Logic in Computer Science

Motivations for temporal logic

One uses temporal logic for specifying and proving properties of reactive systems (e.g. functional reactive programming). One remarkable aspect of the logics we study (LTL and CTL) is that they are *decidable*, contrary to first-order logic.

For Linear Temporal Logic, a proposition is not representing a value 0 or 1 but a(n infinite) binary sequence, e.g. $0, 0, 0, \dots$ or $1, 0, 1, 0, \dots$ One early application of linear temporal logic by A. Church (1957), was for the analysis of circuit. If we introduce the *delay* operation $X(b_0, b_1, \dots) = (b_1, b_2, \dots)$ then the recursive equation $a_0 = 1$ and $X(a) = \neg\ a$ defines the alterning sequence $a_0 = 1,\ a_1 = 0,\ a_2 = 1, \dots$

Linear Temporal Logic

The syntax extends the one of propositional logic by the modalities

$$\varphi\ ::=\ F\ \varphi \mid G\ \varphi \mid X\ \varphi$$

A *model* is now a function $\alpha\ p\ n$ which takes as argument a natural number n and an atomic formula p and produces 0 or 1. We define $\alpha^{(k)}\ p\ n = \alpha\ p\ (n+k)$. We can then define $\alpha \models \varphi$ by induction on φ

- $\alpha \models p$ iff $\alpha\ p\ 0 = 1$
- $\alpha \models \varphi \to \psi$ iff $\alpha \models \varphi$ implies $\alpha \models \psi$
- $\alpha \models \neg\psi$ iff not $\alpha \models \psi$
- $\alpha \models X\ \psi$ iff $\alpha^{(1)} \models \psi$
- $\alpha \models F\ \psi$ iff $\alpha^{(k)} \models \psi$ for some $k \geqslant 0$
- $\alpha \models G\ \psi$ iff $\alpha^{(k)} \models \psi$ for all $k \geqslant 0$

Another way to see this definition is to consider that a model α associates to an atomic formula a binary sequence $b_0, b_1, b_2, \dots$ and then, by induction, a binary sequence to any formula, using the definitions

$$X\ (b_0, b_1, \dots) = (b_1, b_2, \dots)$$

$$F\ (b_0, b_1, b_2, \dots) = (\bigvee_k b_k, \bigvee_{k \geqslant 1} b_k, \bigvee_{k \geqslant 2} b_k, \dots)$$

$$G\ (b_0, b_1, b_2, \dots) = (\bigwedge_k b_k, \bigwedge_{k \geqslant 1} b_k, \bigwedge_{k \geqslant 2} b_k, \dots)$$

where $1 = \bigvee_{k \geqslant n} b_k$ iff $1 = b_k$ for some $k \geqslant n$ and where $1 = \bigwedge_{k \geqslant n} b_k$ iff $1 = b_k$ for all $k \geqslant n$.

The semantics of ψ is the sequence $\alpha \models \psi,\ \alpha^{(1)} \models \psi,\ \alpha^{(2)} \models \psi, \dots$

If we write $\varphi = \psi$ for $\alpha \models \varphi$ iff $\alpha \models \psi$ and $\varphi \leqslant \psi$ for $\alpha \models \varphi \to \psi$ it can then be shown that

1. we have $F\ \varphi = \varphi \vee X\ (F\ \varphi)$ and $\varphi \vee X\ \delta \leqslant \delta$ implies $F\ \varphi \leqslant \delta$. In particular $F\ \varphi$ is the *least* solution δ of the equation $\varphi \vee X\ \delta = \delta$
2. we have $G\ \varphi = \varphi \wedge X\ (G\ \varphi)$ and $\delta \leqslant \varphi \wedge X\ \delta$ implies $\delta \leqslant G\ \varphi$. In particular $G\ \varphi$ is the *greatest* solution δ of the equation $\delta = \varphi \wedge X\ \delta$