

# Hash tables

# Faster than $O(\log n)$ sets and maps?

- Balanced search trees of various kinds and skip lists provides set and map operations with logarithmic complexity. Can we do better?
- For arrays we have constant time random access.
- Let's assume we need to quickly look up the address of any Swedish citizen based on personnummer.
- We can create an array with one element for each possible personnummer and have a quick way to calculate the index. In this array we store the addresses. We get constant time operations.
- This is plausible because the number of possible personnummer isn't a whole lot bigger than the number of Swedish citizens.

# Faster than $O(\log n)$ sets and maps?

- What if google asks us to create a map that maps every word occurring on the internet to a set of url:s to pages where it occurs.
- Then we have a problem since the number of possible words is much larger than the set of occurring words. It's difficult to design the set of keys such that the array doesn't end up very sparsely populated and too big even for data centers.
- Idea: Let's make an array whose size is proportionate to the number of entries and project the keys onto the set of valid indeces.

# Hash tables naively

The plan: take an array of some size  $k$

Define a *hash function* that maps values (set elements or map keys) to indices in the range  $\{0, \dots, k-1\}$

- Example: if the values are integers, hash function might be  $h(n) = n \bmod k$

To find, insert or remove a value  $x$ , put it in index  $h(x)$  of the array

- Avoid searching through the whole array!

This takes constant time!

# Hash tables naively, example

Implementing a set of integers, suppose we take a hash table of size 5 and a hash function  $h(n) = n \bmod 5$

0	1	2	3	4
5		17	8	

This hash table contains {5, 8, 17}

Inserting 14 gives:

0	1	2	3	4
5		17	8	14

Similarly, if we wanted to find 8, we would look it up in index 3

# A problem

This idea doesn't work.

What if we want to insert 12 into the set?

0	1	2	3	4
5		17	8	

We should store 12 at index 2, but there's already something there!

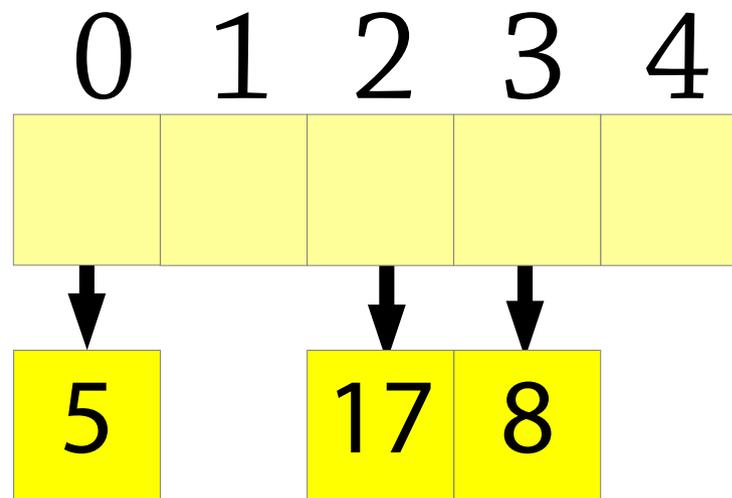
This is called a *collision*

Real hash tables are naive hash tables plus tricks for dealing with and avoiding collisions!

# Handling collisions: chaining

Instead of an array of elements, have an array of *linked lists* (chains)

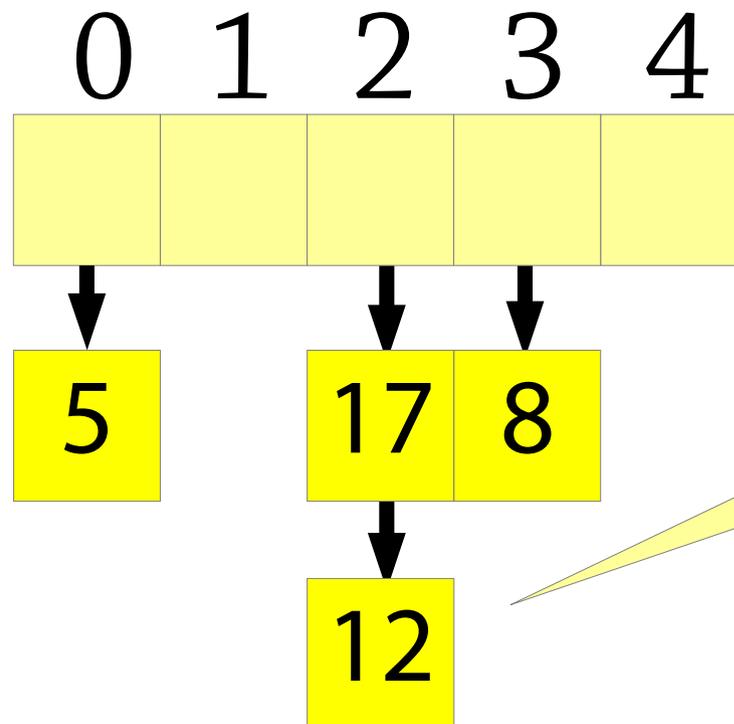
To add an element, calculate its hash and insert it into the list at that index



# Handling collisions: chaining

Instead of an array of elements, have an array of *linked lists* (chains)

To add an element, calculate its hash and insert it into the list at that index



Inserting 12  
into the table

# Performance of chained hash tables

Chained hash tables are fast if the chains are small

- If the size is bounded, operations are  $O(1)$  time

But if the chains get big, everything gets slow

- Can degrade to  $O(n)$  in the worst case

There are two cases when this can happen!

We have to avoid both of them.

# Performance of chained hash tables

## Case one: the hash table is too full

- If we try to store 1,000,000 values in an array of size 5, some chains will be 200,000 long

## Solution: expand the hash table

- If the hash table gets too full (*load factor* too high), allocate a new array about twice as big (*rehashing*)
- load factor = number of elements / size of array

## Problem: $h(x)$ is specific to a particular size of array

- Allow the hash function to return an arbitrary integer (the *hash code* of  $x$ ) and then take it modulo the array size:  
$$h(x) = x.hashCode() \bmod array.size$$
- Hash function of an integer will just be the integer itself

# Performance of chained hash tables

## Case two: the hash function is lousy

- Worst case:  $h(x)$  is a constant function, e.g.  
 $h(x) = 0$
- Then all elements will end up in the same chain!

## The hash function must distribute values evenly

- Each hash bucket has an equal chance of being chosen
- There are no observable patterns, e.g., easy ways to construct two values which always have the same hash

In other words, it should look like the hash function returns a *random* bucket

# Chained hash tables – the theory

We need:

- to resize the hash table when it gets too full
- a hash function which appears to be random (no patterns, equal distribution)

If we do that, the average chain size will be constant and we get *expected*  $O(1)$  performance for insert/lookup/delete!

- Complexity analysis uses probability theory

When should we resize the hash table?

- If the load factor is 3 (number of elements = array size  $\times$  3), each operation needs on average 2.5 comparisons
- Pick some constant load factor, resize when it reaches that

# A slightly awkward problem

In reality, the hash function does not return a random hash code!

- Common hash functions can have patterns

This breaks the nice theory we have. Here is one problem:

- If we double the size of the array when resizing, the array size will always be even
- If we then insert only even numbers into the hash table, only the even buckets will be used

To fix this, we make the array size always be a *prime number* (while roughly doubling it each time) – this masks patterns in the hash function

# Chained hash tables – summary

Start with a naive hash table

Add chaining

Double the size of the array when the load factor is too high...

- ...but make sure the array size is always prime

Now you have a chained hash table!

- $O(1)$  expected complexity for all operations

But how should we design hash functions?

# Designing hash functions

A good hash function should distribute values evenly

- $h(x)$  has a roughly equal chance of being any particular number (up to some large bound)
- That way, all chains will be roughly the same length!
- Also, similar values should not have similar hash codes

Defining good hash functions is a black art!

- Weird heuristics that are semi-backed-up by theory

We'll settle for: unlikely to insert many elements with the same hash

# Defining a good hash function

What is bad about the following hash function on strings?

*Add together the character code of each character in the string*  
(character code of a = 97, b = 98, c = 99 etc.)

- Maps e.g. *bass* and *bart* to the same hash code! ( $s + s = r + t$ )
- Any anagrams will have the same hash code
- Similar strings will be mapped to nearby hash codes – does not distribute strings evenly
- There are many strings with 10 characters or less. All of them will map to numbers 0..2550 (assuming 8 bit ascii characters)

# A hash function on strings

An idea: map strings to integers as follows:

$$128^n + s_0 \cdot 128^{n-1} + s_1 \cdot 128^{n-2} + \dots + s_{n-1}$$

where  $s_i$  is the code of the character at index  $i$

If all characters are ASCII (character code 0 – 127), each string is mapped to a different integer!

# An analogy

Suppose we want to define a hash function for lists of digits from 0-9:

- $[0,9,3,4,2,1]$  etc.

Idea: write out the digits as a single number with a leading 1:

- $\text{hash}([0,9,3,4,2,1]) = 1093421$

(Without the leading 1 we would get the same hash for e.g.  $[0,1]$  and  $[1]$ )

The hash function on strings is doing exactly this, only working in base 128 instead of base 10

# The problem

For performance, we will calculate the hash using machine integers so the calculation

$$128^n + s_0 \cdot 128^{n-1} + s_1 \cdot 128^{n-2} + \dots + s_{n-1},$$

will happen modulo  $2^{32}$  (*integer overflow*)

So the hash will only use the last few characters!

Solution: replace 128 with another number (which is not a power of 2), e.g. 33

$$33^n + s_0 \cdot 33^{n-1} + s_1 \cdot 33^{n-2} + \dots + s_{n-1}$$

This is (almost) what Java uses for strings

# Hash functions

This is called *Bernstein hashing*, it's only one way of defining hash functions

- Bernstein discovered that using 33 as the constant gives good distribution
- Why? Nobody knows, but primes are in general good candidates.

Many hash functions are inspired by random number generation algorithms

- The output of a good hash function should look random so there are many similarities

Often pretty ad hoc!

- Lots of experimentation involved

# Hashing composite values

```
class C { A a; B b; }
```

Use the same approach as for strings!

$$33 \times h(a) + h(b)$$

Since the number of elements is always the same we don't need the leading "1".

# Hash tables in Java

- As we've seen `TreeSet` and `TreeMap` that are search tree implementations of sets and maps.
- In analogy there is `HashSet` and `HashMap` which are hash table implementations.

# Hash functions in Java

- The top class `Object` in Java has a `hashCode()` method.
- This is implemented for all standard types. If you use a standard type as elements in a `HashSet` or keys in a `HashMap` you don't need to worry about the hash function.
- For your own classes you can use the `Objects.hash` method.
- If you provide the instance variables as arguments to this method it will use the hash code for each of them and combine them in a way similar to the example.

# Linear probing

Another way (than chaining) of dealing with collisions is *linear probing*

Linear probing is a kind of probing.

Uses an array of values, like in the naive hash table

If you want to store a value at index  $i$  but it's full, store it in index  $i+1$  instead!

If that's full, try  $i+2$ , and so on

...if you get to the end of the array, wrap around to 0

Probing is also called *open addressing* because the index is not fixed. Chaining is sometimes called *closed addressing*.

# Example of linear probing

Tom Dan Harry Sam Pete

[0]	
[1]	
[2]	
[3]	
[4]	

Name	Hash	Hash % 5
"Tom"	84274	4
"Dan"	68465	0
"Harry"	69496448	3
"Sam"	82879	4
"Pete"	2484038	3

# Example of linear probing

Sam Pete

[0]	Dan
[1]	
[2]	
[3]	Harry
[4]	Tom

Name	Hash	Hash % 5
"Tom"	84274	4
"Dan"	68465	0
"Harry"	69496448	3
"Sam"	82879	4
"Pete"	2484038	3

# Example of linear probing

		<b>Name</b>	<b>Hash</b>	<b>Hash % 5</b>
		"Tom"	84274	4
		"Dan"	68465	0
		"Harry"	69496448	3
		"Sam"	82879	4
		"Pete"	2484038	3

			Pete
	[0]	Dan	
	[1]		
	[2]		
	[3]	Harry	
Sam	[4]	Tom	

# Example of linear probing

		Pete		
		Name	Hash	Hash % 5
		"Tom"	84274	4
		"Dan"	68465	0
		"Harry"	69496448	3
		"Sam"	82879	4
		"Pete"	2484038	3

Sam	[0]	Dan
	[1]	
	[2]	
	[3]	Harry
	[4]	Tom

# Example of linear probing

Pete

[0]	Dan
[1]	Sam
[2]	
[3]	Harry
[4]	Tom

Name	Hash	Hash % 5
"Tom"	84274	4
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# Example of linear probing

Pete

[0]	Dan
[1]	Sam
[2]	
[3]	Harry
[4]	Tom

Name	Hash	Hash % 5
"Tom"	84274	4
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"Pete"	2484038	3

# Example of linear probing

[0]	Dan
[1]	Sam
[2]	Pete
[3]	Harry
[4]	Tom

Name	Hash	Hash % 5
"Tom"	84274	4
"Dan"	68465	0
"Harry"	69496448	3
"Sam"	82879	4
"Pete"	2484038	3

To find "Pete" (hash 3),  
you must start at index 3  
and work your way all the  
way around to index 2

# Searching with linear probing

To find an element under linear probing:

- Calculate the hash of the element,  $i$
- Look at  $array[i]$
- If it's the right element, return it!
- If there's no element there, fail
- If there's a *different* element there, search again at index  $(i+1) \% array.size$

We call a group of adjacent non-empty indices a *cluster*

# Deleting with linear probing

Can't just remove an element...

[0]	Dan
[1]	Sam
[2]	Pete
[3]	Harry
[4]	Tom

Name	Hash	Hash % 5
"Tom"	84274	4
"Dan"	68465	0
"Harry"	69496448	3
"Sam"	82879	4
"Pete"	2484038	3

If we remove Harry,  
Pete will be in the wrong cluster  
and we won't be able to find him

# Deleting with linear probing

Instead, mark it  
as deleted  
(*lazy deletion*)

[0]	Dan
[1]	Sam
[2]	Pete
[3]	XXXXXXXX
[4]	Tom

Name	Hash	Hash % 5
"Tom"	84274	4
"Dan"	68465	0
"Harry"	69496448	3
"Sam"	82879	4
"Pete"	2484038	3

The search algorithm  
should skip over XXXXXXXX

# Deleting with linear probing

It's useful to think of the invariant here:

- chaining: each element is found at the index given by its hash code
- Linear probing: each element is found at the index given by its hash code, *or a later index in the same cluster*

Naive deletion will split a cluster in two, which may break the invariant

Hence the need for an empty value that does not mark the end of a cluster

# Linear probing performance

To insert or find an element under linear probing, you might have to look through a whole cluster of elements

Performance depends on the size of these clusters:

- Small clusters – expected  $O(1)$  performance
- Almost-full array –  $O(n)$  performance
- If the array is full, you can't insert anything!

Thus you need:

- to expand the array and rehash when it starts getting full
- a hash function that distributes elements evenly

Same situation as with linear chaining!

# Linear probing vs chaining

In linear chaining, if you insert many values with the same hash, values with that hash become slower to access but other hashes are unaffected

In linear probing, you get a cluster and values with *nearby* hashes become slower to access too!

As the array gets close to 100% full, you get very long clusters in the hash table and performance becomes dreadful

Linear probing needs a much bigger array than linear chaining for the same performance

But: as you don't need to also create list nodes, you can create a bigger array in the same amount of memory

# Expected number of comparisons

The expected number of comparisons in a successful search is the following:

- Open addressing:  $\frac{1}{2} \left( 1 + \frac{1}{1-L} \right)$
- Chaining:  $1 + \frac{L}{2}$

$L$  is the load factor.

# Linear probing vs chaining

<b>load factor (#elements / array size)</b>	<b>#comparisons (linear probing)</b>	<b>#comparisons (linear chaining)</b>
0 %	1.00	1.00
25 %	1.17	1.13
50 %	1.50	1.25
75 %	<b>2.50</b>	1.38
85 %	3.83	1.43
90 %	5.50	1.45
95 %	10.50	1.48
100 %	—	1.50
200 %	—	2.00
300 %	—	<b>2.50</b>

# Quadratic probing

- Linear probing is not the only kind of probing. There are alternatives that try to improve the tendency of linear probing to create clusters.
- Linear probing looks at index  $(h(x)+i) \% n$  for attempt  $0,1,2,\dots$
- *Quadratic probing* is one of them. Here you look at index  $(h(x)+i^2) \% n$ . The idea is that you shouldn't get stuck locally.

# Re-hashing and lazily deleted entries

- Re-hashing (for both chaining and probing) isn't to create a larger array and copy the contents of the old one (like dynamic arrays). The hash value changes so you should insert each element in the new hash table the normal way.
- When inserting elements in a hash table with linear probing, you should re-use an index with a lazily deleted element if you come across one.

# Summary of hash table design

Several details to consider:

- *Rehashing*: resize the array when the load factor is too high
- *A good hash function*: need an even distribution
- *Collisions*: either chaining or probing
  - Other alternatives to linear probing, e.g. quadratic probing
  - Some sort of probing seems to be fastest

In return:

- *Expected* (average)  $O(1)$  performance if the hash function is random (there are no patterns)
- Better performance in practice than BSTs
- Disadvantage: hash tables are *unordered* so you can't get the elements in increasing order

Theoretical foundations of hash *functions* are a bit uncertain, but heuristics work well in practice