrix multiplications using 4 x 4 matrices
  - Hardware pipeline works with 4 dimensional representations
  - For orthographic viewing, we can maintain w=0 for vectors and w=1 for points
  - For perspective we need a *perspective division*

Rotation about the z axis

- Rotation about z axis in three dimensions leaves all points with the same z
  - Equivalent to rotation in two dimensions in planes of constant z

$$x'=x \cos \theta - y \sin \theta$$
$$y' = x \sin \theta + y \cos \theta$$
$$z' = z$$

  - or in homogeneous coordinates

$$\mathbf{p'} = \mathbf{R_Z}(\theta)\mathbf{p}$$

Rotation Matrix

$$\mathbf{R} = \mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation about x and y axes

- Same argument as for rotation about *z* axis
  - For rotation about *x* axis, *x* is unchanged
  - For rotation about *y* axis, *y* is unchanged

$$\mathbf{R} = \mathbf{R}_x(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{R}_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

Expand or contract along each axis (fixed point of origin)

$x' = s_x x$

$y' = s_y x$

$z' = s_z x$

$\mathbf{p}' = \mathbf{S}\mathbf{p}$

$$\mathbf{S} = \mathbf{S}(s_x, s_y, s_z) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
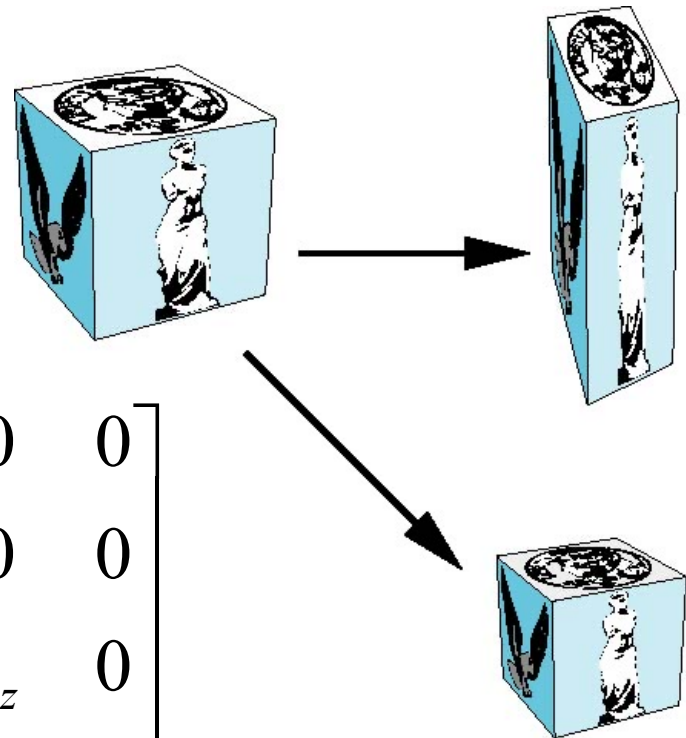
Reflection

corresponds to negative scale factors

$s_x = -1 \ s_y = 1$

original

$s_x = -1 \ s_y = -1$

$s_x = 1 \ s_y = -1$
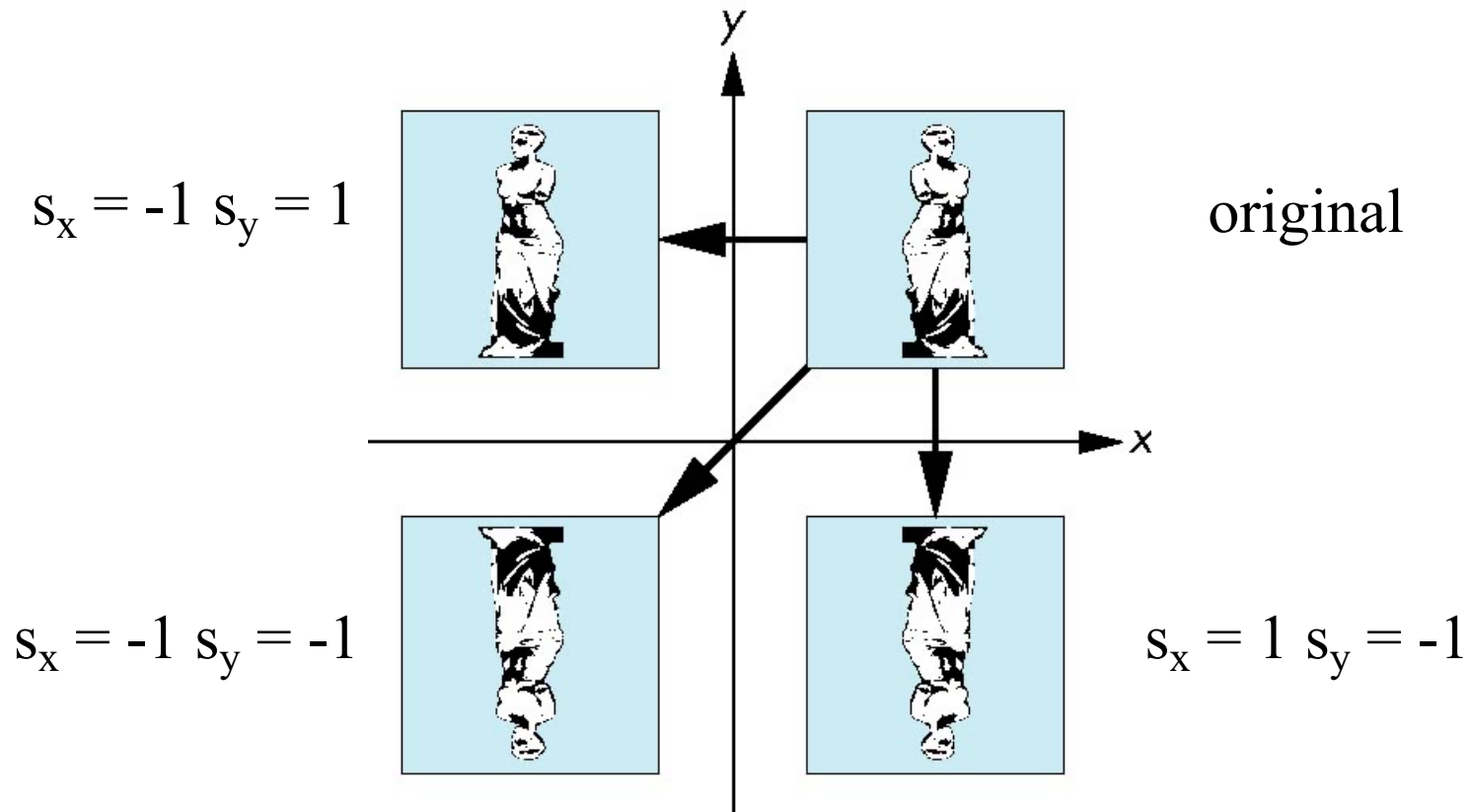
Inverses

- Although we could compute inverse matrices by general formulas, we can use simple geometric observations

  - Translation: $\mathbf{T}^{-1}(d_x, d_y, d_z) = \mathbf{T}(-d_x, -d_y, -d_z)$
  - Rotation: $\mathbf{R}^{-1}(\theta) = \mathbf{R}(-\theta)$

    - Holds for any rotation matrix
    - Note that since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$

    $\mathbf{R}^{-1}(\theta) = \mathbf{R}^{T}(\theta)$

  - Scaling: $\mathbf{S}^{-1}(s_x, s_y, s_z) = \mathbf{S}(1/s_x, 1/s_y, 1/s_z)$

Concatenation

- We can form arbitrary affine transformation matrices by multiplying together rotation, translation, and scaling matrices

- Because the same transformation is applied to many vertices, the cost of forming a matrix **M**=**ABCD** is not significant compared to the cost of computing **Mp** for many vertices **p**

- The difficult part is how to form a desired transformation from the specifications in the application

Order of Transformations

- Note that matrix on the right is the first applied
- Mathematically, the following are equivalent

$$\mathbf{p}' = \mathbf{ABCp} = \mathbf{A(B(Cp))}$$

- Note many references use column matrices to represent points. In terms of column matrices

$$\mathbf{p}'^{\mathrm{T}} = \mathbf{p}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

General Rotation About the Origin

A rotation by θ about an arbitrary axis can be decomposed into the concatenation of rotations about the *x*, *y*, and *z* axes

$$\mathbf{R}(\theta) = \mathbf{R}_z(\theta_z)\, \mathbf{R}_y(\theta_y)\, \mathbf{R}_x(\theta_x)$$

$\theta_x\, \theta_y\, \theta_z$ are called the Euler angles

Note that rotations do not commute
We can use rotations in another order but
with different angles
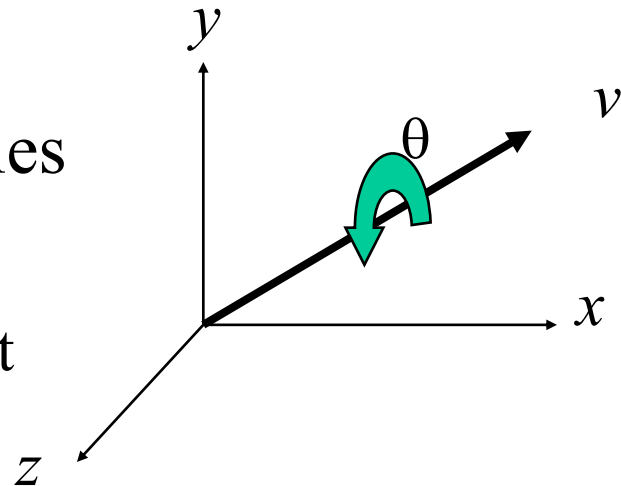
Rotation About a Fixed Point other than the Origin

Move fixed point to origin

Rotate

Move fixed point back

$\mathbf{M} = \mathbf{T}(p_f)\, \mathbf{R}(\theta)\, \mathbf{T}(-p_f)$
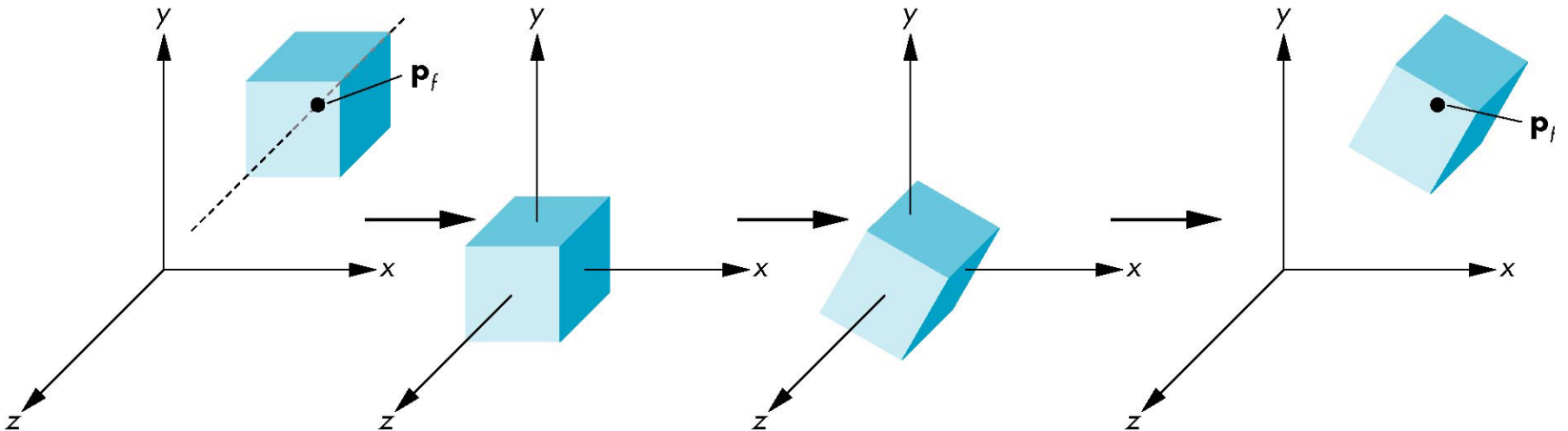
Instancing

- In modeling, we often start with a simple object centered at the origin, oriented with the axis, and at a standard size

- We apply an *instance transformation* to its vertices to

  Scale

  Orient

  Locate
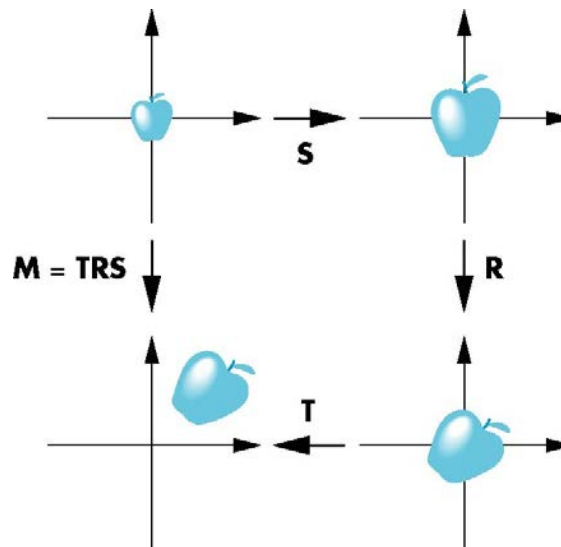
Shear

- Helpful to add one more basic transformation
- Equivalent to pulling faces in opposite directions
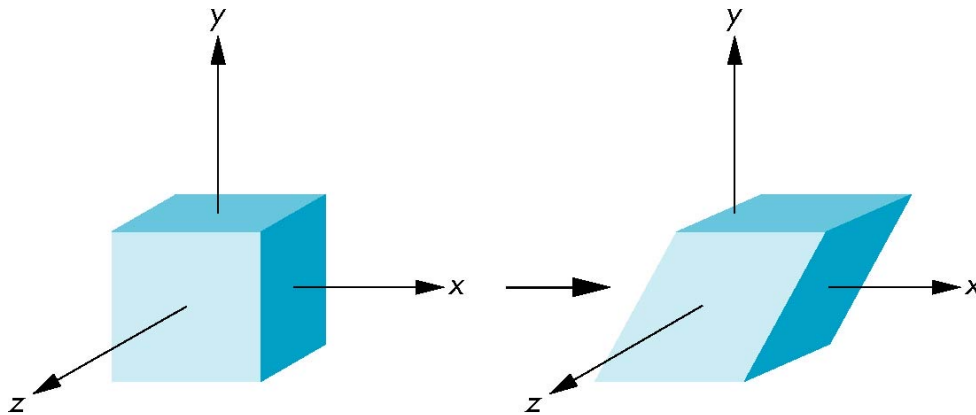
Shear Matrix

Consider simple shear along *x* axis

x' = x + y cot θ
y' = y
z' = z

$$\mathbf{H}(\theta) = \begin{bmatrix} 1 & \cot \theta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
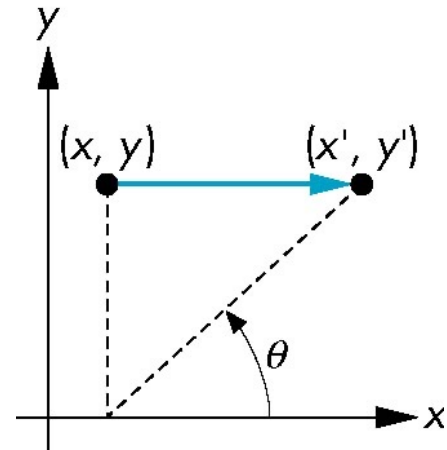
Computer Viewing

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Objectives

- Introduce the mathematics of projection

Computer Viewing

- There are three aspects of the viewing process, all of which are implemented in the pipeline,
  - Positioning the camera
    - Setting the model-view matrix
  - Selecting a lens
    - Setting the projection matrix
  - Clipping
    - Setting the view volume
    - (default is unit cube, R^3, [-1,1])

Default Projection

Default projection is orthogonal
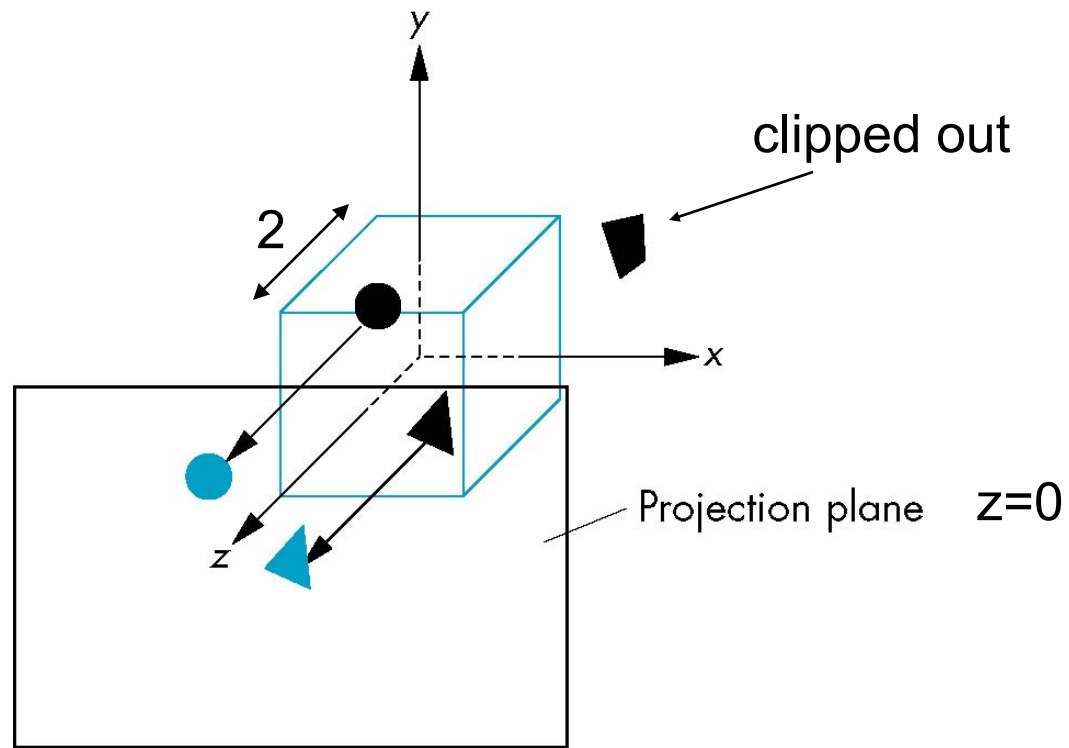
Moving the Camera Frame

- If we want to visualize object with both positive and negative z values we can either
  - Move the camera in the positive z direction
    - Translate the camera frame
  - Move the objects in the negative z direction
    - Translate the world frame
- Both of these views are equivalent and are determined by the model-view matrix

Moving the Camera

- We can move the camera to any desired position by a sequence of rotations and translations

- Example: side view
  - Rotate the camera
  - Move it away from origin
  - Model-view matrix $C = TR$
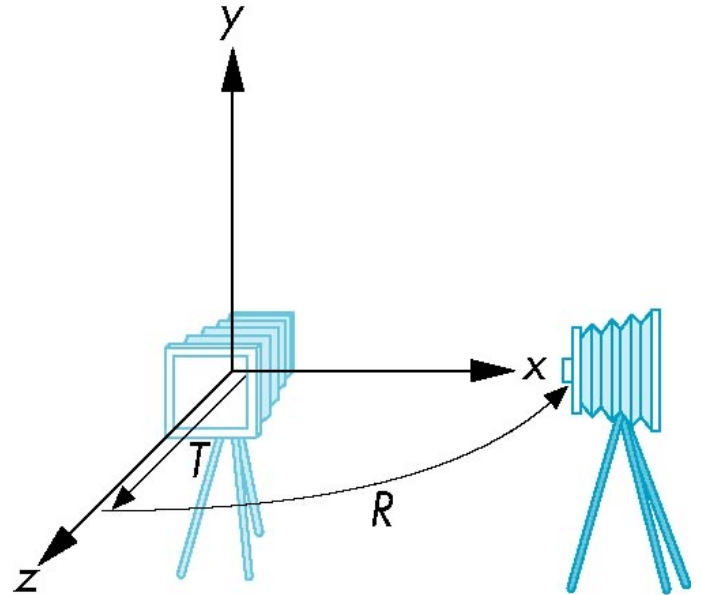
OpenGL Orthogonal Viewing
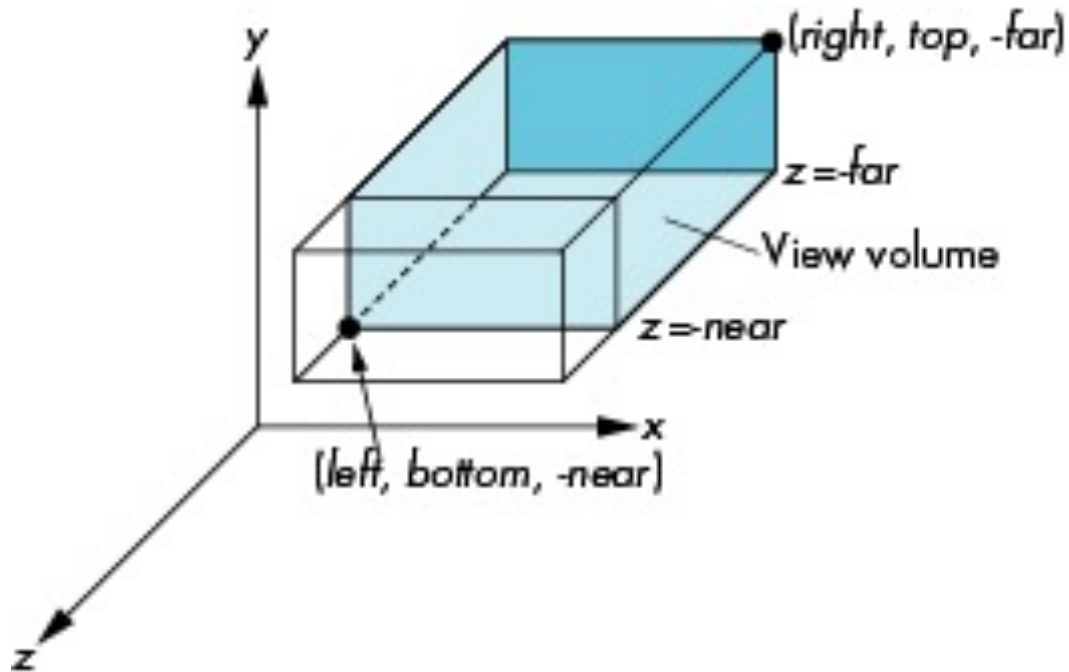
near and far measured from camera

OpenGL Perspective
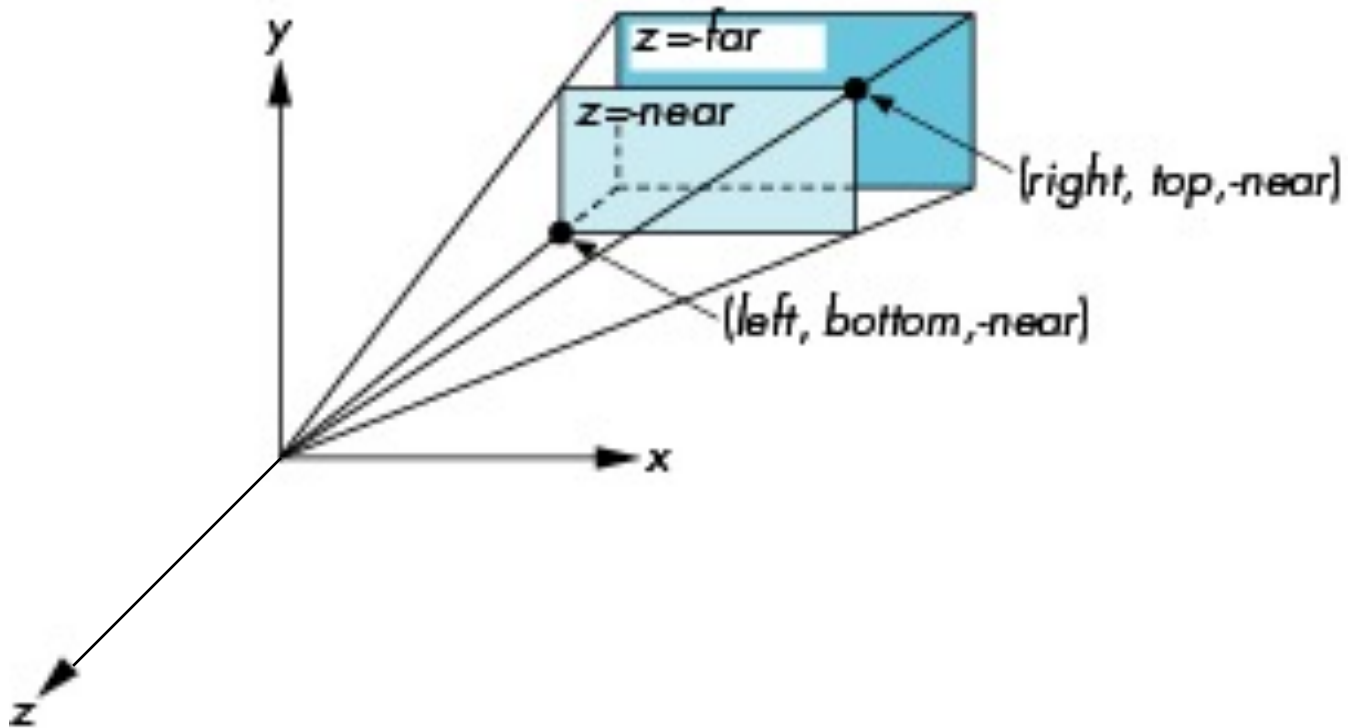
Using Field of View

- Parameters **fovy, aspect, near, far** often provides a better interface

front plane

aspect = w/h
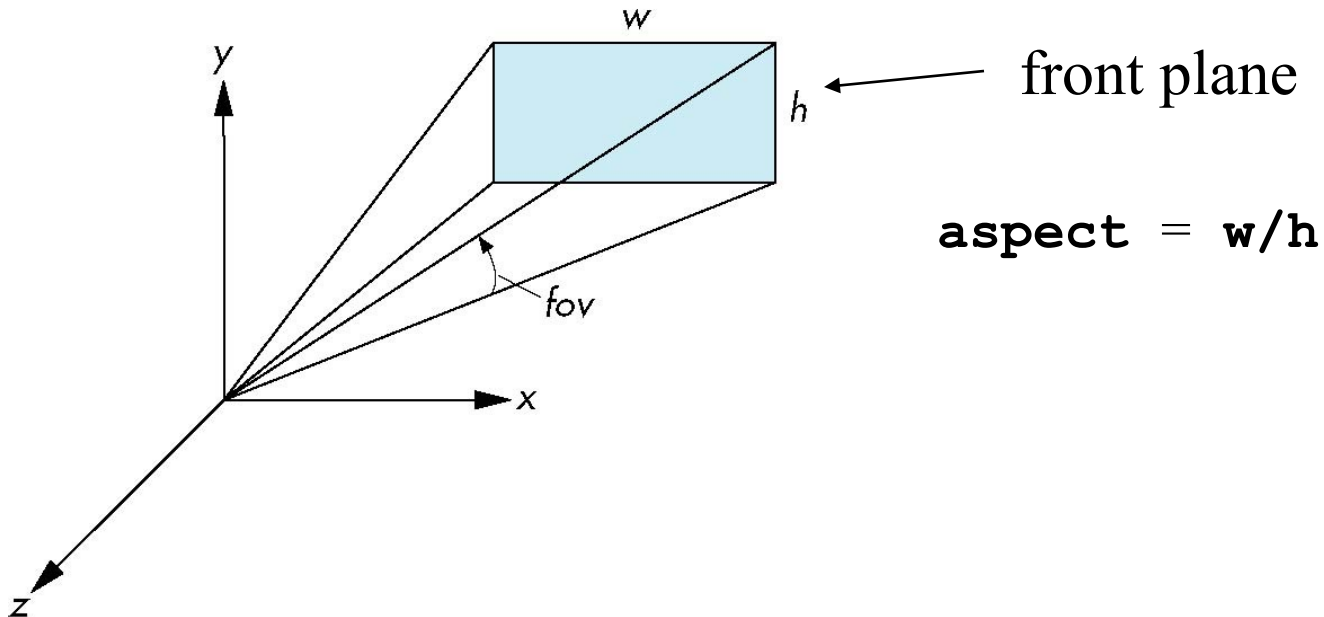
Projections explained differently

- Read the following slides about orthogonal and perspective projections by your selves
- They present the same thing, but explained differently

Projections and Normalization

- The default projection in the eye (camera) frame is orthogonal

- For points within the default view volume

$$x_p = x$$
$$y_p = y$$
$$z_p = 0$$

- Most graphics systems use *view normalization*
  - All other views are converted to the default view by transformations that determine the projection matrix
  - Allows use of the same pipeline for all views

Homogeneous Coordinate Representation

default orthographic projection

$$\mathbf{p}_p = \mathbf{M}\mathbf{p}$$

$x_p = x$
$y_p = y$
$z_p = 0$
$w_p = 1$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In practice, we can let $\mathbf{M} = \mathbf{I}$ and set the z term to zero later

Simple Perspective

- Center of projection at the origin
- Projection plane $z = d,\ d < 0$
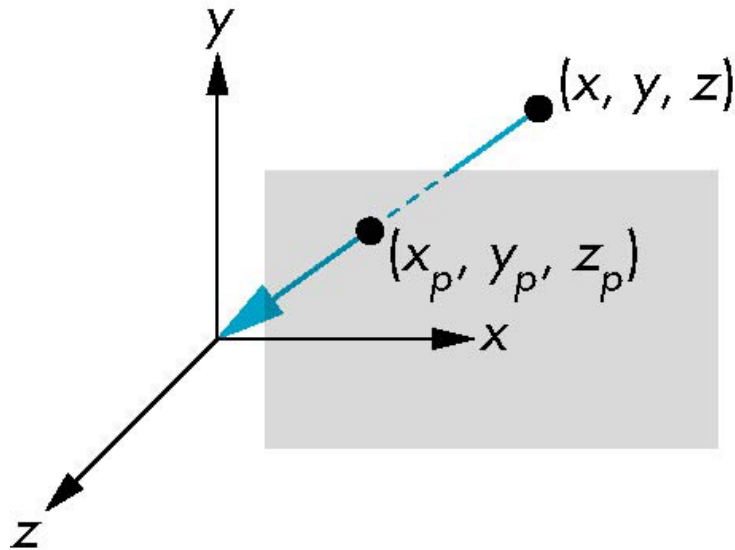
Perspective Equations

Consider top and side views
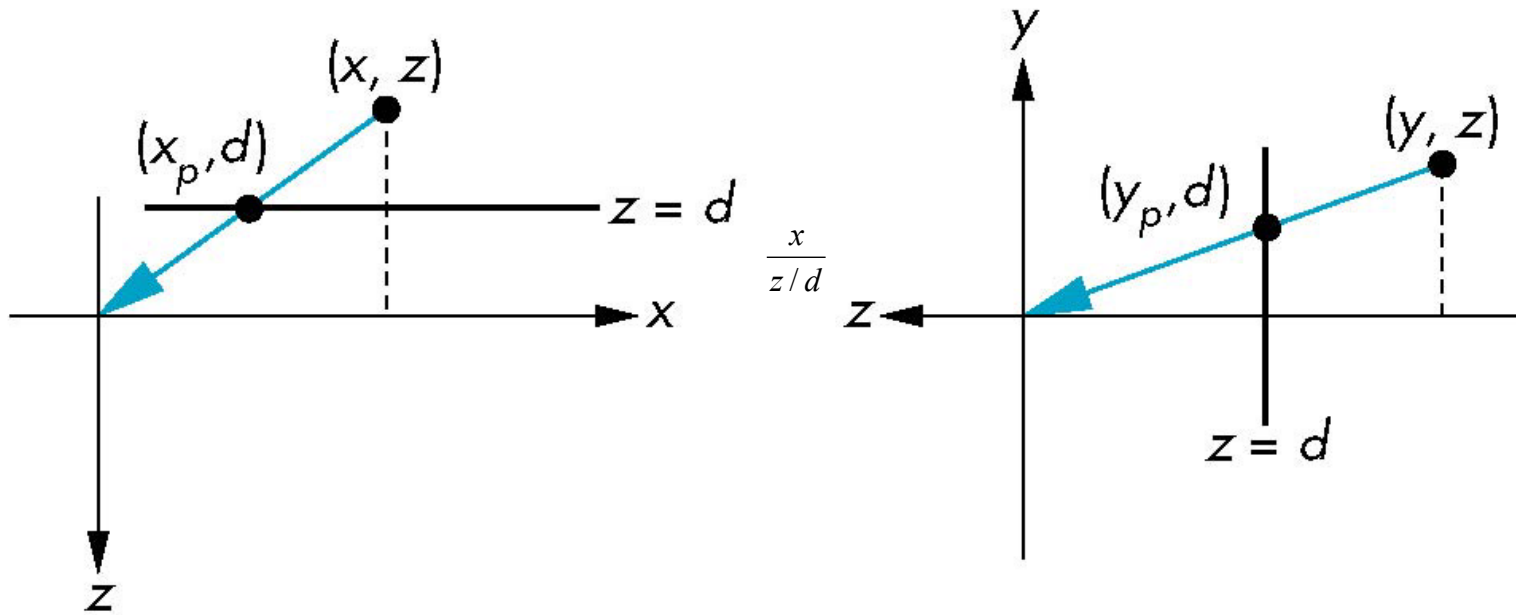
$$x_p = \frac{x}{z/d} \qquad y_p = \frac{y}{z/d} \qquad z_p = d$$

Homogeneous Coordinate Form

consider $\mathbf{q} = \mathbf{Mp}$ where

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/d & 0 \end{bmatrix}$$

$$\mathbf{q} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow \mathbf{p} = \begin{bmatrix} x \\ y \\ z \\ z/d \end{bmatrix}$$

Perspective Division

- However $w \neq 1$, so we must divide by w to return from homogeneous coordinates

- This *perspective division* yields

$$x_p = \frac{x}{z/d} \qquad y_p = \frac{y}{z/d} \qquad z_p = d$$

the desired perspective equations

- We will consider the corresponding clipping volume with the OpenGL functions

Normalization

- Rather than derive a different projection matrix for each type of projection, we can convert all projections to orthogonal projections with the default view volume

- This strategy allows us to use standard transformations in the pipeline and makes for efficient clipping

Pipeline View

modelview transformation → projection transformation → perspective division

nonsingular

4D → 3D

→ clipping → projection →

against default cube

3D → 2D
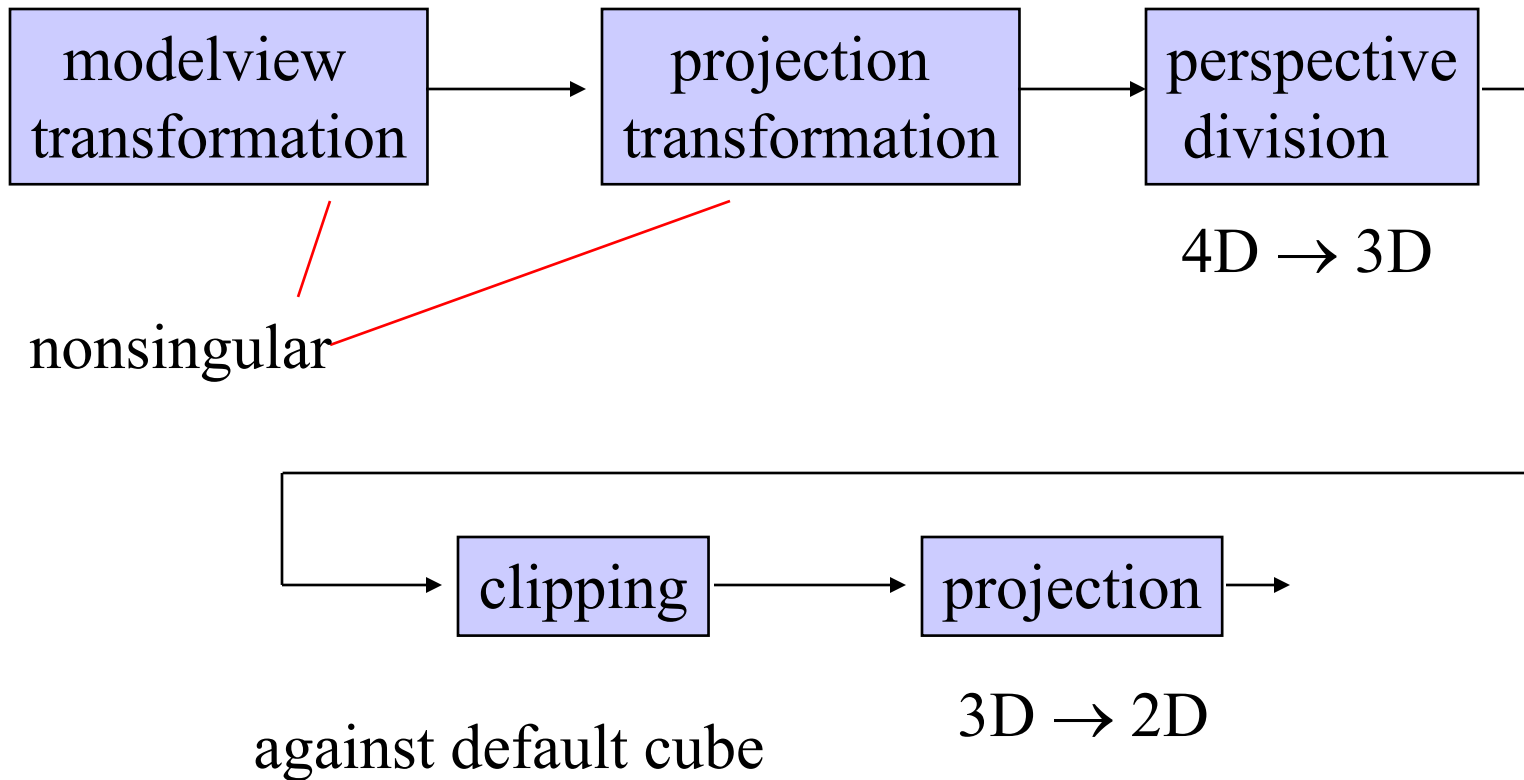
Notes

- We stay in four-dimensional homogeneous coordinates through both the modelview and projection transformations
  - Both these transformations are nonsingular
  - Default to identity matrices (orthogonal view)
- Normalization lets us clip against simple cube regardless of type of projection
- Delay final projection until end
  - Important for hidden-surface removal to retain depth information as long as possible

Orthogonal Normalization

normalization $\Rightarrow$ find transformation to convert specified clipping volume to default
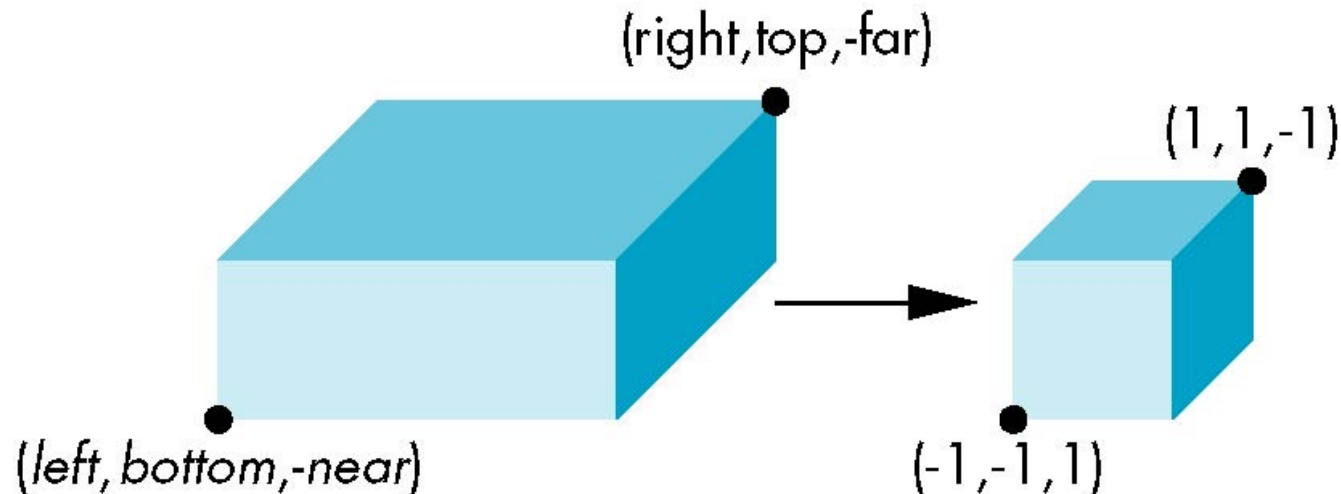
Orthogonal Matrix

- Two steps
  - Move center to origin

    T(-(left+right)/2, -(bottom+top)/2,(near+far)/2))
  - Scale to have sides of length 2

    S(2/(left-right),2/(top-bottom),2/(near-far))

$$\mathbf{P} = \mathbf{ST} = \begin{bmatrix} \dfrac{2}{right-left} & 0 & 0 & -\dfrac{right-left}{right-left} \\ 0 & \dfrac{2}{top-bottom} & 0 & -\dfrac{top+bottom}{top-bottom} \\ 0 & 0 & \dfrac{2}{near-far} & \dfrac{far+near}{far-near} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
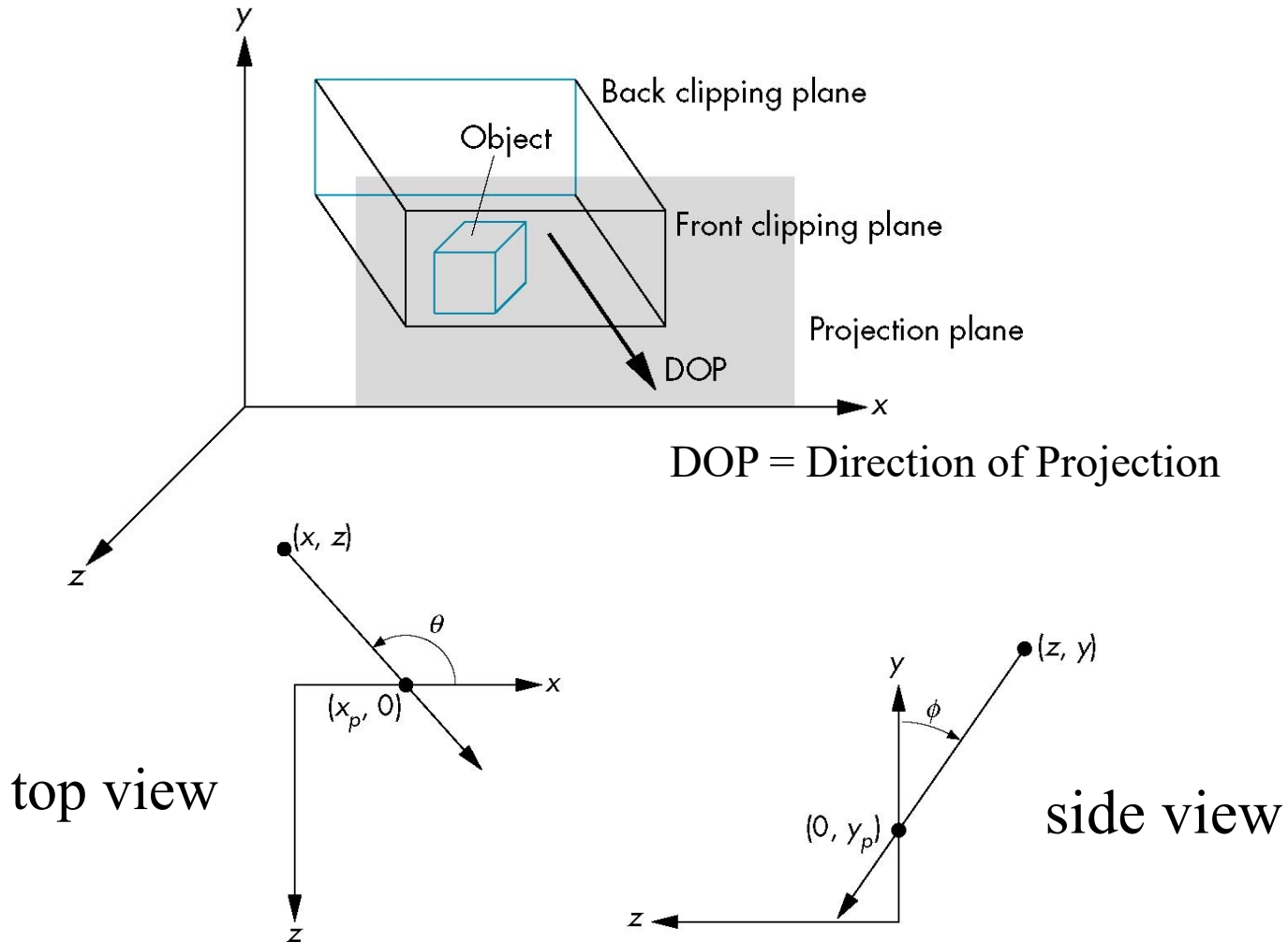
Final Projection

- Set $z = 0$

- Equivalent to the homogeneous coordinate transformation

$$\mathbf{M}_{orth} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Hence, general orthogonal projection in 4D is

$$\mathbf{P} = \mathbf{M}_{orth}\mathbf{ST}$$

General Shear

DOP = Direction of Projection

top view

side view

Shear Matrix

xy shear (*z* values unchanged)

$$\mathbf{H}(\theta,\phi) = \begin{bmatrix} 1 & 0 & -\cot\theta & 0 \\ 0 & 1 & -\cot\phi & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection matrix

$$\mathbf{P} = \mathbf{M}_{\text{orth}}\,\mathbf{H}(\theta,\phi)$$

General case: $\mathbf{P} = \mathbf{M}_{\text{orth}}\,\mathbf{STH}(\theta,\phi)$

Effect on Clipping

- The projection matrix **P** = **STH** transforms the original clipping volume to the default clipping volume

object

top view

$z = 1$

DOP

DOP

$x = -1$

$x = 1$

far plane

$z = -1$

clipping
volume

near plane

distorted object
(projects correctly)
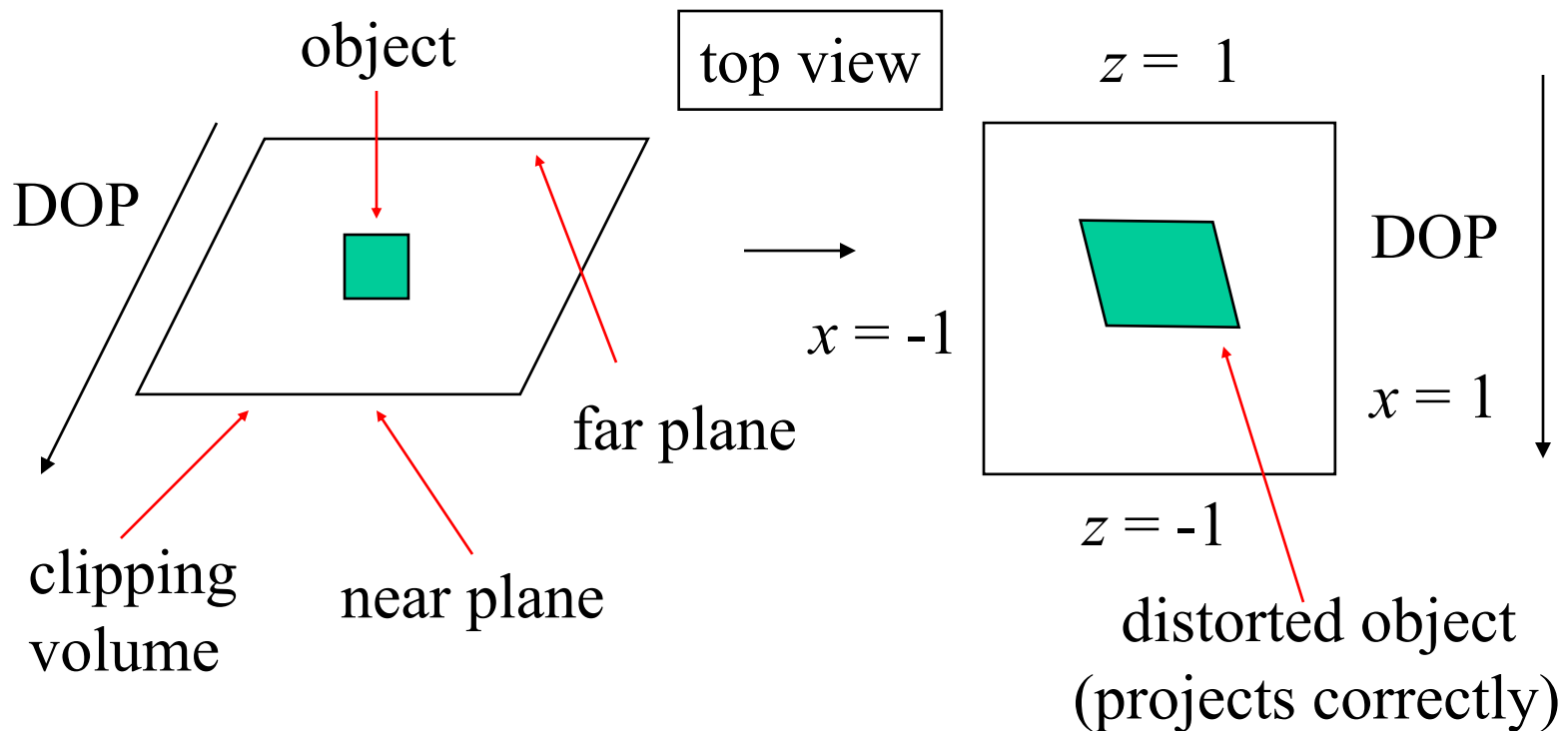
Simple Perspective

Consider a simple perspective with the COP (=center of projection) at the origin, the near clipping plane at $z = -1$, and a 90 degree field of view determined by the planes

$x = \pm z, y = \pm z$
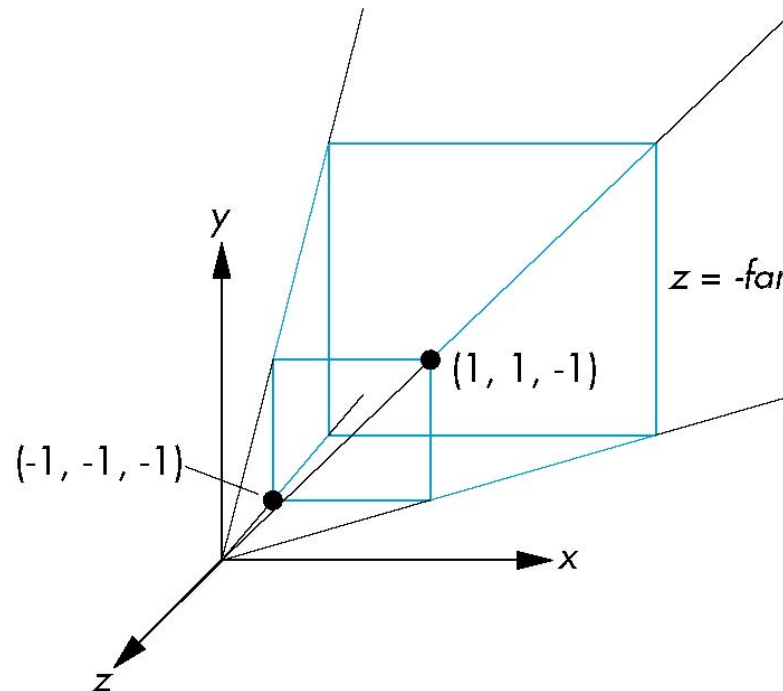
Perspective Matrices

Simple projection matrix in homogeneous
coordinates

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Note that this matrix is independent of the far
clipping plane

Generalization

$$\mathbf{N} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

after perspective division, the point $(x, y, z, 1)$ goes to

$x'' = x/z$
$y'' = y/z$
$Z'' = -(\alpha + \beta/z)$

which projects orthogonally to the desired point regardless of α and β

Picking α and β

If we pick

$$\alpha = \frac{\text{near} + \text{far}}{\text{far} - \text{near}}$$

$$\beta = \frac{2\,\text{near} * \text{far}}{\text{near} - \text{far}}$$
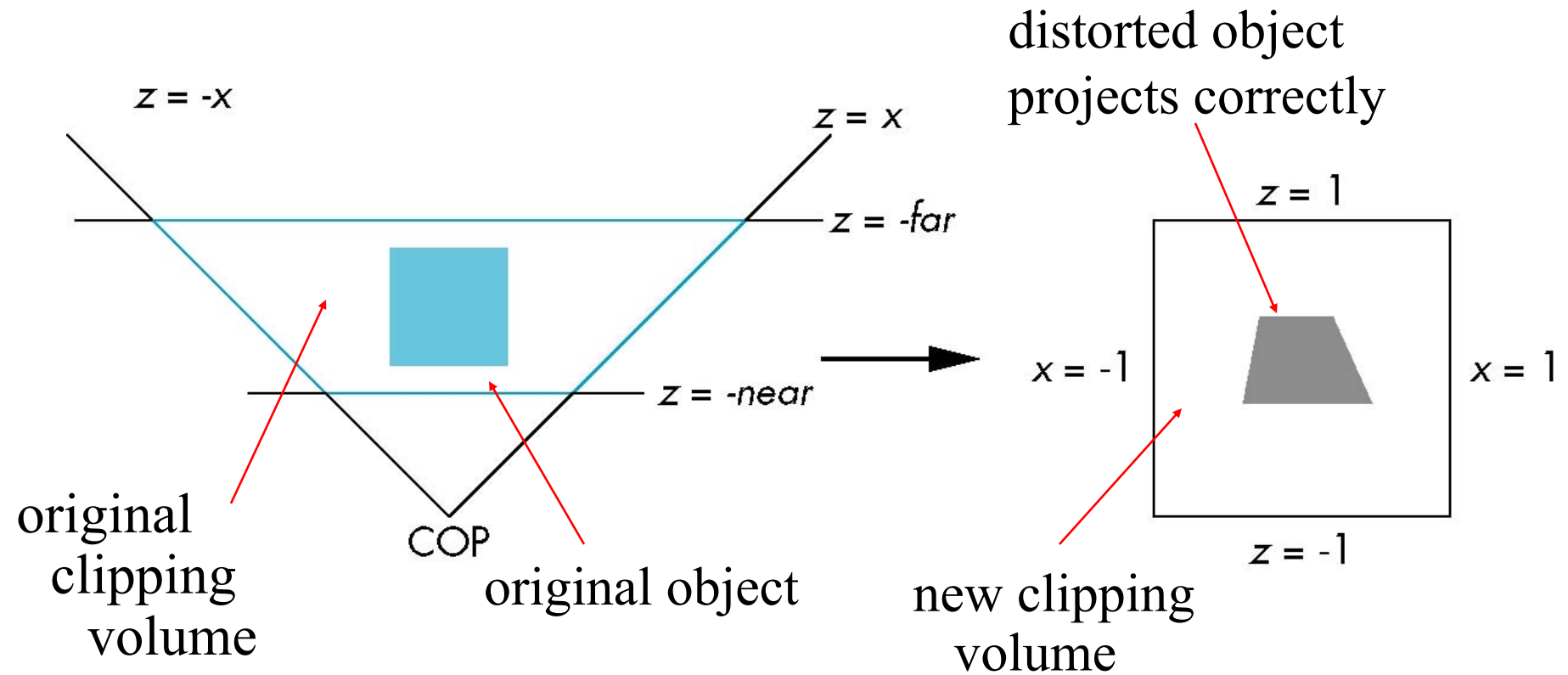
the near plane is mapped to $z = -1$
the far plane is mapped to $z = 1$
and the sides are mapped to $x = \pm\,1,\, y = \pm\,1$

Hence the new clipping volume is the default clipping volume

Normalization Transformation

Normalization and Hidden-Surface Removal

- Although our selection of the form of the perspective matrices may appear somewhat arbitrary, it was chosen so that if $z_1 > z_2$ in the original clipping volume then the for the transformed points $z_1' > z_2'$

- Thus hidden surface removal works if we first apply the normalization transformation

- However, the formula $z'' = -(\alpha + \beta/z)$ implies that the distances are distorted by the normalization which can cause numerical problems especially if the near distance is small

OpenGL Perspective

- Unsymmetric viewing frustum possible:

$z = z_{min}$

$(x_{max}, y_{max}, z_{max})$

$(x_{min}, y_{min}, z_{max})$

COP
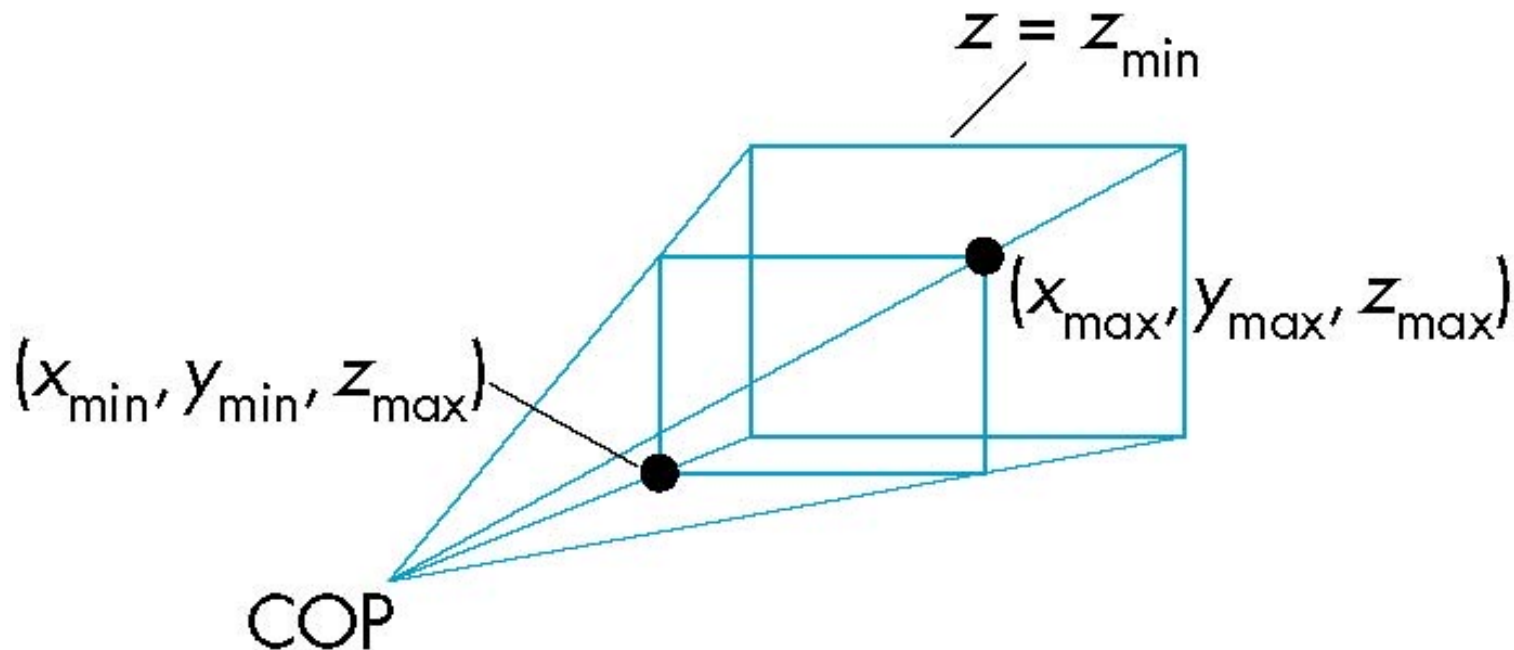
OpenGL Perspective Matrix

- The normalization by a perspective projection requires an initial shear to form a right viewing pyramid, followed by a scaling to get the normalized perspective volume. Finally, the perspective matrix results in needing only a final orthogonal transformation

$$\mathbf{P} = \mathbf{NSH}$$

our previously defined
perspective matrix

shear and scale

Why do we do it this way?

- Normalization allows for a single pipeline for both perspective and orthogonal viewing
- We stay in four dimensional homogeneous coordinates as long as possible to retain three-dimensional information needed for hidden-surface removal and shading
- We simplify clipping