Exercise set 1. Introduction to probability

Christos Dimitrakakis: chrdimi@chalmers.se

Note: In the following exercises, $\mathcal{S}$ will be the sample space.

Time and scoring The numbers indicate expected time to complete the exercise. If you take more time than indicated, please note that. A ! indicates that it may require some extra thought. A ? indicates that this is an open question.

This particular set of exercises does not *count towards your grade. It is good to complete them, nevertheless.*

Exercise 1 (5)**.** *Show that for any sets A, B, D:*

$$A \cap (B \cup D) = (A \cap B) \cup (A \cap D).$$

Show that

$$(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}, \quad \text{and} \quad (A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$$

Exercise 2 (10)**.** *Prove that any probability measure P has the following properties:*

1. *$P(A^{\complement}) = 1 - P(A)$.*
2. *If $A \subset B$ then $P(A) \leq P(B)$.*
3. *For any sequence of events $A_1, \ldots, A_n$*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i) \qquad \text{(union bound)}$$

Hint: Recall that If $A_1, \ldots, A_n$ are *disjoint* then $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$ and that $P(\emptyset) = 0$

Definition 1. A random variable $X \in \{0, 1\}$ has Bernoulli distribution with parameter $p > [0, 1]$, written $X \sim \mathcal{B}ern(p)$, if

$$p = \mathbb{P}(X = 1) = 1 - \mathbb{P}(X = 0).$$

The probability function of X can be written as

$$f(x \mid p) = \begin{cases} p^x (1-p)^{1-x}, & x \in \{0, 1\} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2. A random variable $X \in \{0, 1\}$ has a binomial distribution with parameters $p > [0, 1]$, $n \in \mathbb{N}$ written $X \sim \mathcal{Binom}(p, n)$, if the probability function of X is

$$f(x \mid n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, 1, \dots, n\} \\ 0, & \text{otherwise.} \end{cases}$$

If $X_1, \dots, X_n$ is a sequence of Bernoulli random variables with parameter p, then $\sum_{i=1}^n X_i$ has a binomial distribution with parameters n, p.

Exercise 3 (10)**.** *Let* $X \sim \mathcal{Bern}(p)$

1. *Show that* $\mathbb{E}\, X = p$
2. *Show that* $\mathbb{V}\, X = p(1-p)$
3. *Find the value of* p *for which* X *has the greatest variance.*

Exercise 4 (10)**.** *In a few sentences, describe your views on the usefulness of probability.*

- *Is it the only formalism that can describe both random events and uncertainty?*
- *Would it be useful to separate randomness from uncertainty?*
- *What would be desirable properties of an alternative concept?*

Finally, some questions about this unit:

1. Did you find the material interesting?
2. Did you find it potentially useful?
3. How much did you already know?
4. How much had you already seen but did not remember in detail?
5. How much have you seen for the first time?
6. Which aspect did you like the most?
7. Which aspect did you like the least?
8. Feel free to add any further comments.