# Sample solutions for the examination of Models of Computation (DIT310/TDA183/TDA184) from 2017-04-11

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1. (a) $\mathit{Bool} \to \mathit{Bool}$ is countable, $\mathbb{N} \to \mathit{Bool}$ is not countable.

   (b) $\mathit{Bool} \to \mathbb{N}$ is countable, because it is in bijective correspondence with $\mathbb{N} \times \mathbb{N}$, which is countable.

2. $\mathbf{rec}\ x = f\ x$.

3. No. We can prove this by reducing the halting problem (which is not $\chi$-decidable) to $f$.

   If $f$ is $\chi$-decidable, then there is a closed $\chi$ expression $\underline{f}$ witnessing the computability of $f$. We can use this expression to construct a closed $\chi$ expression $\underline{\mathit{halts}}$ (written using a mixture of concrete syntax and meta-level notation):

   $$\underline{\mathit{halts}} = \lambda\, p.\ \underline{f}\ \ulcorner \lambda\_.\ (\lambda\_.\ \ulcorner 0 \urcorner)\ \llcorner p \lrcorner \urcorner$$

   This expression witnesses the decidability of the halting problem. Note that, for any closed expression $e \in \mathit{Exp}$,

   $$\begin{aligned}
   &\llbracket \underline{\mathit{halts}}\ \ulcorner e \urcorner \rrbracket &=\\
   &\llbracket \underline{f}\ \ulcorner \lambda_.\ (\lambda_.\ \ulcorner 0 \urcorner)\ e \urcorner \rrbracket &=\\
   &\ulcorner\ \mathbf{if}\ \llbracket (\lambda_.\ (\lambda_.\ \ulcorner 0 \urcorner)\ e)\ \ulcorner 7 \urcorner \rrbracket = \ulcorner 0 \urcorner\ \mathbf{then}\ \mathsf{true}\ \mathbf{else}\ \mathsf{false} \urcorner &=\\
   &\ulcorner\ \mathbf{if}\ \llbracket (\lambda_.\ \ulcorner 0 \urcorner)\ e \rrbracket = \ulcorner 0 \urcorner\ \mathbf{then}\ \mathsf{true}\ \mathbf{else}\ \mathsf{false} \urcorner. &
   \end{aligned}$$

   We have two cases to consider:

   - If $e$ is a closed $\chi$ expression that terminates with a value, then $\llbracket (\lambda\_.\ \ulcorner 0 \urcorner)\ e \rrbracket = \ulcorner 0 \urcorner$, and thus $\llbracket \underline{\mathit{halts}}\ \ulcorner e \urcorner \rrbracket = \ulcorner \mathsf{true} \urcorner$.
   - If $e$ is a closed $\chi$ expression that does not terminate with a value, then $\llbracket (\lambda\_.\ \ulcorner 0 \urcorner)\ e \rrbracket \neq \ulcorner 0 \urcorner$, and thus $\llbracket \underline{\mathit{halts}}\ \ulcorner e \urcorner \rrbracket = \ulcorner \mathsf{false} \urcorner$.

4. Yes. This function is constantly $\mathsf{false}$, because the expression $\mathsf{apply}\ e\ \ulcorner 7 \urcorner$ is not equal to $\ulcorner 0 \urcorner$ (which has $\mathsf{const}$ as its head constructor), no matter what $e$ is. Thus the $\chi$-decidability of the function is witnessed by the $\chi$ program $\lambda\_.\ \mathsf{False}()$.

5. (a) If the machine is run with 110 as the input string, then the following configurations are encountered:

    - $(s_0, [], [1,1,0])$.
    - $(s_2, [\underline{1}], [1,0])$.
    - $(s_2, [1,\underline{1}], [0])$.
    - $(s_3, [1,1,\underline{1}], [])$.
    - $(s_4, [1,\underline{1}], [1,0])$.
    - $(s_4, [\underline{1}], [1,1,0])$.
    - $(s_4, [], [\underline{1},1,1,0])$.
    - $(s_5, [], [1,1,1,0])$.

    The last configuration above is a halting one, with the head over the leftmost square, so the resulting string is 1110.

   (b) No. If the input is $0 \in \mathbb{N}$, i.e. the string 0, then the machine terminates successfully with the string 1, which does not correspond to a natural number.

6. The following lemma (where $PRF^-_n$ is the variant of $PRF_n$ obtained by removing $\mathsf{rec}$) implies that *is-zero* is not computable, because $0 \leq 1$ but *is-zero* $0 = 1 \nleq 0 =$ *is-zero* $1$:

   **Lemma.** *For any $n \in \mathbb{N}$, $f \in PRF^-_n$, and $\rho_1, \rho_2 \in \mathbb{N}^n$, we have that if $\rho_1 \leq \rho_2$, then $[\![f]\!]\ \rho_1 \leq [\![f]\!]\ \rho_2$. Similarly, for any $m, n \in \mathbb{N}$, $fs \in (PRF^-_m)^n$, and $\rho_1, \rho_2 \in \mathbb{N}^m$, we have that if $\rho_1 \leq \rho_2$, then $[\![fs]\!]^\star\ \rho_1 \leq [\![fs]\!]^\star\ \rho_2$.*

   *Proof.* Let us prove the two statements simultaneously, using induction on the structure of $f$ and $fs$. There are four cases for the first statement:

   - $\mathsf{zero}$: $[\![\mathsf{zero}]\!]\ \rho_1 = 0 \leq 0 = [\![\mathsf{zero}]\!]\ \rho_2$.
   - $\mathsf{suc}$: In this case $\rho_1 = \mathsf{nil}, n_1$ and $\rho_2 = \mathsf{nil}, n_2$ for some $n_1, n_2 \in \mathbb{N}$ with $n_1 \leq n_2$. We get that $[\![\mathsf{suc}]\!]\ \rho_1 = 1 + n_1 \leq 1 + n_2 = [\![\mathsf{suc}]\!]\ \rho_2$.
   - $\mathsf{proj}\ i$: $[\![\mathsf{proj}\ i]\!]\ \rho_1 = \mathit{index}\ \rho_1\ i \leq \mathit{index}\ \rho_2\ i = [\![\mathsf{proj}\ i]\!]\ \rho_2$.
   - $\mathsf{comp}\ f\ gs$: Note first that, by one inductive hypothesis, $[\![gs]\!]^\star\ \rho_1 \leq [\![gs]\!]^\star\ \rho_2$. Another inductive hypothesis lets us conclude that

   $$\begin{aligned}[\![\mathsf{comp}\ f\ gs]\!]\ \rho_1 &= [\![f]\!]\ ([\![gs]\!]^\star\ \rho_1) \\ &\leq [\![f]\!]\ ([\![gs]\!]^\star\ \rho_2) = [\![\mathsf{comp}\ f\ gs]\!]\ \rho_2.\end{aligned}$$

   Finally there are two cases for the second statement:

   - $\mathsf{nil}$: $[\![\mathsf{nil}]\!]^\star\ \rho_1 = \mathsf{nil} \leq \mathsf{nil} = [\![\mathsf{nil}]\!]^\star\ \rho_2$.
   - $fs, f$: Two separate inductive hypotheses let us conclude that

   $$\begin{aligned}[\![fs,f]\!]^\star\ \rho_1 &= [\![fs]\!]^\star\ \rho_1, [\![f]\!]\ \rho_1 \\ &\leq [\![fs]\!]^\star\ \rho_2, [\![f]\!]\ \rho_2 = [\![fs,f]\!]^\star\ \rho_2.\end{aligned}$$ □